Unequal connections

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Abstract

Connections seem to matter both at the individual level, in terms how well a person, a firm, or a country does, as well as for the aggregate performance of the system. The advantages that accrue from connections suggest that individuals will find it attractive to invest in forming links with others. However, an individual's links influence the payoffs of other players and hence their incentives to form links. These considerations lead us to formulate a strategic model of link formation.

We develop general methods to study spillovers across a player's own links and the links of others in the context of network models. Our analysis shows that individual incentives sharply restrict the set of permissible networks and that equilibrium networks have simple architectures. Our results also suggest that negative spillovers from the links of others lead to an unequal distribution of links, with some players having many links while others are poorly connected. By contrast, positive spillovers from the links of others lead to networks in which different players have a similar number of links. We present several applications to illustrate the scope of the analysis.

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1 Introduction

Connections seem to matter both at the individual level as well as in the aggregate. For example, it has been argued that better connected managers are higher achievers in organizations and that well-connected workers earn higher wages, that firms use collaborations to gain competitive advantage vis-a-vis competitors, social connections shape returns from individual investments in human capital, and countries which are better linked may exploit their ties to bargain for better terms in particular contexts.[1] Similarly, the aggregate effects of differences in connections can be substantial; for example, the outcome of competition among firms which have different number of collaborative ties, and hence different cost structures, is quite different from the outcome where all firms are symmetric. Given the substantial advantages that accrue from having connections, it seems natural that individuals will invest in forming links with others. This consideration motivates an examination of the implications of link formation activity for the architecture of networks. We also study the economic circumstances that give rise to unequal connections.

We develop a general model of link formation to address these issues. In our model, ex-ante identical players choose to form links with other players. Links are bilateral and costly and are formed only if both players agree to put in the required resources. The links of all the players together define a network. A network in turn defines a structure within which the social and economic interaction takes place. We suppose that for every network there is a well-defined payoff for each player. We use the notion of pair-wise stable equilibrium to solve the game.[2] We assume throughout that the costs of link formation are the same for every link and that they are linearly increasing in the number of links.

The idea of inequality is complex and has inspired a vast literature (for a survey see Sen (1998)). In this paper we shall be concerned with two aspects of inequality in networks. The first notion of inequality relates to the distribution of links across players in a network; this type of inequality can be measured by traditional concepts of range and Gini-coefficient. The second notion of inequality focuses on the nature of the relationship between linked individuals: we shall say that a link is unequal if it connects two individuals with different number of links. In this interpretation a network is unequal if the links that exist in it are unequal.

To get a first impression of the issues involved let us consider the benchmark case with no spillovers across links, i.e., the marginal returns to additional links are independent of the

[1] There is a large body of theoretical and empirical work on this subject. See e.g., Burt (1992) on careers of professional managers, Montgomery (1991) on wage inequality in labour markets, Delapierre and Mytelka (1998) and Hagedoorn and Schakenraad (1990) on collaborations among firms, Durlauf (2000) on the memberships approach to inequality, and Siedman (2001) on international trading ties. There is an extensive literature in sociology on issues of power and inequality in social networks; see e.g., Wasserman and Faust (1994).

[2] A pair-wise stable equilibrium network is one which is supported by a Nash equilibrium strategy profile, and in addition satisfies the property that there is no pair of players who would be better off by forming an additional link. See sections 2 and 6 for a discussion on alternative solution concepts in network games.

network structure. The network formation process then yields a simple answer: if these marginal returns are larger (smaller) than the constant marginal costs then we get the complete (empty) network. Thus, in the absence of spillovers all the players have the same number of links, and this number is either zero or the maximum possible. This finding leads us to explore the role of spillovers in generating inequality. We study settings with spillovers across a player's own links as well as the spillovers generated by the links formed among other players. The formulation of spillovers in a network context is complicated by the fact that indirect effects of links can travel across networks in a variety of different ways. One of the main contributions of the paper is the development of general methods to study these different possibilities.

We first analyze games in which the payoffs depend on the number of links a player has and the number of links of the rest of the population. We refer to this as a *playing the field* game. A variety of economic applications – examples include Cournot firms forming cost-reducing collaboration links and firms sharing knowledge in a patent race – satisfy this property (see examples 1 and 2 in section 5). In this setting, marginal returns to a player from an additional link are the same irrespective of whom he connects with; thus the identity of the partners is not critical per se.

For playing the field games, we begin by studying *positive spillovers* across own links: given any network, the marginal returns to an additional link are increasing in the number of links formed by the player. Our analysis shows that equilibrium networks have the *dominant group* architecture (Proposition 3.1). Figure 1 illustrates the set of equilibrium networks for $n = 5$. The dominant group network has two groups of players, one group consists of a clique of completely connected players (which is referred to as a complete component) and another group which consists of isolated players. The intuition behind this result is as follows: Consider a network in which two players i and j have formed some links. This means that marginal returns from links are larger than the marginal costs of forming them. Given that the cost of forming links remains constant and marginal returns from links are increasing, it then follows that the two players have an incentive to form a link with each other. Thus every pair of players who have any links must also be linked with each other. This argument does not apply to isolated players who may not have an incentive to link with anyone. Hence, every (non-empty) network will have at most one non-singleton component, and the rest of the players will be isolated. This result shows that individual incentives sharply restrict the range of possible architectures. In a 5 player game, there are 34 distinct architectures (Harary, 1972) but only 5 architectures – which can be parameterized in terms of the size of the dominant group – can arise in equilibrium (see Figure 1).[3] We also examine how spillovers from the links of the rest of poplulation play a role in determining the size of the dominant group. If marginal returns are increasing in the total links of others, i.e., there are positive spillovers across links of others, then the dominant group will include all players, while if spillovers are negative then the dominant group is likely to include only a strict subset of the players.

[3]Two networks have the same architecture if one can be derived from the other by permuting the strategies of the players.

We continue by considering the case of *negative spillovers* across own links: the marginal returns from an additional link are decreasing in the number of links a player has. In this setting, a wide range of symmetric networks, with everyone having the same number of links, can arise in equilibrium. In our analysis of asymmetric networks – where players have unequal number of links – we find that the architecture of equilibrium networks depend crucially on the nature of spillovers across the links of others. If these spillovers are positive then equilibrium networks have the following property: *there is a direct link between all non-maximally connected players* (Proposition 3.2). This property rules out networks such as the dominant group and star networks.[4] Figure 2 illustrates equilibrium networks in this case. By contrast, if there are negative spillovers across third party links then a wide variety of asymmetric architectures such as the star and dominant group can arise. Figure 3 presents some examples of equilibrium networks in this case. The intuition behind this finding is as follows: as a player builds up links, the marginal returns from links go down and he will eventually stop forming additional links. The returns to others from forming links, however, may go down even more sharply due to negative spillovers and thus an unequal distribution of links across players can be sustained. This argument also suggests why sharp differences in the number of links are difficult to sustain when spillovers of a player's links on other players are positive.

We next examine games in which marginal returns from a link depend crucially on the level of connections of the partner in the network. In particular, the marginal links of player i from a link with player j depend on the number of links of player i as well as the number of links of player j. We refer to this as a game with *local spillovers.* A variety of economic applications – examples include market sharing agreements between firms not to enter each other's home market and free trade agreements between countries to eliminate tariffs – satisfy the local spillovers property (see examples 4 and 5 in Section 5). The payoffs can be increasing or decreasing in each of the two arguments; hence there are four types of local spillover games in all. If marginal returns are increasing in the number of own as well as partner's links then an equilibrium network is likely to be either empty or complete (Proposition 4.1 and the subsequent discussion). By contrast, if spillovers are negative with respect to own links and the links of others then a variety of asymmetric networks, including stars, can arise in equilibrium (Proposition 4.6). Figure 6 illustrates these equilibrium networks. We also show that in the two cases of mixed spillovers, equilibrium networks have similar architectures – with a set of isolated players and a collection of unequal sized groups of completely linked players – which display moderate asymmetry only (Propositions 4.3 and 4.5). Figure 5 illustrates these networks.

We now discuss our findings from the perspective of inequality. We note first that both the links distribution perspective as well as the unequal relations approach imply that symmetric networks are equal. The ranking of asymmetric networks can be quite different in the two approaches. For example, the dominant group network has an unequal distribution of links but these links are between players with an equal number of links hence this network architecture

[4] A star is a network architecture in which there is a player who is linked to each of the other players, while the latter do not have any links between them.

is unequal according to the first approach but not according to the latter approach. Keeping these considerations in mind, an inspection of the above results suggests the following general pattern with regard to inequality. Consider first the case of positive spillovers across own links. In this setting, negative spillovers from the links of others facilitate the emergence of an unequal distribution of links in equilibrium networks. However, these links are usually between players with a similar number of links. Consider next the case of negative spillovers across own links. Here, negative spillovers from the links of others lead to relatively more inequality as compared to positive spillovers. Moreover, in this case equilibrium networks have the feature that links are often unequal: linked players have unequal number of links.

The conditions on payoffs that we identify are simple and appealing from an economic point of view. In section 5 we present several economic applications to illustrate the scope of the enquiry. These examples apply the above results, illustrate the existence of equilibrium and demonstrate that the characterization results reported above are tight in most cases.

Our paper is intimately related to the research on interaction structure and economic performance (see e.g., Bala and Goyal (1998), Benabou (1993), Ellison and Fudenberg (1993), Glaezer, Sacredote and Schienkman (1996), Goyal (1996), and Morris (2000). Also refer to Granovetter (1985) for a discussion of the role of embeddedness in economic transactions and Schelling (1975) for early work in economics on the role of social interaction.) This literature on interaction structure and economic performance has highlighted the different ways in which structure, broadly construed, affects key economic outcomes. This research is perhaps the principal motivation underlying the research on a theory of network formation. In this paper we take the view that in many interesting settings – since connections deliver large payoff advantages – individual entities consciously decide on whether to form links/ties with others. Thus social and economic structures arise out of individual incentives. In making these choices, individuals trade off the benefits of forming links against the costs of doing so.[5]

Our paper is thus a contribution to the theory of of network formation. This is currently a very active field of research; see e.g., Aumann and Myerson (1989), Bala and Goyal (2000), Boorman (1975), Calvo (2002), Dutta, van den Nouweland and Tijs (1995), Kranton and Minehart (2001), Jackson and Watts (2002), Jackson and Wolinsky (1996), and van den Nouweland and Slikker (2001)).[6] The present paper makes three contributions to this body

[5]There is also a literature in physics on the architecure of empirically observed networks such as the world wide web, the network of citations and networks of co-authors. In this literature, the focus is on statistical properties such as scale-free distribution of links. There is also a small body of work on mechanical models of network evolution which generate unequal distributions of links (see Albert, Jeong and Barabasi (2000) and the references given there). By contrast, in the present paper, we examine how strategic link formation by rational players gives rise to unequal connections.

[6]Traditionally, group formation has been studied with the help of coalition formation models (see Bloch (1997) for a survey of this literature, and Bloch (1996) and Yi (1997), for related papers on coalition formation with externalities. For a discussion on the relationship between networks and coalitions, refer to Jackson and Wolinsky (1996). Jackson (2001) provides a survey of research on the relationship between stability and efficiency in network models, while Jackson and Watts (2002) study the stochastic stability of pair-wise stable networks in some economic examples.

of work. The *first* contribution is the general formulation of positive and negative spillovers. We illustrate the generality of our formulation by presenting a wide range of economic applications in which individual incentives are appropriately reflected in the dependence of marginal returns on specific features of the network, such as the number of own links, the total number of links of the rest of the players, and the links of the potential partner. Some of these applications are novel while others have been taken from other papers (see below for citations). These applications typically make fairly strong assumptions on the functional forms, and deal with particular examples. Our paper locates these applications within a common framework and thereby clarifies the essential economic factors at work. The *second* contribution of our paper is the characterization of equilibrium network architectures in a broad class of games. This characterization helps us understand the relationship between different types of spillovers and inequality in networks. The *third* contribution of our paper is of a methodological nature: our results illustrate that in a broad class of games, fairly limited knowledge of the network is sufficient for deriving restrictions on permissible networks. This is evident in the case of playing the field games, where the knowledge of the number of own links and the aggregate number of links in the rest of the population suffices to derive strong restrictions on network architectures.

The paper is organized as follows. Section 2 presents the model. Section 3 studies situations in which marginal returns depend only on the number of links of a player. Section 4 analyzes situations in which marginal returns from a link depend crucially on the number of links of the potential partner as well. Section 5 illustrates the scope of the results by applying the results to a variety of social and economic examples. Section 6 presents extensions of the basic model – a model of local and global spillovers, a study of strongly stable networks and network formation with one-sided links. Section 7 concludes. All the proofs are given in an Appendix at the end of the paper.

2 The Model

Link formation game: Let $N = \{1, 2, ..., n\}$ denote a finite set of ex-ante identical players. We shall assume that $n \geq 3$. Every player makes an announcement of intended links. An intended link $s_{i,j} \in \{0, 1\}$, where $s_{i,j} = 1$ means that player i intends to form a link with player j, while $s_{i,j} = 0$ means that player i does not intend to form such a link. Thus a strategy of player i is given by $s_i = \{\{s_{i,j}\}_{j \in N \setminus \{i\}}\}$. Let S_i denote the strategy set of player i. A link between two players i and j is formed if and only if $s_{i,j} = s_{j,i} = 1$. We denote the formed link by $g_{i,j}$. A strategy profile $s = \{s_1, s_2, ..., s_n\}$ therefore induces a network $g(s)$. For expositional simplicity we shall often omit the dependence of the network on the underlying strategy profile. A *network* $g = \{(g_{i,j})\}$, is a formal description of the pair-wise links that exist between the players. We let $\mathcal{G}$ denote the set of all networks (the set of all undirected networks with n vertices.) Also let $\mathcal{N}_i(g) = \{j \in N : j \neq i,\ g_{i,j} = 1\}$ be the set of players with whom player i has a link in the network g, and let $\eta_i(g) = |\mathcal{N}_i(g)|$ be the cardinality of this set.[7]

[7]See Dutta, van den Nouweland and Tijs (1995) for an earlier model of link announcments.

Given a strategy profile $s = \{s_1, s_2, ..., s_n\}$, the (net) payoffs to a player are given by

$$\Pi_i(s_i, s_{-i}) = \pi_i(g(s)) - \eta_i(g(s))f \tag{1}$$

where f is the cost incurred by each player when a link is formed.[8] We study the architecture of networks that are strategically stable. Our notion of strategic stability is a refinement of Nash equilibrium. A strategy profile $s^* = \{s_1^*, s_2^*, ..., s_n^*\}$ is said to be a Nash equilibrium if $\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*)$, for all $s_i \in S_i$, and for all $i \in N$. In our model, a link requires that both players acquiesce in the formation of the link. It is then easy to see that an empty network is always a Nash equilibrium. More generally, for any pair i and j, it is always a mutual best response for the players to offer to form no link. To avoid this potential coordination problem we supplement the idea of Nash equilibrium with the requirement of pair-wise stability. An equilibrium network is said to be pair-wise stable if any pair of players have no incentive to form a link that does not exist in the network. We borrow this idea from Jackson and Wolinsky (1996). We will focus on pair-wise stable equilibrium networks.

Definition 2.1 *A network g is a pair-wise stable equilibrium network if the following conditions hold:*

1. *There is a Nash equilibrium strategy profile which supports g.*
2. *For $g_{i,j} = 0$, $\pi_i(g + g_{i,j}) - \pi_i(g) > f \implies \pi_j(g + g_{i,j}) - \pi_j(g) < f$*

In what follows, for expositional simplicity we shall use the short form – pws-equilibrium – while referring to pair-wise stable equilibrium networks.[9] The notion of pair-wise stability is clearly very mild and there are different ways in which we can enrich the set of deviations allowed to players. There are two main reasons for using this solution concept. The first reason is that it is tractable and the second reason is that it yields sharp restrictions on the set of permissible networks and this generates a range of insights which appear to be robust. For example one possible extension would be to have coordinated deviations by any subset of players. Clearly the deviation we consider is then a special case and the set of equilibrium under the stronger notion will be a refinement. We discuss this extension in section 6. Another interesting extension would allow for transfers across players. It is worth noting that the equilibrium requirement presented above requires that players individually have no incentives to form additional links. i.e., no transfers are permitted across players. The stability notion can be modified to check for links that are not in the joint interests of a pair of players. Our conjecture is that allowing for transfers will not alter the results on

[8] In this paper we assume that costs of link formation are constant across all links and focus on effects of network structure on the gross returns function. Clearly, an analogous analysis can be carried out if we assume that returns from links are constant while the costs of links vary as a function of the network.

[9] We note that the pair-wise stability requirement is different from the refinement of perfection. The latter rules out Nash equilibrium in weakly dominated strategies; the requirement of pair-wise stability is a local condition and so in settings with negative spillovers, there can exist perfect equilibrium networks which are not pair-wise stable.

symmetric equilibrium networks, but will have an impact on the stability of some asymmetric networks.[10]

2. Networks: Given a network g, $g + g_{i,j}$ denotes the network obtained by replacing $g_{i,j} = 0$ in network g by $g_{i,j} = 1$, while $g - g_{i,j}$ denotes the network obtained by replacing $g_{i,j} = 1$ in network g by $g_{i,j} = 0$. There exists a *path* between i and j in network g if either $g_{i,j} = 1$ or if there is a distinct set of players $\{i_1, \ldots, i_n\}$ such that $g_{i,i_1} = g_{i_1,i_2} = g_{i_2,i_3} = \cdots = g_{i_n,j} = 1$. A network is *connected* if there exists a path between any pair $i, j \in N$. A network, $g' \subset g$, is a *component* of g if for all $i, j \in g'$, $i \neq j$, there exists a path in g' connecting i and j, and for all $i \in g'$ and $k \in g$, $g_{i,k} = 1$ implies $k \in g'$. A component $g' \subset g$ is *complete* if $g_{i,j} = 1$ for all $i, j \in g'$.

We shall say that a network is symmetric if every player has the same number of links, i.e., $\eta_i(g) = \eta(g)$ $\forall i \in N$. We refer to η as the *degree* of the network. The *complete* network, g^c, is a symmetric network in which $\eta = n-1$, $\forall i \in N$, while the *empty* network, g^e, is a symmetric network in which $\eta = 0$, $\forall i \in N$. We shall say that a network is asymmetric if there is at least one pair of players who have a different number of links. Let $N_1(g), N_2(g), ..., N_m(g)$ be a partition of players, corresponding to the number of links that players have, i.e., $i, j \in N_k(g)$, $k = 1, 2, ..m$, if and only if $\eta_i(g) = \eta_j(g)$. We note that k here refers to the order in the partition and not the precise number of links that players have. Our analysis will highlight two asymmetric network architectures: *inter-linked stars* and *dominant group* networks. An inter-linked stars architecture has at least two members in the above partition, and the minimally and maximally linked groups, respectively, satisfy the following two conditions: (i). $\eta_i(g) = n-1$ for $i \in N_m(g)$ and (ii). $\mathcal{N}_i(g) = N_m(g)$ for $i \in N_1(g)$. The star network is a special case of such an architecture with $|N_m(g)| = 1$ and $|N_1(g)| = n-1$. A *dominant group* architecture is characterized by one complete non-singleton component and a set of singleton players. Thus there are two groups, $N_1(g)$ and $N_2(g)$, with the property that $\eta_i(g) = 0$, for $i \in N_1(g)$, while $\eta_j(g) = |N_2(g)| - 1$, for $j \in N_2(g)$.

We present two alternative approaches to inequality in networks. The first approach focuses on the distribution of links. By way of illustration, we compute the range and Gini-coefficient of different network architectures. The range for a network g is defined as

$$R(g) = \max_{i \in N} \eta_i(g) - \min_{i \in N} \eta_i(g)$$

The range of a symmetric network is 0 and a higher range is to be interpreted as an increase in inequality. The range of a network with a k member dominant group is $k-1$. This is clearly increasing with the size of the dominant group but equals O once the dominant group contains all players (and the network is therefore symmetric). The range of an inter-linked

[10]For instance, in the study of networks of cost-reducing collaboration between firms, we found that introducing transfers across firms led to a greater amount of asymmetry (see Goyal and Joshi, 2002).

star with k central players is $n-1-k$. This is decreasing in k, with the maximum value attained for the star.

We compute the Gini-coefficient next. Let $\mu(g)=\sum_{i\in N}\eta_i(g)/n$ be average number of links in a network g. We can renumber the players so that player 1 has the largest number of links, player 2 the second largest and so on until player n who has the smallest number of links. We define the Gini coefficient of the network g as follows:

$$G(g)=1+\frac{1}{n}-\frac{2}{n^2\mu(g)}\left[\eta_1(g)+2\eta_2(g)+...+n\eta_n(g)\right]$$

The first thing to note is that $G(g)=0$ for symmetric networks and higher values of the coefficient signify a greater level of inequality. The Gini coefficient for a dominant group network with a dominant group of size k is given by

$$G(g^{kd})=\frac{n-k}{n} \tag{2}$$

We note that this is decreasing in k and attains the value of 0 when the dominant group includes all players (and the network is complete). The Gini coeffient for the inter-linked stars with k central players is given by

$$G(g^{ks})=\frac{n+1}{n}-\frac{k+1}{n}-\frac{n-k}{2n-1-k} \tag{3}$$

We note that the Gini measure is initially increasing and then decreasing in the number of central players (attaining a value of 0 when all players are central and the network is complete). We also note that $G(g^{kd})>G(g^{ks})$, for all $0<k<n$; thus a dominant group network with k members of the dominant group is more unequal than a inter-linked star with k central players. However, there is no general ranking between dominant group networks and inter-linked stars; the dominant group with $n/2+1$ or more players has a smaller Gini coefficient than the star.

We turn next to the unequal connections notion of inequality. Let $\eta(g)=\sum_{i\in N}\eta_i(g)/2$ be the total number of links in a network g. Define $u(i,j;g)=|\eta_i(g)-\eta_j(g)|$ to be the level of inequality in a connection and let

$$u(g)=\frac{2}{\sum_i\eta_i(g)}\sum_{i,j:g_{i,j}=1}u(i,j;g) \tag{4}$$

define the level of unequal connectedness in a network. It is easy to see that for every symmetric network and dominant group network $u(g)=0$. Moreover, the maximal level of inequality in a link is $n-2$, and therefore the star attains the maximal value of $u(g)$ across all networks.

3 Playing the field games

In this section we consider a class of network formation games in which the marginal returns from links for a player can be expressed in terms of the number of links of the player and the aggregate number of links of the rest of the players. Formally, let g_{-i} be the network obtained by deleting player i and all his links from the network g. Then the payoffs of player i respect the playing the field property if:

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \Phi(\eta_i(g), \sum_{j \neq i} \eta_j(g_{-i})) \qquad (A.1)$$

In section 5 below we present two economic examples – cost-reducing collaboration among firms in quantity competition and collaboration among competitors in a patent race – that satisfy this property.

We start by examining the case where a player's marginal payoffs are strictly increasing in his own links.

Definition 3.1 *The payoff function of player i displays positive spillovers with respect to own links (PSOL) if the marginal returns $\Phi(\eta_i(g), \sum_{j \neq i} \eta_j(g_{-i}))$ are strictly increasing with respect to $\eta_i(g)$.*

In the context of playing the field games, positive spillovers with respect to own links have powerful implications for the architecture of pws-equilibrium networks.

Proposition 3.1 *Suppose the payoffs of each player satisfy (A.1) and (PSOL). Let g be an pws-equilibrium network. If g is symmetric then it is empty or complete. If g is asymmetric then it has the dominant group architecture.*

Figure 1 illustrates the set of pws-equilibrium networks under (PSOL).

— Insert Figure 1 somewhere here —

The proofs of all results are given in the Appendix. The first step in the proof establishes a generalized transitivity property: suppose g is an equilibrium network. If players i and j have a link in g then they also have a link with each other. This statement is a consequence of the constant costs of links and the positive spillovers hypothesis. This property implies that if g is a symmetric network then it must be empty or complete. This property also implies that if two players are in a component then they must have a link with each other, in other words every (non-singleton) component must be complete. Moreover, any network with two non-singleton components will be vulnerable to a deviation by players from each of the two components. Hence an equilibrium network can have at most one singleton component.

These arguments establish that an equilibrium network will have at most one non-singleton component and that this component will be complete, in other words, an equilibrium network has a dominant group architecture.

This result shows that in the context of playing the field games, the property of positive spillovers with respect to own links sharply restricts the architecture of networks that can arise in a pws-equilibrium. In particular, only the two trivial symmetric networks – the empty and the complete – can arise in a pws-equilibrium. Moreover, in the class of asymmetric networks there is a unique architecture that can arise in a pws-equilibrium: the dominant group network. What can we say about the size of the dominant group in a pws-equilibrium? This will depend on the nature of spillovers from the links of others, in other words on the effect of $\sum_{j \neq i} \eta_j(g_{-i})$. If the marginal returns, $\Phi(.,.)$, are increasing with respect to the links of others then players will be encouraged to form links as others form links, and this suggests that an equilibrium network will be either empty or complete. On the other hand, if the marginal returns are negatively affected by third party links then a range of dominant group can be sustained in a pws-equilibrium, depending on the costs of forming links.

We now take up the case where marginal returns to an additional link are decreasing with respect to own links.

Definition 3.2 *The payoff function of player i satisfies negative spillovers with respect to own links (NSOL) if the marginal returns $\Phi(\eta_i(g), \sum_{j \neq i} \eta_j(g_{-i}))$ are decreasing in $\eta_i(g)$.*

In this case, we find it helpful to analyze the two cases of positive spillovers and negative spillovers with respect to third party links separately. We start with the case where marginal returns are increasing in the links of others.

Definition 3.3 *The payoff function of a player satisfies positive spillovers with regard to third party links (PSTP) if the marginal returns $\Phi(\eta_i(g), \sum_{j \neq i} \eta_j(g_{-i}))$ are increasing in $\sum_{j \neq i} \eta_j(g_{-i})$.*

Our analysis of network formation in the presence of negative spillovers with respect to own links and positive spillovers with respect to third party links is summarized in the following result.

Proposition 3.2 *Suppose the payoff function of every player satisfies (A.1), (NSOL) and (PSTP). Let g be a pws-equilibrium network. For a given value of f, a symmetric g can have more than one degree. If g is asymmetric then it has the property that all non-maximally linked players are mutually linked.*

Figure 2 illustrates the set of equilibrium networks under (NSOL) and (PSTP).

— Insert Figure 2 somewhere here —

The intuition for the first part of the result is as follows: suppose that spillovers across others' links are zero. Then every player has diminishing returns from links and it is easy to see that an equilibrium number of links (roughly) corresponds to the point where marginal returns are equal to the cost of links. Thus a network with everyone having the same number of links is a natural candidate for equilibrium. Now it is easy to see that additonal equilibria can be generated due to positive spillovers from others links (this is simply an outward shifting of the individual marginal returns curve, while the costs remain constant.) We now turn to the asymmetic networks. Suppose that $1 \in N_m(g)$ and $2, 3 \in N_k(g)$, with $k \neq m$. It follows that $\Phi(\eta_1(g) - 1, x) \geq f$, where $x = \sum_{j \neq 1} \eta_j(g_{-1})$. However, the marginal returns to player 2 from forming a link with player 3 are strictly greater than this since $\eta_2(g) \leq \eta_1(g) - 1$, $\sum_{j \neq 1} \eta_j(g_{-1}) < \sum_{l \neq 2} \eta_l(g_{-2})$. These inequalities hold by hypothesis on the number of links of 1 and 2. Taken along with (NSOL) and (PSTP) they imply that player 2 has a strict incentive to form a link with 3. Similar argument applies for player 3. Since players i and j were arbitrary, this shows that every pair of non-maximally connected players will be mutually linked.

The *first* part of the above result shows that symmetric networks are easily sustained in a pws-equilibrium, and that for the same value of f, different degree networks can arise. This is in marked contrast to the situation under increasing returns where symmetric networks are in general not sustainable in a pws-equilibrium. The *second* part of the result rules out networks such as stars and dominant group architectures with two or more isolated players. It also rules out inter-linked stars. However, positive spillovers do allow for asymmetrically-sized complete components. One may interpret this result as saying that (NSOL) and (PSTP) together imply that networks can be at most moderately asymmetric. This is perhaps most visible with regard to the unequal connections idea. Figure 2 suggests that almost all links in an equilibrium network are between players who have the same number of links.

We finally take up the case where links of third parties have a negative spillover on a player's marginal returns.

Definition 3.4 *The payoff function of a player satisfies negative spillovers with regard to third party links (NSTP) if the marginal returns* $\Phi(\eta_i(g), \sum_{j \neq i} \eta_j(g_{-i}))$ *are decreasing in* $\sum_{j \neq i} \eta_j(g_{-i})$.

We first take up the case of symmetric networks.

Proposition 3.3 *Suppose payoffs of every player satisfy (A.1) and (NSOL) and (NSTP). For a given value of* f*, symmetric networks of at most one degree* k *can be sustained in a pws-equilibrium.*

For a symmetric network of degree k to be an equilibrium it must be the case that $\Phi(k, (n-2)(k)) \leq f \leq \Phi(k-1, (n-2)k)$. It is clear that, under the hypotheses (NSOL) and (NSTP), two or more degrees cannot simultaneously satisfy this requirement. We explore next the nature of asymmetric pws-equilibrium networks. We have been unable to provide a general characterization here. The following example illustrates some of the issues that arise.

Example 3.1: Suppose $n = 3$ and individual payoffs are given as follows:

$$\pi_i(g) = \alpha\sqrt{\eta_i(g)} - \beta \sum_{j \neq i} \eta_j(g)\eta_i(g), \tag{5}$$

where $\alpha > \beta > 0$. It is possible to verify that this payoff function satisfies (NSOL) and (NSTP). This is done as follows:

$$\Phi(\eta_i(g), \sum_{j \neq i} \eta_j(g_{-i})) \;=\; \alpha(\sqrt{\eta_i(g)+1} - \sqrt{\eta_i(g)}) - \beta \sum_{j \neq i} \eta_j(g_{-i}) - \beta - \beta\eta_i(g) \tag{6}$$

Clearly, this is decreasing in η_i and also decreasing in $\sum_{j \neq i} \eta_j(g_{-i})$. The pws-equilibrium networks are characterized as follows: *(a). If* $f < 0.41\alpha - 5\beta$ *then the unique pws-equilibrium is the complete network. (b). If* $0.41\alpha - 5\beta < f < 0.41\alpha - 3\beta$ *then the unique pws-equilibrium is the star network. (c). If* $0.41\alpha - 3\beta < f < \alpha - \beta$ *then the unique pws-equilibrium is the partially connected network. (d). If* $\alpha - \beta < f$ *then the unique pws-equilibrium is the empty network.*

This characterization follows from simple computations of payoffs in the different networks.[11]

△

Figure 3 illustrates some equilibrium networks for the above example for the case of $n = 5$.[12]

— Insert Figure 3 somewhere here —

The above figure shows that negative spillovers from others' links imply that poorly linked players no longer have an incentive to form links and the property of non-maximally connected players having a direct link which holds under positive third party spillovers, Proposition 3.2, no longer holds. This implies that sharply asymmetric networks such as stars can arise in equilibrium. Recall that a star displays an unequal distribution of links as well as maximal unequal connectedness!

In this section we have analyzed a class of network formation games in which the payoffs satisfy the playing the field property, i.e., the marginal returns to a link are a function

[11] It is easily checked that $\Pi_i(g^c) = 1.41\alpha - 8\beta - 2f$, for all $i \in N$, while $\Pi_i(g^e) = 0$, for all $i \in N$. In the partially connected network, the payoffs of the linked players are $\Pi_i(g^{pc}) = \alpha - \beta - f$, while the payoffs of the isolated player are $\Pi_j(g^{pc}) = 0$. Finally, in the star, the payoffs of the central player are given by $\Pi_i(g^s) = 1.41\alpha - 4\beta - 2f$, while the payoffs of the peripheral players are given by $\Pi_j(g^s) = \alpha - 3\beta - f$.

[12] In Figure 3, the line network is a pws-equilibrium for $0.41\alpha - 8\beta < f < 0.41\alpha - 6\beta$; the star network is a pws-equilibrium over the range $0.41\alpha - 8\beta < f < 0.27\alpha - 4\beta$; the interlinked star with two center players is a pws-equilibrium for $0.32\alpha - 13\beta < f < 0.27\alpha - 10\beta$; the interlinked star with three center players is a pws-equilibrium for $0.27\alpha - 16\beta < f < 0.27\alpha - 14\beta$.

of the number of links of the player in question and the aggregate number of links of the rest of the players. Our analysis suggests that simple restrictions on the marginal returns yield a sharp characterization of pws-equilibrium networks. Our *first* observation is that it is difficult to sustain (non-trivial) symmetric networks in the presence of positive spillovers with respect to own links. By contrast, a variety of symmetric networks arise naturally in a pws-equilibrium under negative spillovers with respect to own links. Our *second* remark pertains to asymmetric networks. Here we found that the nature of spillovers across third party links plays a crucial role. We first analyzed positive spillovers across own links. In this case, if the spillovers across third party links are negative, then this leads to the dominant group network. On the other hand, if spillovers across third party links are positive then asymmetries are difficult to sustain and typically a network is either empty or complete. The nature of asymmetries under negative spillovers with respect to own links are also similarly affected by the spillovers from third party links. If these spillovers are positive, then we found that only a moderate amount of asymmetry can arise in a pws-equilibrium (dominant groups with two or more isoalted players and stars cannot arise in a pws-equilibrium). On the other hand, if spillovers from third party links are negative, then significant asymmetry can reappear in a pws-equilibrium; for example, stars can arise in a pws-equilibrium. Thus spillovers from third party links are crucial to an understanding of the extent of the inequality that can arise in networks.

4 Local spillovers

In this section we consider a class of games in which the marginal returns of a player from a link are a function both of the number of links of the player and that of the potential partner. In this setting marginal returns are given by:

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \psi(\eta_i(g), \eta_j(g)) \qquad (A.2)$$

In contrast to playing the field models, the identity of the potential partner now plays a crucial role. We refer to these games as games of *local spillovers*. These spillovers capture the indirect benefits or costs transmitted across links when a player forges a link and gains access to the partner's links and concedes access to own links. Spillovers are called local because marginal returns of a player are not influenced by the number of links of those who are linked to the player and the potential partner. In section 5 we present three economic applications – information sharing in a public good problem, market-sharing agreements between firms, and free-trade agreements between countries – where the marginal payoffs satisfy the local spillovers property. We note that (A.2) places restrictions on the marginal returns and in general allows for individual payoffs to depend on the entire network. The market sharing example (see section 5 below) illustrates this point clearly.

We distinguish between positive and negative spillovers from own or partner's links based respectively on whether marginal payoffs from a link are influenced favorably or adversely by these links. There are thus four possible cases in all. We take them up in turn.

We start by studying the case where the marginal returns satisfy positive spillovers with respect to own links (PSOL) as well as the links of the potential partners (PSPL), i.e., $\psi(\eta_i, \eta_j)$ is increasing in both η_i and η_j. In section 5, we present a pure public good game where the payoffs satisfy these properties. In this setting, pws-equilibrium networks are characterized as follows:

Proposition 4.1 *Suppose payoffs of every player satisfy (A.2), (PSOL) and (PSPL). Let g be a pws-equilibrium network. If g is symmetric then it is either empty or complete. If g is an asymmetric network, then it has at most one non-singleton component and this component is either complete or has an inter-linked stars architecture. Moreover, if $\eta_i(g) \neq \eta_j(g)$ then $|\eta_i(g) - \eta_j(g)| \geq 2$.*

Figure 4 illustrates pws-equilibrium networks under (PSOL) and (PSPL).

— Insert Figure 4 somewhere here —

The proofs of all the results are given in the Appendix. There are four steps in the proof. *First*, we show that if g is symmetric then it is either empty or complete. Note that if a network with degree k (where $0 < k < n-1$) is a pws-equilibrium, then $\psi(k-1, k-1) \geq f$. However, (PSOL) and (PSPL) imply that $\psi(k, k) > \psi(k-1, k-1) \geq f$, and this contradicts the hypothesis that g is a pws-equilibrium. The *second* step in the proof shows that there is at most one non-singleton component in a pws-equilibrium network g. If there are two non-singleton components, then we can always find a pair of players, i and j, one in each component, who have an incentive to form a link with each other. In establishing this, we exploit the fact that since g is a pws-equilibrium, the marginal returns from current links for i and j must exceed the costs of forming links. We use this fact in combination with (PSOL) and (PSPL) to infer that the returns from the next link, the link with each other, will exceed the costs of forming links as well. We now turn to the structure of the non-singleton component. If this component is symmetric, then the above argument shows that it must be complete. The only other possibility is that it is asymmetric and then there are maximally and minimally connected groups of players. The *third* step in the proof shows that a maximally connected player in the non-singleton component must have a link with all players. We start by showing that a maximally linked player is linked with all minimally linked players. We then use (PSOL) and (PSPL) to argue that the maximally linked player and every non-minimally linked player have an incentive to form a link with each other. The *final* step in the argument shows that a minimally linked player forms links only with the maximally connected players. This step uses the idea that if a minimally linked player is linked with a non-maximally connected player i, then this player i must also have an incentive to link with everyone else, and moreover this incentive will be reciprocated, implying that the player i will be directly linked with every other player. This however shows that i is maximally linked, contradicting the initial hypothesis.

Proposition 4.1 shows that positive spillovers with respect to own and partners links have powerful implications for the nature of pws-equilibrium networks. In particular, it is worth contrasting this result with the results for playing the field games. Recall that in the playing the field game, if there are positive spillovers across own links, then a pws-equilibrium network is either empty, complete or a dominant group network. The above result shows that, in games with local spillovers, if there are positive spillovers across own links as well as partner's links, then an additional network architecture can emerge in a pws-equilibrium: the inter-linked star.

How likely is it that these networks will actually arise in equilibrium? In this connection we would briefly like to discuss the role of richer deviations by players.[13] Take an equilibrium network with a dominant group, and suppose that player i is a typical player in the dominant group, while player j lies outside the dominant group. This network is stable because at least one of the two players, i and j, does not wish to form a link with each other. The reason for this is that player j has very few (zero) links and marginal returns are increasing in both own links as well as partner's links. However, let us look at the incentives to form links between player j and every player in the dominant group. It can be checked that the marginal returns curve for the isolated player j lies above the marginal returns curve for player i who is in the dominant group (due to PSPL). Hence player j would be willing to form links with all of the dominant group players at the same time. On the other hand, since player i will now be linked to all the members of the dominant group, an individual player in the dominant group will also be willing to form a link with player j (due to (PSOL) and (PSPL)). This suggests that the dominant group network is not stable against deviations by players which involve the formation of a large set of links simultaneously. A similar argument can be used to show that the inter-linked stars network is not stable to a deviation in which one peripheral player forms links with each of the other peripheral players. These considerations suggest that if players are able to coordinate and carry out coordinated deviations, then under conditions (PSOL) and (PSPL), an incentive compatible network will be either empty or complete.

We next study the case where the marginal returns exhibit positive spillovers with respect to own links (PSOL) but negative spillovers with respect to the links of the potential partners (NSPL), i.e., $\psi(\eta_i, \eta_j)$ is increasing in η_i and decreasing in η_j. In section 5, we present a market sharing game in which firms form links with a view to not entering each other's market and show that it satisfies these properties.

In order to characterize symmetric pws-equilibrium networks, it is useful to distinguish between two cases:

(M) $\psi(\eta + 1, \eta + 1) > \psi(\eta, \eta)$ for $\eta \in \{0, 1, 2, ..., n - 2\}$.

(M') $\psi(\eta + 1, \eta + 1) < \psi(\eta, \eta)$ for $\eta \in \{0, 1, 2, ..., n - 2\}$.

Proposition 4.2 *Suppose that payoffs of every player satisfy (A.2), (PSOL) and (NSPL). Let g be an pws-equilibrium network. (i). If g is symmetric and (M) holds, then it is either*

[13] We discuss the notion of strongly stable networks that allow coordinated deviations by any coalition of players in somewhat greater detail in Section 6.

empty or complete, while if (M') holds, then for generic values of f, g has a unique degree which depends on the value of f. (ii). If g is asymmetric and $g_{i,j} = 1$, then $g_{i,k} = 1$ for all players $k \neq i, j$ such that $\eta_i(g) \leq \eta_k(g) < \eta_j(g)$.

Figure 5 illustrates the set of pws-equilibrium networks under (PSOL) and (NSPL).[14]

— Insert Figure 5 somewhere here —

For a symmetric network of degree $k \in \{1, 2, , n-2\}$ to be an equilibrium it must be the case that $\psi(k, k) \leq f \leq \psi(k-1, k-1)$. It is clear that, under the hypotheses (A.2), (PSOL), (NSPL) and (M), these inequalities cannot be simultaneously satisfied. The second part of the statement about symmetric networks can be easily verified. The intuition behind the asymmetric networks result can be seen by considering a star network, with n is the central player. If players 1 and 2 who have zero links, respectively, are willing to link with the central player n who has many links then surely they would be willing to link with each other since they now have more links (1 link each as compared to zero before) and the other player (1 or 2 as the case may be) has fewer links as compared to the center of the star.

We make some remarks in connection with Proposition 4.2. *First*, we note an aspect of the result: in an equilibrium network g, if $g_{i,l} = g_{j,k} = 1$ and $\eta_i, \eta_j < \min\{\eta_l, \eta_k\}$ then $g_{i,j} = 1$. Thus networks such as stars and inter-linked stars cannot arise in equilibrium. *Second*, we note that if the positive spillovers of own links dominate the negative spillovers of partners links then the only symmetric networks that can arise in pws-equilibrium are the empty and the complete ones. By contrast, if the negative spillovers dominate the positive spillovers then a variety of symmetric networks can arise in pws-equilibrium. *Third*, we note that players are more keen to form links with poorly linked players and the marginal return is increasing in own links. This implies that the star and interlinked star cannot be sustained in pws-equilibrium. More generally it suggests that if there are positive spillovers across own links and negative spillovers across partner's links, then poorly linked players will have an incentive to link with each other and this will mitigate inequality of links in the network.

Proposition 4.2 restricts the range of networks significantly, but we would prefer to have a characterization in terms of architectures directly. This leads us to study stronger restrictions on payoffs, and motivates the following stronger condition:

(SM) $\psi(k+1, l+1) > \psi(k, l)$ for all $k, l \in \{0, 1, 2, ..., n-2\}$.

We are able to obtain the following sharp characterization if payoffs satisfy this condition.

Proposition 4.3 *Suppose that payoffs of every player satisfy (A.2), (PSOL), (NSPL) and (SM). Let g be an pws-equilibrium network. If g is symmetric then it is either empty of complete. If g is asymmetric then it consists of a group of isolated players and a collection of complete and unequal sized components.*

[14] We note that the network with one link cannot be stable for generic values of f and is omitted from the figure.

The main step in the argument here shows that every (non-singelton) component must be complete. This step taken in combination with the earlier result, Proposition 4.2, completes the proof. The arguments underlying the complete component step and the role of assumption (SM) can be understood with the help of the following example. Consider a network with 6 players: players 1-4 are fully linked with each other, player 5 is linked with 1, 2 and 6, while player 6 is linked with 3,4 and 5, respectively. It may be verified that this network satisfies the restrictions derived in Proposition 4.2. In this network, player 5 does not have an incentive to link with players 3 or 4 since they have more links as compared to player 1,2, and 6. A similar argument applies to the incentives of player 6. Next consider the situation under (SM). Suppose that the above network is an equilibrium. Then player 5 is willing to form a link with 1 which implies that $f \leq \psi(2,3)$. However, if (SM) is satisfied then $\psi(2,3) < \psi(3,4)$ and it follows that player 5 has an incentive to form a link with players 3 and 4 as well; so the above network is no longer an equilibrium.

We next study the case where the marginal returns exhibit negative spillovers with respect to own links (NSOL) but positive spillovers with respect to the links of the potential partners (PSPL), i.e., $\psi(\eta_i, \eta_j)$ is decreasing in η_i and increasing in η_j.As in the previous subsection, it is useful to distinguish between cases (M) and (M') in order to characterize pws-equilibrium networks.

Proposition 4.4 *Suppose that the payoffs of every player satisfy (A.2), (NSOL) and (PSPL). Let g be an pws-equilibrium network. (i) If g is symmetric and (M) holds, then g is either empty or complete, while if (M') holds, then for generic values of f, g has a unique degree. (ii) If g is asymmetric and $g_{i,j} = 1$, then $g_{i,k} = 1$ for all players $k \neq i, j$ such that $\eta_i(g) \leq \eta_k(g) < \eta_j(g)$. If in addition (M) holds then every pair of maximally connected players is also mutually linked.*

We would like to draw attention to the somewhat surprising fact that the architecture of equilibrium networks in the two off-diagonal cases is very similar: *if $g_{i,j} = 1$, then $g_{i,k} = 1$ for all players $k \neq i, j$ such that $\eta_i(g) \leq \eta_k(g) < \eta_j(g)$.* It is worth noting that the arguments in the two cases are somewhat different. We elaborate on the distinction with the example of the star network. Let players 1 and 2 be peripheral players and player n be the center of the star. The argument in the case of (NSOL) and (PSPL) goes as follows. If player n is willing to form links with players who have zero links otherwise, then they surely have an incentive to form a link with each other in a star since peripheral players have fewer links themselves as compared to player n and their potential partner (player 1 or 2 as the case may be) has 1 link. By contrast, recall that under (PSOL) and (NSPL), the argument goes as follows. If players 1 and 2 who have zero links, respectively, are willing to link with the central player who has many links in a star then surely they would be willing to link with each other since they now have more links (1 link each as compared to zero before) and the other player (i or k as the case may be) has fewer links as compared to the center of the star.

We would like to obtain a sharper characterization in terms of network architectures and this motivates an examination of stronger assumptions on equilibrium architectures. We use condition (SM) to obtain the following characterization.

Proposition 4.5 *Suppose that payoffs of every player satisfy (A.2), (NSOL), (PSPL) and (SM). Let g be an pws-equilibrium network. If g is symmetric then it is either empty of complete. If g is asymmetric then it consists of a group of isolated players and a collection of complete and unequal sized components.*

The proof of this uses arguments analogous to those of Proposition 4.3 and is given in the appendix.

We finally study the case where the marginal returns satisfy negative spillovers with respect to own links (NSOL) as well as the links of the potential partners (NSPL), i.e., $\psi(\eta_i, \eta_j)$ is decreasing in both η_i and η_j. In section 5, we present a free trade agreements game in which the payoffs satisfy these properties.

In this setting, pws-equilibrium networks are characterized as follows:

Proposition 4.6 *Assume that $n \geq 4$. Suppose that the payoffs of each player network satisfy (A.2), (NSOL) and (NSPL). Suppose g is a pws-equilibrium network. Then for generic values of f the following properties hold. (i) If g is symmetric then it has a unique degree k. (ii) If g is asymmetric then it has at most one singleton component. If i is this singleton player, then the network g_{-i} is symmetric and has a unique degree. If g is asymmetric but has no singleton components then it satisfies the following property: if $g_{i,j} = 1$ for some $i, j \in N_q(g)$, $q \in \{2, ..., m\}$, then $g_{k,l} = 1$ for all $k, l \in N_1(g) \cup \ldots \cup N_{q-1}(g)$; if $g_{i,j} = 0$ for some $i, j \in N_q(g)$, $q \in \{1, ..., m-1\}$, then $g_{p,r} = 0$ for all $p, r \in N_{q+1} \cup \ldots \cup N_m(g)$.*

Figure 6 illustrates the set of pws-equilibrium networks under (NSOL) and (NSPL).

– Insert Figure 6 somewhere here –

The proof of this result is given in the Appendix. We briefly discuss the arguments used. In the *first* step, we establish uniqueness of a symmetric pws-equilibrium. Here we exploit (NSOL) and (NSPL) to obtain the inequality $\psi(\eta, \eta) \leq \psi(\eta - 1, \eta - 1)$, for $\eta \geq \eta'$. If g of degree η is a pws-equilibrium, then it must be true that $\psi(\eta, \eta) \leq f \leq \psi(\eta - 1, \eta - 1)$. Similarly, if g' of degree $\eta' < \eta$ is a pws-equilibrium, then $\psi(\eta', \eta') \leq f \leq \psi(\eta' - 1, \eta' - 1)$. Given that $\eta > \eta'$, it is easy to see that this contradicts the above inequality for generic values of f. The *second* step shows that there cannot be two isolated players in an (asymmetric) pws-equilibrium network. The reason for this is that if there is a pair of players with a link, then their marginal return must exceed the costs of forming a link. However, the marginal returns to a link for a pair of isolated players weakly exceeds this marginal return, and so they must have an incentive to form a link (for generic values of f). Let player i be the isolated player in g; the *third* step in the proof shows that g_{-i} is symmetric and has a unique degree. We begin by showing that if g_{-i} is asymmetric then the minimally linked player in g_{-i} and player i have an incentive to form a link. We then use a variant of the argument in step 1 to conclude that g_{-i} must have a unique degree. This leaves the case of a pws-equilibrium g in which there are no singleton components. We exploit (NSOL) and (NSPL) to derive the stated properties.

Proposition 4.6 says that dominant group networks with two or more singleton players as well as inter-linked stars with two or more centers are ruled out under (NSOL) and (NSPL). However, a star can be an equilibrium, and this suggests that unequal connections as well as an unequal distribution of links can arise in equilibrium. Finally, the characterization says that complete components of unequal size can also arise in equilibrium.

5 Applications

In this section we present economic examples to illustrate the scope of the analysis.

Example 1: (Cost-reducing collaboration between firms, Goyal and Joshi, 2002): Consider a homogeneous product Cournot duopoly consisting of n ex-ante symmetric firms who face the linear inverse demand: $p = \alpha - \sum_{i \in N} q_i$, $\alpha > 0$. The firms are initially symmetric with zero fixed costs and identical constant returns-to-scale cost functions. Bilateral collaborations lower marginal costs of production along the following lines: $c_i(g) = \gamma_0 - \gamma\eta_i(g)$, $i \in N$, where γ_0 is a positive parameter representing a firm's marginal cost if it has no links. In this case, firm i's marginal costs are *linearly* declining in the number of links it has with other firms. Given any network g, the Cournot equilibrium output can be written as:

$$q_i(g) \; = \; \frac{(\alpha - \gamma_0) + n\gamma\eta_i(g) - \gamma \sum_{j \neq i} \eta_j(g)}{(n+1)} \;, \quad i \in N \tag{7}$$

In order to ensure that each firm produces a strictly positive quantity in equilibrium, we will assume that $(\alpha - \gamma_0) - (n-1)(n-2)\gamma > 0$. The Cournot profits for firm $i \in N$ are given by $\pi_i(g) = q_i^2(g)$. The marginal returns to player i from an additional link are:

$$\Phi(\eta_i(g), \sum_{j \neq i} \eta_j(g_{-i})) \; = \; \frac{(n-1)\gamma}{(n+1)^2} \left[\lambda(n) + 2(n-1)\gamma\eta_i(g) - 2\gamma \sum_{j \neq i} \eta_j(g_{-i}) \right] \tag{8}$$

where $\lambda(n) = 2(\alpha - \gamma_0) + (n-1)\gamma$. An inspection of marginal payoffs reveals that the cost-reducing collaboration game is a playing the field game and satisfies (PSOL). What is the architecure of equilibrium networks? Proposition 4.1 in Goyal and Joshi (2002) proves existence of equilibrium and also shows that a wide range of dominant group sizes can be sustained in equilibrium. Proposition 3.1 in the present paper shows that the dominant group architecture is a property of a general class of network formation games with positive spillovers with respect to own links.

Example 2: (Patent races) Consider n firms who are racing to innovate a new product or process. The race is conducted in continuous time. The firm which succeeds in innovating first wins a patent which prevents the innovation from imitation or duplication for perpetuity. The value of the patent is V, which we set equal to 1, without loss of generality. Loosing firms get a payoff of 0. All firms use the same discount rate ρ. Suppose that firms are inelastically endowed with one unit of R&D capability (or technical know-how). Firms race to innovate by forming bilateral links with other firms; these links represent agreements to mutually share R&D capability or technical information. Let $\tau(\eta_i(g))$ denote the random time at which firm i innovates in a network g in which firm i has established $\eta_i(g)$ bilateral links. We assume that τ has an exponential distribution:

$$Pr\{\tau(\eta_i(g)) \leq t\} \;=\; 1 - e^{-\eta_i(g)t} \tag{9}$$

Thus we are presenting a variation on the basic model of the memoryless patent race model (see Dasgupta and Stiglitz, 1980), where firms can collaborate to speed up their R&D. As firm i establishes more links, it increases the probability of innovating successfully before time t. In addition to this technological uncertainty, there is also market uncertainty: any of the rival $n-1$ firms may successfully innovate before firm i. Assuming that the distribution of the time of innovation is stochastically independent for the firms, the probability that firm i is the first to successfully innovate by time t is:

$$Pr\{\tau(\eta_i(g)) \in [t, t+dt],\; \tau(\eta_j(g)) > t \;\forall j \neq i\} \;=\; \eta_i(g)e^{-t\sum_{j=1}^{n}\eta_j(g)}dt \tag{10}$$

There are both benefits and costs from establishing bilateral links. The benefit to i from linking with j is that it increases the probability of i innovating successfully before time t; the cost is that the probability that j will innovate successfully before time t also increases. This tension between benefits and costs will determine the equilibrium architecture of the network. In the network g, the (expected) payoff to firm i is given by:

$$\pi_i(g) \;=\; \int_0^\infty e^{-\rho t}\eta_i(g)e^{-t\sum_{j=1}^{n}\eta_j(g)}dt \;=\; \frac{\eta_i(g)}{\rho + \sum_{j=1}^{n}\eta_j(g)} \tag{11}$$

The marginal payoffs to player i from a link are given by:

$$\Phi(\eta_i(g), \sum_{j\neq i}\eta_j(g_{-i})) \;=\; \frac{\rho + \sum_{j\neq i}\eta_j(g_{-i})}{[\rho + 2\eta_i(g) + \sum_{j\neq i}\eta_j(g_{-i})][\rho + 2\eta_i(g) + \sum_{j\neq i}\eta_j(g_{-i}) + 2]} \tag{12}$$

Collaborating in a patent race game is a playing the field game which satisfies (NSOL) and (NSTP) for sufficiently large values of ρ.

Recall that our analysis in section 3 tells us that in the presence of (NSOL) and (NSTP), equilibrium networks can have a variety of architectures. Indeed, it is possible to show that in a 3 firms example all the four possible networks can all arise in equilibrium, for different values of costs of forming links.

Example 3: (Provision of a pure public good) There are n persons, each of whom is deciding on what output share, x_i to produce of a pure public good. Given each person's output, the utility of person i is: $u_i(x) = x_i + \sum_{j \neq i} x_j$. A collaboration link between two persons can be interpreted as an agreement to share knowledge about the production of a public good. Let $f > 0$ be the fixed investment required from each person in such a link. In any network g in which person i has a neighborhood of size $n_i(g) = \eta_i(g) + 1$, the cost of producing output x_i is given by:

$$C_i(x_i) = \frac{1}{2}\left(\frac{x_i}{n_i(g)}\right)^2 \tag{13}$$

Given any network g from the first stage, person i will choose output to maximize utility net of production costs. This yields an optimal output of $x_i(g) = n_i^2(g)$. Therefore, the reduced form gross payoff of person i is:

$$\pi_i(g) = \frac{1}{2}n_i^2(g) + \sum_{j \neq i} n_j^2(g) \tag{14}$$

The marginal returns to person i from an additional link $g_{i,j}$, can be written as follows:

$$\psi(\eta_i(g), \eta_j(g)) = \frac{3}{2} + n_i(g) + 2n_j(g). \tag{15}$$

We note that the marginal returns to person i from a link $g_{i,j}$ are a function only of his own links and the links of player j. Moreover, is easy to see that they are increasing in $n_i(g)$ and $n_j(g)$. Thus this is a local spillovers game which satisfies (PSOL) and (PSPL). We can therefore use Proposition 4.1 to obtain a characterization of equilibrium networks in this case.

We shall consider interlinked stars with two types of players, those who are linked to everyone and a second group of players who are only linked to this former set. Recall that $N_m(g)$ ($N_1(g)$) is the collection of players which has the largest (smallest) number of links in a network g. For the class of inter-linked stars that we are looking at, we can set $m = |N_m(g)|$ and $n - m = |N_1(g)|$. Also define $x_m = (n+1)/2 + (m-1)(2n-1)/(n-1) + (n-m)(2m+$

$1)/(n-1)$, $y_m = m/2 + 2n$, and $z_m = 3m + 9/2$. Straightforward computations then yield the following result: *(a) An inter-linked star with* $|N_m(g)| = m$*, where* $m \in \{1, 2, .., n-2\}$*, is a pws-equilibrium if and only if* $z_m \leq f \leq \min\{x_m, y_m\}$*. (b) A dominant-group network of size* k*, where* $k \in \{2, ..., n-1\}$ *is a pws-equilibrium if and only if* $7/2 + k < f < 5k/2 - 1/2$*. The complete network is a pws-equilibrium if* $f < 5n/2 - 1/2$*, while the empty network is a pws-equilibrium if* $f > 9/2$*.*[15]

Example 4: (Market Sharing, Belleflamme and Bloch, 2001). Consider n ex-ante symmetric firms and associate with each firm i a homogeneous product market i. Before engaging in competition in these markets, the firms can form collaboration links with each other. A collaboration link between i and j is an agreement between the two firms to stay out of each others market. If firm i has $\eta_i(g)$ links in a network g, then there are $n - \eta_i(g)$ active firms in market i who compete as Cournot oligopolists. The Cournot profits earned by a firm that is active in market k is given by $\lambda(n - \eta_k(g))$. Therefore, the gross payoff to firm i in a network g is given by:

$$\pi_i(g) \;=\; \lambda(n - \eta_i(g)) \;+\; \sum_{k;\; g_{i,k}=0} \lambda(n - \eta_k(g)) \tag{16}$$

The marginal gross payoff to i from establishing a link with j is given by:

$$\psi(\eta_i(g), \eta_j(g)) \;=\; [\lambda(n - \eta_i(g) - 1) - \lambda(n - \eta_i(g))] - \lambda(n - \eta_j(g)) \tag{17}$$

Assume that Cournot profits in market i are decreasing and convex in the number of firms active in the market. An inspection of the above marginal returns expression reveals that this is a local spillovers game that satisfies (PSOL) and (NSPL).

Proposition 2.3 in Bloch and Belleflamme (2001) proves existence of strongly stable networks – which are Nash networks that satisfy 2 player coalition stability – and provides a characterization as well. Given a network, g, let $m(g_l)$ be the size of a component g_l. We state this result here for easy reference.

Bloch and Belleflamme (2001): *Suppose that individual firm profits are decreasing and log-convex in the number of firms in a market. Then a network is a pws-equilibrium if and only if it consists of a group of isolated firms and distinct complete components* $g_1, g_2, ..g_L$*, such that (i)* $\pi(N - m(g_l) + 1) \geq \pi(N) + (m(g_l) - 1)\pi(N - m(g_l) + 2)$ *for all* l*, and (ii)* $m(g_l) \neq m(g_{l'})$*, if* $l \neq l'$*.*

It can be checked that the log-convexity assumption implies that condition (M) is satisfied. Hence the above result tells us that pws-equilibrium exist in interesting games which satisfy (PSOL) and (NSPL) and (M). A comparison of the above result with Figure 5A suggests that two things: one, that the characterization of pws-equilibrium is almost tight and two, that the

[15] The computations are available from the authors upon request.

architectural properties identified by the above result obtain for all games with local spillovers which satisfy (PSOL) and (NSPL) and condition (M). A later result, Proposition 4.3 actually derives an exact architecture characterization result, under the stronger assumption, (SM).

Example 5: (Free trade agreements among countries, Goyal and Joshi, 1999). Suppose there are n countries. In each country there is one firm producing a homogeneous good and competing as a Cournot oligopolist in all countries. We let the output of firm j in country i be denoted by Q_i^j. The total output in country i is given by $Q_i = \sum_{j \in N} Q_i^j$. In each country $i \in N$, a firm faces an identical inverse linear demand given by:

$$P_i \; = \; \alpha - Q_i \, , \quad \alpha > 0 \tag{18}$$

All firms have a constant and identical marginal cost of production, $\gamma > 0$. We assume that $\alpha > \gamma$. Let the initial pre-agreement import tariff in each country be $T > \alpha$. Countries can form agreements which lower the tariff to 0. The natural interpretation of such an agreement is as a *bilateral free trade agreement.* We suppose that tariffs remain prohibitively high between countries that do not have a bilateral free-trade agreement. The assumption that $T > \alpha$ ensures that a firm i sells in country j if and only if there is a trade agreement between the two countries. Therefore, $n_i(g) = \eta_i(g) + 1$ is the number of firms active in country i given the network g. If firm i is active in market j, then its output is given by $Q_j^i = (\alpha - \gamma)/(n_j(g) + 1)$. The social welfare of country i is given by:

$$S_i(g) \; = \; \frac{1}{2} \left[\frac{(\alpha - \gamma) n_i(g)}{n_i(g) + 1} \right]^2 + \sum_{j \in N_i(g)} \left[\frac{\alpha - \gamma}{n_j(g) + 1} \right]^2 \tag{19}$$

The marginal return from an additional free trade agreement is given by:

$$\psi(n_i(g), n_j(g)) = (\alpha - \gamma)^2 \left[\frac{(2n_i^2(g) - 5)}{2(n_i(g) + 2)^2 (n_i(g) + 1)^2} + \frac{1}{(n_j(g) + 2)^2} \right] \tag{20}$$

Since $n_k(g) = \eta_k(g) + 1$, the marginal returns to country i from a link $g_{i,j}$ are actually a function of $\eta_i(g)$ and $\eta_j(g)$. In particular, the marginal returns are declining with respect to $\eta_i(g)$, for $\eta_i(g) \geq 2$, and declining with respect to $\eta_j(g)$, for $\eta_j(g) \geq 0$. Thus, this is a game with local spillovers which (almost) satisfies (NSOL) and (NSPL). We can therefore use Proposition 4.6 to characterize equilibrium network architectures.

Let $n = 5$ and consider Figure 6. The computations reveal that all the symmetric networks and a subset of the asymmetric networks identified in this figure – with two complete components of 2 and 3 players respectively, with four players in a component (with degree 2) and one isolated player, a complete component with four players and one isolated player, three players have 3 links each and one player has 2 links, and finally the case in which four players have 4 links each while one player has 3 links – can arise as equilibrium in the trade game.[16]

[16]We can also indicate the parameter value; without loss of generality set $(\alpha - \gamma)^2 = 1$. The empty

6 Extensions

In this section we briefly discuss three issues: (1) the nature of networks under local and global spillovers, (2) an alternate solution concept of strong stability, and (3) network formation under one-sided link formation.

1. Games with local and global spillovers: In some interesting settings, marginal returns from an additional link depend on the links of the player and his potential partner, as well as on the level of connectedness of the rest of the players. These situations may be seen as combining elements of the two classes of games we have analyzed in the paper so far. We will supplement the local spillovers model by having marginal returns depend on the number of links of the player and his potential partner and the level of connectedness of the rest of the network.[17] Formally, we shall say that given a network g, the marginal returns to player i from a link between players i and j are given by:

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \rho(\eta_i(g), \eta_j(g), g_{-i-j}) \tag{A.3}$$

$\mathcal{G}$ is a partially ordered set with the ordering relation $\geq$ defined as follows: for $g = \{(g_{i,j})\}$, $g' = \{(g'_{i,j})\}$ in $\mathcal{G}$, $g \geq g'$ if $g_{i,j} \geq g'_{i,j}$ $\forall i, j \in N$. The strict ordering relation is defined as follows: for $g = \{(g_{i,j})\}$, $g' = \{(g'_{i,j})\}$ in $\mathcal{G}$, $g > g'$ if $g_{i,j} \geq g'_{i,j}$ $\forall i, j \in N$ and $g_{i,j} > g'_{i,j}$ for some $i, j \in N$. We can use this partial ordering on graphs to study the effects of positive global spillovers on marginal returns.

We shall say that the payoffs of a player exhibit positive global spillovers if $\rho(\eta_i(g), \eta_j(g), g_{-i-j})$ is increasing in g_{-i-j}. We start by noting that the main conclusions of Proposition 4.1 carry over if there are positive global spillovers in this sense. However, in general global spillovers are difficult to analyze. We illustrate this by considering an example with negative global and local spillovers. Note that negative local and global spillovers are modelled as follows: $\rho(\eta_i(g), \eta_j(g), g_{-i-j})$ is decreasing in $\eta_i(g)$, $\eta_j(g)$ and g_{-i-j}, respectively.

Example: Let $n = 4$. Suppose the payoff function of the players in the network g is given by:

$$\pi_i(g) \;=\; \alpha\sqrt{\eta_i(g)} - \beta \sum_{l \in N} \eta_l^2(g)\eta_i(g), \quad i \in N \tag{21}$$

network for $f > 0.069$; the symmetric network of degree 2 for if $0.056 < f < 0.073$, while the complete network for $f < 0.043$. The network with two complete components with 2 and 3 players, respectively, for $0.05 < f < 0.069$; the network with a symmetric component of degree 2 and one isolated player for $0.056 < f < 0.073$; the dominant group network with one isolated player for $0 < f < 0.056$; the network where 4 players have three links and one player has 2 links for $0.04 < f < 0.056$; finally, the network where 4 players have 3 links and one player has four links for $0.042 < f < 0.044$.

[17] The co-author model in Jackson and Wolinsky (1996) and the job-market contacts model in Calvo (2001), are two instances of games where marginal returns depend on players links, the potential partner's links and the links of the current partners. The games we consider below do not cover these examples strictly speaking but the analysis below illustrates some of the complications that arise in a general analysis of games with global spillovers.

where $\alpha > \beta > 0$. The marginal payoffs are given by:

$$\pi_i(g + g_{i,j}) - \pi_i(g) = \alpha \left\{ \sqrt{\eta_i(g) + 1} - \sqrt{\eta_i(g)} \right\}$$
$$-\beta \sum_{l \in N} \eta_l^2(g) - 2\beta \{\eta_i(g) + \eta_j(g) + 1\} \{\eta_i(g) + 1\} \quad (22)$$

It can be verified that payoffs satisfy the local and global negative spillovers property. In this example, the symmetric pws-equilibria are characterized as follows: (a). The empty network is a pws-equilibrium for $\alpha - 2\beta < f$. (b). The symmetric network of degree 1, g^1, is a pws-equilibrium for $0.41\alpha - 16\beta < f \leq \alpha - 4\beta$. (c). The symmetric network of degree 2, g^2, is a pws-equilibrium for $0.32\alpha - 46\beta < f \leq 0.41\alpha - 22\beta$. (d). The complete network is a pws-equilibrium for $0.32\alpha - 56\beta \geq f$.[18]

We now provide a complete characterization of asymmetric networks. (a). The dominant group network with one isolated player is a pws-equilibrium for $0.32\alpha - 30\beta < f \leq 0.41\alpha - 18\beta$ while the dominant group with two isolated players is a pws-equilibrium for $\alpha - 4\beta < f \leq \alpha - 2\beta$. (b). The interlinked star network in which $N_1(g) = \{i\}$, $N_2(g) = \{j, k\}$ and $N_3(g) = \{l\}$ is a pws-equilibrium for $0.32\alpha - 42\beta < f \leq 0.32\alpha - 34\beta$; the interlinked star network in which $N_1(g) = \{i, j\}$ and $N_2(g) = \{k, l\}$ is a pws-equilibrium for $0.32\alpha - 56\beta < f \leq 0.32\alpha - 42\beta$. The star network, however, is not a pws-equilibrium. (c). The "line" network, in which two players have two links and two players have one link, is a pws-equilibrium for $0.41\alpha - 22\beta < f \leq 0.41\alpha - 16\beta$. (d) The non-singleton component may be incomplete: the network with one isolated player and the other three players in a star component is a pws-equilibrium for $0.41\alpha - 14\beta < f \leq 0.41\alpha - 10\beta$. △

Two observations follow from the above example. *First*, we see that symmetric networks and highly asymmetric networks can coexist as equilibria under negative spillovers. This is mainly due to global negative spillovers: the marginal payoffs to players i and j from a link are adversely affected as the network becomes more connected; therefore, if some players have established a large number of links, then they can deter other players from forming links. This is closely related to our second observation. The above example shows that in the presence of negative local and global spillovers, all network architectures except the star can be sustained as pws-equilibria (over some cost range). Recall that in the local spillovers setting, negative spillovers with respect to own and partners links implied that in an equilibrium network (Proposition 4.6), there is at most one isolated player and in such a network the rest of the players constitute a symmetric network. In the presence of global negative spillovers, this equilibrium restriction no longer holds. For instance, a dominant group network with two isolated players can be an equilibrium; similarly, a network with one isolated player and three players forming a star can arise in equilibrium. Thus the set of equilibria expand significantly once we supplement local negative spillovers with global ones.

[18]This characterization of symmetric networks can be derived by computing the payoffs for each player in the different networks. For example, $\Pi_i(g^e) = 0$ while $\Pi_i(g^c) = 1.72\alpha - 108\beta - 3f$. Similarly, $\Pi_i(g^1) = \alpha - 4\beta - f$ and $\Pi_i(g^2) = 1.41\alpha - 32\beta - 2f$.

2. Strong Stability: In this paper, we have utilized the notion of *pair-wise stable* equilibrium. As we pointed out earlier, while this requirement is relatively mild, it sharply restricts the number of architectures that can emerge in equilibrium in most cases. There are, however, two instances where the pws-criterion is not as discriminating and consequently a large number of architectures remain as possible candidates for equilibrium. These correspond to the playing-the-field model under (NSOL) and (NSTP) and the local spillovers model under (NSOL) and (NSPL). It may seem that a stronger notion of stability may allow us to further restrict the set of possible equilibrium networks. In this regard, we briefly consider the notion of strongly stable equilibrium networks - henceforth ss-equilibrium networks - put forward by Jackson and van den Nouweland (2001) as an alternate solution concept.

Let g be a Nash network and let $S \subseteq N$ be any coalition of the set of players. A network g' is obtainable from g through deviations by the coalition S if:

(i) $g_{i,j} = 1$ in g' and $g_{i,j} = 0$ in g implies that $i, j \in S$.

(ii) $g_{i,j} = 0$ in g' and $g_{i,j} = 1$ in g implies that $\{i, j\} \cap S \neq \emptyset$.

This definition describes changes in a network that can be effected by a coalition S without requiring the consent of players in $N \backslash S$.

Definition 6.1 *A network g is a ss-equilibrium network if the following conditions hold:*

1. *There is a Nash equilibrium strategy profile which supports g.*
2. *Consider any $S \subseteq N$ and any g' obtainable from g through deviations by S. For any $i \in S$ such that $\Pi_i(g') > \Pi_i(g)$, there exists $j \in S$ such that $\Pi_j(g') < \Pi_j(g)$.*

Note that a ss-equilibrium network must also be a pws-equilibrium, i.e. strong stability refines the set of pws-equilibrium networks. This follows by restricting S to all 2-player coalitions, $\{i, j\}$, and restricting the network g' obtainable from g to networks of the form $g' = g + g_{i,j}$.

Our characterization results in sections 3 and 4 identify particular network architectures in most classes of games that we studied. However, there are two settings in which the results were weak: the first one was the playing the field game where payoffs satisfy (NSOL) and (NSTP) while the second one was a local spillovers game where the payoffs satisfy (NSOL) and (NSPL). We now discuss how the richer set of coalitional deviations permitted by strong stability refine the set of pws-equilibria in the latter case. Recall that the pws-equilibria for this case are given by Figure 6. Consider for example the star network in this figure. The coalition $S = \{2, 5\}$ can obtain the network $g' = g - g_{1,2} - g_{1,5} + g_{2,5}$ from g. The gross marginal loss to the spokes from deleting their link with the center is $\psi(0, 2)$ while the gross marginal gain to the spokes from linking with each other is $\psi(0, 0) > \psi(0, 2)$. The aggregate cost of forming links has remained unchanged since each spoke in S has simply substituted one link with another. Therefore, players in S have a profitable deviation from g. Applying the same argument to the other pws-equilibrium networks in Figure 6 yields the smaller set of pws-equilibrium networks in Figure 7 as *possible* ss-equilibria.

— Insert Figure 7 somewhere here —

Since we have not placed any restrictions on aggregate payoffs, it is not possible at this level of generality to verify whether each of the architectures in Figure 7 is immune to all possible coalitional deviations.

3. One-sided link formation: In the analysis so far we have assumed that link formation is two-sided. It is instructive to briefly discuss the nature of networks if link-formation is one-sided. This discussion will help clarify the role of the cost-sharing rule – that the costs must be equally shared by the two players – which is implicit in our analysis so far. Our main point is that the argument concerning the relation between negative spillovers across others' links and unequal networks is robust to the precise model of link formation.

We illustrate this by considering a setting in which link formation is one-sided while the gains to link formation are two-sided. Consider the playing the field game. Suppose marginal returns exhibit positive spillovers with respect to own links. It can be checked that under this assumption an equilibrium network will be an inter-linked star, an empty network or the complete network. Moreover, if spillovers from the links of the rest of the players are positive then the complete network will be the unique equilibrium, while if these spillovers are negative (and significant) then the equilibrium network is an inter-linked star, with a strict subset of the population forming links with everyone else. Thus positive spillovers from others' links lead to symmetric networks while negative spillovers can generate networks in which players have an unequal number of links.

7 Concluding Remarks

It is widely felt that connections matter both at the individual level as well as in the aggregate. In particular, well-connected individual entities are seen to be at an advantage as compared to their less connected cohorts. Given the advantages that arise out of connections it seems reasonable to suppose that individuals will devote resources in forming links with others. Moreover, the decisions on link formation trade off the costs of forming links against the benefits of doing so. We develop a general model of networks to study the incentives of individuals to form links and the implications of this link formation activity for the architecture of networks. In particular, we are interested in identifying the economic circumstances which give rise to an unequal distribution of links across individuals.

Our results provide a characterization of equilibrium network architecures in a broad class of games. We find the interaction of individual incentives narrows down the range of networks sharply, and that equilibrium networks have simple architectures. Turning to the issue of inequality, we first show that spillovers across links are essential for generating unequal connections. We then examine the effects of positive and negative spillovers across own links and the links of others. In a setting where spillovers across a player's own links are positive, negative spillovers from others' links lead to a network with two groups of players: a clique of fully connected players and a group of isolated players. In a setting where spillovers across a player's own links are negative, negative spillovers from others' links can lead to the emergence of stars. In the former case the distribution of links is unequal but links connect individuals with an equal number of links; in the latter case, the distribution of links is unequal and in addition the links connect individuals who maintain an unequal number of links. We present several economic applications to illustrate the scope of the analysis.

8 Appendix

Proof of Proposition 3.1: The first step in the proof is to show the following transitivity property of pws-equilibrium networks: *Suppose that g is a pws-equilibrium network. Consider any two players $i, j \in N$. If $g_{i,k} = g_{j,l} = 1$ for some $k, l \in N\backslash\{i, j\}$, then $g_{i,j} = 1$.* Assume to the contrary that for some $i, j, k, l \in N$, $g_{i,k} = g_{j,l} = 1$ in a pws-equilibrium network g but $g_{i,j} = 0$. By the definition of pws-equilibrium:

$$\begin{aligned} \pi_i(g) - \pi_i(g - g_{i,k}) &= \Phi(\eta_i(g) - 1, \sum_{j \neq i} \eta_j(g_{-i})) \geq f \\ \pi_j(g) - \pi_j(g - g_{j,l}) &= \Phi(\eta_j(g) - 1, \sum_{k \neq j} \eta_k(g_{-j})) \geq f \end{aligned} \tag{23}$$

Since each player's marginal payoffs are strictly increasing in own links:

$$\begin{aligned} \pi_i(g + g_{i,j}) - \pi_i(g) &= \Phi(\eta_i(g), \sum_{k \neq i} \eta_k(g_{-i})) > \Phi(\eta_i(g) - 1, \sum_{k \neq i} \eta_k(g_{-i})) \geq f \\ \pi_j(g + g_{i,j}) - \pi_j(g) &= \Phi(\eta_j(g), \sum_{k \neq j} \eta_k(g_{-j})) > \Phi(\eta_j(g) - 1, \sum_{k \neq j} \eta_k(g_{-j})) \geq f \end{aligned} \tag{24}$$

This implies that players i and j have an incentive to form a link and, therefore, g is not a pws-equilibrium.

We now take up the case of symmetric networks. For high enough f, the empty network is trivially stable. Now consider a non-empty pws-equilibrium symmetric network $g \neq g^c$ with degree $0 < \eta < n - 1$. Since g is symmetric, we can find two players i and j such that $g_{i,l} = g_{j,m} = 1$ for some $l, m \in N\backslash\{i, j\}$ but $g_{i,j} = 0$. However, from step 1 of the proof, this contradicts the hypothesis that g is a pws-equilibrium.

Next consider the class of asymmetric networks. We first show that any asymmetric network can have at most one non-singleton component. Let $C_1(g)$ and $C_2(g)$ be two non-singleton components in g. Suppose that $i, j \in C_1$ with $g_{i,j} = 1$ and $l, m \in C_2$, with $g_{l,m} = 1$. From the definition of a component, $g_{i,l} = 0$. However, from step 1 of the proof, this contradicts the hypothesis that g is a pws-equilibrium.

Next we show that the non-singleton component must be complete. If it is incomplete, then there are players i and j in the component such that $\eta_i(g) \geq 1$, $\eta_j(g) \geq 1$ and $g_{i,j} = 0$. Step 1 then shows that this contradicts the pws-equilibrium hypothesis concerning g.

△

Proof of Proposition 3.2: We illustrate the first part of the result by an example. Suppose $n = 4$ and individual payoffs are given as follows:

$$\pi_i(g) = \alpha\sqrt{\eta_i(g)} + \beta \sum_{j \neq i} \eta_j(g)\eta_i(g), \tag{25}$$

where $\alpha > \beta > 0$. It follows that:

$$\Phi(\eta_i(g), \sum_{j \neq i} \eta_j(g_{-i})) \;= \alpha(\sqrt{\eta_i(g) + 1} - \sqrt{\eta_i(g)}) + \beta \sum_{j \neq i} \eta_j(g_{-i}) + \beta + \beta\eta_i(g) \tag{26}$$

It is easily verified that the payoff function satisfies (NSOL) for α sufficiently greater than β. Further, it satisfies (PSTP) since an increase in $\sum_{j \neq i} \eta_j(g_{-i})$ leads to an increase in Φ. The symmetric pws-equilibrium networks are characterized as follows: *(a). If $\alpha + \beta < f$, then the empty network is a pws-equilibrium network. (b). If $0.41\alpha + 5\beta < f \leq \alpha + 3\beta$, then the symmetric network of degree 1, g^1, is a pws-equilibrium. (c). If $0.32\alpha + 9\beta < f \leq 0.41\alpha + 7\beta$ then the symmetric network of degree 2, g^2, is a pws-equilibrium. (d). If $f \leq 0.32\alpha + 11\beta$ then the complete network is a pws-equilibrium.*[19]

We next consider the case of asymmetric networks. Here we show the following property: suppose g is an asymmetric pws-equilibrium network. If $i, j \notin N_m(g)$, then $g_{i,j} = 1$. Take some player $k \in N_m(g)$. It follows from the pws-equilibrium hypothesis that

$$\Phi(\eta_k(g) - 1, \sum_{l \neq k} \eta_l(g_{-k})) \geq f. \tag{27}$$

Suppose $i, j \notin N_m(g)$, and $g_{i,j} = 0$. Then it follows that either $\Phi(\eta_i(g) + 1, \sum_{l \neq i} \eta_l(g_{-i})) \leq f$, or $\Phi(\eta_j(g), \sum_{l \neq j} \eta_l(g_{-j})) \leq f$, or both. Since $\eta_i(g) < \eta_k(g)$, it follows that $\sum_{l \neq i} \eta_l(g_{-i}) >$

[19]This characterization can be derived by computing the payoffs to different players in the different networks. The payoffs in the complete network are given by $\Pi_i(g^c) = 1.72\alpha + 27\beta - 3f$. The payoffs in the empty network are given by $\Pi_i(g^e) = 0$. In the symmetric network of degree 1, $\Pi_i(g^1) = \alpha + 3\beta - f$, while in the symmetric network of degree 2, $\Pi_i(g^2) = 1.41\alpha + 12\beta - 2f$. The characterization stated above can be derived using these payoffs. If we set $\alpha = 40$ and $\beta = 1$ then g^1, g^2 and g^c are pws-equilibrium networks over the range $21.8 < f \leq 23.4$.

$\sum_{l \neq k} \eta_l(g_{-k})$. Hence it follows that

$$\begin{aligned} \Phi(\eta_i(g), \sum_{l \neq i} \eta_l(g_{-i})) &> \Phi(\eta_i(g), \sum_{l \neq k} \eta_l(g_{-k})) \\ &\geq \Phi(\eta_k(g) - 1, \sum_{l \neq k} \eta_l(g_{-k})) \geq f. \end{aligned} \tag{28}$$

In the above expression, the first inequality follows from (PSTP), while the second inequality follows from noting that $\eta_k(g) > \eta_i(g)$ and applying (NSOL). The final inequality follows from the pws-equilibrium hypothesis. Similar reasoning establishes that:

$$\Phi(\eta_j(g), \sum_{l \neq j} \eta_l(g_{-j})) > f \tag{29}$$

Hence, i and j have a strict incentive to form a link, which contradicts our starting hypothesis that g is a pws-equilibrium.

△

Proof of Proposition 3.3: Suppose symmetric networks g and g' of degree k and k', respectively, are both pws-equilibrium networks and that $k < k'$. Then it follows that,

$$\Phi(k, \sum_{j \neq i} \eta_j(g_{-i})) \leq f, \quad \Phi(k-1, \sum_{j \neq i} \eta_j(g_{-i})) \geq f \tag{30}$$

and moreover that,

$$\Phi(k'-1, \sum_{j \neq i} \eta_j(g'_{-i})) \geq f. \tag{31}$$

Combining these equations yields us the following:

$$f \leq \Phi(k'-1, \sum_{j \neq i} \eta_j(g'_{-i})) < \Phi(k'-1, \sum_{j \neq i} \eta_j(g_{-i})) \leq \Phi(k, \sum_{j \neq i} \eta_j(g_{-i})) \leq f. \tag{32}$$

The first inequality follows from the hypothesis that g' is an pws-equilibrium network, the second inequality follows from (NSTP), while the third inequality follows from the hypothesis that $k < k'$ and (NSOL). The final inequality follows from the hypothesis that g is a pws-equilibrium. This generates a contradiction which completes the proof. △

Proof of Proposition 4.1: Consider the symmetric case first. Let g be a symmetric pws-equilibrium network with degree k, where $0 < k < n-1$. Then it follows that there exist players i and j with $g_{i,j} = 0$, whose payoffs satisfy the following conditions:

$$\begin{aligned} \psi(\eta_i(g - g_{i,l}), \eta_l(g - g_{i,l})) &= \psi(\eta_i(g) - 1, \eta_l(g) - 1) \geq f \\ \psi(\eta_j(g - g_{j,p}), \eta_p(g - g_{j,p})) &= \psi(\eta_j(g) - 1, \eta_p(g) - 1) \geq f \end{aligned} \tag{33}$$

for some $l, p \neq i, j$. Since g is symmetric, $\eta_j(g) > \eta_l(g) - 1$, and $\eta_i(g) > \eta_p(g) - 1$. Thus we have the following implication:

$$\begin{aligned} \psi(\eta_i(g), \eta_j(g)) &> \psi(\eta_i(g) - 1, \eta_j(g)) > \psi(\eta_i(g) - 1, \eta_l(g) - 1) \geq f \\ \psi(\eta_j(g), \eta_i(g)) &> \psi(\eta_j(g) - 1, \eta_i(g)) > \psi(\eta_j(g) - 1, \eta_p(g) - 1) \geq f \end{aligned} \tag{34}$$

where the first inequality follows from (PSOL), the second inequality from (PSPL), and the third inequality from (33). This implies that players i and j have an incentive to form a link and therefore g is not a pws-equilibrium, a contradiction that completes the proof.

We now consider asymmetric networks. We first prove that a pws-equilibrium network can have at most one non-singleton component. Let C_1 and C_2 be two non-singleton components in g. Suppose that $i, j \in C_1$ with $g_{i,j} = 1$ and $l, m \in C_2$, with $g_{l,m} = 1$. Moreover, let $\eta_i(g) \geq \eta_p(g) \forall p \in N$. Since g is pws-equilibrium it follows that:

$$\psi(\eta_l(g) - 1, \eta_m(g) - 1) \geq f, \;\; \psi(\eta_m(g) - 1, \eta_l(g) - 1) \geq f \tag{35}$$

Since $\eta_i(g) \geq \eta_p(g)\ \forall p \in N$, it follows that $\eta_i(g) > \eta_m(g) - 1$. Therefore, using (PSPL), (PSOL) and (35) respectively:

$$\begin{aligned} \psi(\eta_i(g), \eta_l(g)) &> \psi(\eta_i(g), \eta_l(g) - 1) > \psi(\eta_m(g) - 1, \eta_l(g) - 1) \geq f \\ \psi(\eta_l(g), \eta_i(g)) &> \psi(\eta_l(g), \eta_m(g) - 1) > \psi(\eta_l(g) - 1, \eta_m(g) - 1) \geq f \end{aligned} \tag{36}$$

Thus players i and l have an incentive to form an additional link, contradicting the hypothesis that the network g is a pws-equilibrium. This proves that there can be at most one non-singleton component, $C(g)$, in a pws-equilibrium network g. Let $N_1(C(g)), ..., N_m(C(g))$ be a partition of players according to their number of links in $C(g)$. The argument from above shows that if g is symmetric then it must be complete. We finally consider a component $C(g)$, which is asymmetric. We show that $\mathcal{N}_i(g) = N_m(C(g))$ for all $i \in N_1(C(g))$, i.e. $C(g)$ is an inter-linked star. Suppose not. Then there is some $i \in N_1(C(g))$ such that $g_{i,j} = 1$ for $j \notin N_m(C(g))$. Since g is a pws-equilibrium:

$$\psi(\eta_i(g) - 1, \eta_j(g) - 1) \geq f, \;\; \psi(\eta_j(g) - 1, \eta_i(g) - 1) \geq f \tag{37}$$

Since $j \notin N_m(C(g))$, there exists a player k in $C(g)$ such that $g_{j,k} = 0$. Note that $\eta_k(g) > \eta_i(g) - 1$. We now show that j and k have an incentive to form a link with each other:

$$\begin{aligned} \psi(\eta_j(g), \eta_k(g)) &> \psi(\eta_j(g) - 1, \eta_k(g)) > \psi(\eta_j(g) - 1, \eta_i(g) - 1) \geq f \\ \psi(\eta_k(g), \eta_j(g)) &> \psi(\eta_i(g) - 1, \eta_j(g)) > \psi(\eta_i(g) - 1, \eta_j(g) - 1) \geq f \end{aligned} \tag{38}$$

Therefore g is not a pws-equilibrium, a contradiction that completes the proof for inter-linked stars.

We finally prove the statement about the difference in number of links. Assume to the contrary that in a pws-equilibrium network g there exists players i and j in some $N_r(g)$ and $N_{r+1}(g)$ respectively, $r = 1, 2, ..., m-1$ such that $\eta_j(g) - \eta_i(g) = 1$. Since j has strictly more links than i, there must exist some player $k \neq i, j$ such that $g_{i,k} = 0$ and $g_{j,k} = 1$. From the equilibrium hypothesis:

$$\psi(\eta_j(g) - 1, \eta_k(g) - 1) \ \geq \ f, \quad \psi(\eta_k(g) - 1, \eta_j(g) - 1) \ \geq \ f \tag{39}$$

But now it follows that players i and k have an incentive to form a link:

$$\begin{aligned} \psi(\eta_i(g), \eta_k(g)) = \psi(\eta_j(g) - 1, \eta_k(g)) > \psi(\eta_j(g) - 1, \eta_k(g) - 1) \geq f \\ \psi(\eta_k(g), \eta_i(g)) = \psi(\eta_k(g), \eta_j(g) - 1) > \psi(\eta_k(g) - 1, \eta_j(g) - 1) \geq f \end{aligned} \tag{40}$$

This contradicts the hypothesis that g is a pws-equilibrium and that completes the proof of the proposition.

△

Proof of Proposition 4.2: (i). Suppose that a pws-equilibrium network g is symmetric of degree $1 \leq \eta < n-1$. Then for any player in g:

$$\psi(\eta, \eta) \ \leq \ f \ \leq \ \psi(\eta - 1, \eta - 1) \tag{41}$$

It is immediate that this condition violates (M). However, it does satisfy (M'), and it is clear that symmetric networks of different degrees can arise in pws-equilibrium. However, for a generic value of f, at most one degree can be an pws-equilibrium. To see this suppose that two symmetric networks, g and g' with degree η and η', respectively, are pws-equilibrium networks. Suppose that $\eta' > \eta$. It then follows that

$$\begin{aligned} \psi(\eta, \eta) \ \leq \ f \ \leq \ \psi(\eta - 1, \eta - 1) \\ \psi(\eta', \eta') \ \leq \ f \ \leq \ \psi(\eta' - 1, \eta - 1). \end{aligned} \tag{42}$$

If $\eta' > \eta + 1$, then this violates (M'), while if $\eta' = \eta + 1$ then the conditions are satisfied only for non-generic values of f. This proves the result.

(ii). Since g is a pws-equilibrium, it follows that $\psi(\eta_i(g) - 1, \eta_j(g) - 1) \geq f$. However:

$$\psi(\eta_i(g), \eta_k(g)) > \psi(\eta_i(g) - 1, \eta_k(g)) \geq \psi(\eta_i(g) - 1, \eta_j(g) - 1) \geq f. \tag{43}$$

where the first inequality follows from the (PSOL), the second inequality follows from the hypothesis $\eta_k(g) \leq \eta_j(g) - 1$ and (NSPL), and the third inequality follows from the hypothesis that g is an equilibrium network. Similarly, for player k, we note that

$$\psi(\eta_k(g), \eta_i(g)) > \psi(\eta_k(g) - 1, \eta_i(g)) \geq \psi(\eta_i(g) - 1, \eta_j(g) - 1) \geq f. \tag{44}$$

where the first inequality follows from (PSOL), the second inequality follows the hypothesis that $\eta_i(g) \leq \eta_k(g) < \eta_j(g)$ and (PSOL) and (NSPL), while the third inequality follows from the hypothesis that g is an equilibrium network. Thus players i and k have a strict incentive to form a link. This proves (ii).

$\triangle$

Proof of Proposition 4.3: We note that the conclusions of Proposition 4.2 remain valid as before since (SM) is stronger than (M). Thus the statement about symmetric networks in particular follows. We now derive an additional property of asymmetric networks which exploits (SM): *If* $g_{i,j} = 1$*,* $\eta_j > \eta_i$ *then* $g_{i,k} = 1$*, if* $\eta_i \leq \eta_k \leq \eta_j$. The first step considers the incentives of player i.

$$\psi(\eta_i, \eta_k) > \psi(\eta_i - 1, \eta_k - 1) \geq \psi(\eta_i - 1, \eta_j - 1) \geq f. \tag{45}$$

The first inequality follows from (SM), the second inequality follows from the hypothesis $\eta_k \leq \eta_j$ and (NSPL), while the third inequality follows from the equilibrium hypothesis. We next consider the incentives of player k.

$$\psi(\eta_k, \eta_i) > \psi(\eta_i - 1, \eta_i) \geq \psi(\eta_i - 1, \eta_j - 1) \geq f. \tag{46}$$

The first inequality follows from (PSOL), the second from (NSPL), while the final inequality follows from the equilibrium hypothesis. Thus players i and k have an incentive to form a link.

We now show that every non-singleton component $C(g)$ in complete. Clearly a component with 2 players is complete; suppose therefore that $|C(g)| > 2$ and it is incomplete. Proposition 4.2 implies that $C(g)$ is asymmetric. Let player j be a maximally connected player in this component who has a link with a non-maximally connected player i (note that such players must exist), i.e., $g_{i,j} = 1$ and $\eta_j > \eta_i$. Then there must exist some player k such that $g_{j,k} = 1$ but $g_{i,k} = 0$. Since $\eta_k \leq \eta_j$ the claim above implies that $g_{i,k} = 1$. Since k was arbitrary, this implies that $\eta_j \leq \eta_i$, a contradiction that proves the claim that $C(g)$ is complete. A similar argument holds for any non-singleton component. Finally, we note that an equilibrium cannot have two (non-singleton) components of equal size; this follows from (SM) and arguments in Proposition 4.2. The proof is complete.

$\triangle$

Proof of Proposition 4.4: The proof of part (i) is identical to that for symmetric networks in Proposition 4.2 and is omitted. We present the proof of (ii).

We first take up the first statement in (ii). Since g is a pws-equilibrium, it follows that $\psi(\eta_i(g) - 1, \eta_j(g) - 1) \geq f$ and $\psi(\eta_j(g) - 1, \eta_i(g) - 1) \geq f$. However:

$$\psi(\eta_i(g), \eta_k(g)) \geq \psi(\eta_j(g) - 1, \eta_k(g)) > \psi(\eta_j(g) - 1, \eta_i(g) - 1) \geq f. \tag{47}$$

where the first inequality follows from the hypotheses that $\eta_i(g) < \eta_j(g)$ and (NSOL), the second inequality follows from the hypothesis $\eta_k(g) > \eta_i(g)$ and (PSPL), and the third inequality follows from the hypothesis that g is an equilibrium network. Similarly, for player k, we note that

$$\psi(\eta_k(g), \eta_i(g)) \geq \psi(\eta_j(g) - 1, \eta_i(g)) > \psi(\eta_j(g) - 1, \eta_i(g) - 1) \geq f. \tag{48}$$

where the first inequality follows from the hypotheses that $\eta_j(g) > \eta_k(g)$ and (NSOL), the second inequality follows the (PSPL) hypothesis, while the third inequality follows from the hypothesis that g is an equilibrium network. Thus players i and k have a strict incentive to form a link. This proves the first statement in (ii).

We now turn to the second statement in (ii). Let players $i, j \in N_m(g)$; since g is asymmetric $\eta_i(g) = \eta_j(g) > 0$. Since g is a pws-equilibrium, it follows that, $\psi(\eta_i(g) - 1, \eta_k(g) - 1) \geq f$ for some $k \in N$. Similarly, it is true that $\psi(\eta_j(g) - 1, \eta_l(g) - 1) \geq f$ for some $l \in N$. Next note that:

$$\psi(\eta_i(g), \eta_j(g)) > \psi(\eta_i(g) - 1, \eta_j - 1) \geq \psi(\eta_i(g) - 1, \eta_k(g) - 1) \geq f. \tag{49}$$

where the first inequality follows from condition (M), the second inequality follows from the hypothesis $\eta_j(g) \geq \eta_k(g)$ and (PSPL), while the final inequality follows from the equilibrium hypothesis. A similar argument holds for player j. This proves the second statement in (ii).

△

Proof of Proposition 4.5: We note that the conclusions of Proposition 4.2 remain valid as before since (SM) is stronger than (M). Thus the statement about symmetric networks in particular follows. We next derive the following stronger restriction on asymmetric networks which exploits (SM): *If* $g_{i,j} = 1$ *and* $\eta_j > \eta_i$, *then* $g_{i,k} = 1$ *for* $\eta_i \leq \eta_k \leq \eta_j$. The proof of this claim involves slightly different arguments than the analogous claim in Proposition 4.3 and is presented here for the sake of completeness. The first step considers the incentives of player i.

$$\psi(\eta_i, \eta_k) > \psi(\eta_j - 1, \eta_k - 1) \geq \psi(\eta_j - 1, \eta_i - 1) \geq f. \tag{50}$$

The first inequality follows from (NSOL) and (PSPL), the second inequality follows from (PSPL), while the third inequality follows from the equilibrium hypothesis. We next consider the incentives of player k.

$$\psi(\eta_k, \eta_i) > \psi(\eta_k - 1, \eta_i - 1) \geq \psi(\eta_j - 1, \eta_i - 1) \geq f. \tag{51}$$

The first inequality follows from (SM), the second from (NSOL), while the final inequality follows from the equilibrium hypothesis. Thus players i and k have an incentive to form a link.

We can now use arguments analogous to those of Proposition 4.3 to complete the proof of the result.

△

Proof of Proposition 4.6: (i). Consider the class of symmetric equilibria. If η and η' are any two degrees, $\eta \geq \eta'$, then using (NSOL) and (NSPL) respectively:

$$\psi(\eta, \eta) \ \leq \ \psi(\eta', \eta) \ \leq \ \psi(\eta', \eta') \tag{52}$$

There are three possible cases to consider:

(a) $f < \psi(n-2, n-2)$. Note that the complete network is a pws-equilibrium over this range of costs: players cannot add any links nor do they have an incentive to delete any links. We now argue that no other degree can be sustained as an pws-equilibrium over this range. Consider any degree $\eta < (n-1)$. In g^η, there will exist players i and j such that $g_{i,j} = 0$. However, noting that $\eta \leq (n-2)$, it follows that $\psi(\eta, \eta) \geq \psi(n-2, n-2) > f$. Therefore players i and j have an incentive to form a link; thus g^η is not a pws-equilibrium.

(b) $f > \psi(0, 0)$. In this case, the empty network is a pws-equilibrium because players have no incentive to add a link, and there are no links to delete. Consider any degree of symmetry $\eta > 0$. In g^η, there will exist players i and j such that $g_{i,j} = 1$. However, $\psi(\eta-1, \eta-1) \leq \psi(0, 0) < f$. Therefore players i and j have an incentive to delete their link; thus g^η is not a pws-equilibrium.

(c) $\psi(n-2, n-2) \leq f \leq \psi(0, 0)$. In this case, we can find a degree η' such that $\psi(\eta', \eta') \leq f \leq \psi(\eta'-1, \eta'-1)$, where at least one inequality is strict. Then $g^{\eta'}$ is a pws-equilibrium because over this range of costs players have no incentive to delete their links or to form additional links. Now consider any $\eta \neq \eta'$. If $\eta > \eta'$, then $f \geq \psi(\eta', \eta') \geq \psi(\eta-1, \eta-1)$, and players in g^η have an incentive to delete their links for generic values of f. Similarly, if $\eta < \eta'$, then $\psi(\eta, \eta) \geq \psi(\eta'-1, \eta'-1) \geq f$, and unlinked players in g^η have an incentive to form a link for generic values of f.

(ii). We first prove that an unconnected asymmetric pws-equilibrium network can have at most one isolated player. Suppose to the contrary that i and j are two isolated players in a pws-equilibrium network g. Since $g \neq g^e$, there must exist players k and l such that $g_{k,l} = 1$ in g. Further, since g is an pws-equilibrium network, $\psi(\eta_k(g)-1, \eta_l(g)-1) \geq f$ and $\psi(\eta_l(g)-1, \eta_k(g)-1) \geq f$. It now follows that:

$$\psi(\eta_i(g), \eta_j(g)) \ = \ \psi(0, 0) \ \geq \ \psi(\eta_k(g)-1, \eta_l(g)-1) \ \geq \ f \tag{53}$$

and similarly for player j. Therefore for generic values of f, players i and j have an incentive to form a link, a contradiction.

We establish next that if i is isolated in a pws-equilibrium network g, then the network g_{-i} is symmetric. Suppose to the contrary that g_{-i} is asymmetric. In g_{-i} there exists players j and l with the minimum and maximum number of links respectively such that $0 < \eta_j(g) < \eta_l(g)$. Since $n \geq 4$, there also exists a player k in g_{-i} such that $\eta_j(g) \leq \eta_k(g) \leq \eta_l(g)$ with $g_{j,k} = 0$ and $g_{k,l} = 1$. Since g is a pws-equilibrium network, $\psi(\eta_k(g)-1, \eta_l(g)-1) \geq f$ and

$\psi(\eta_l(g)-1, \eta_k(g)-1) \geq f$. Further, $\eta_l(g)-1 \geq \eta_j(g)$ and $\eta_k(g)-1 \geq \eta_i(g)$. Using (NSOL) and (NSPL) respectively:

$$\begin{aligned} \psi(\eta_j(g), \eta_i(g)) \ &\geq\ \psi(\eta_l(g)-1, \eta_i(g)) \ \geq\ \psi(\eta_l(g)-1, \eta_k(g)-1) \ \geq\ f \\ \psi(\eta_i(g), \eta_j(g)) \ &\geq\ \psi(\eta_k(g)-1, \eta_l(g)) \ \geq\ \psi(\eta_k(g)-1, \eta_l(g)-1) \ \geq\ f \end{aligned} \tag{54}$$

Therefore, for generic values of f, players i and j have an incentive to form a link. This contradicts the hypothesis that g is a pws-equilibrium network.

Finally we demonstrate the uniqueness property. Let g be a pws- equilibrium network with an isolated player l and $\eta > 0$ denote the degree of symmetry of g_{-l}. Suppose f is the cost of forming links. Any two players i and j in g_{-l} who are linked must satisfy $\psi(\eta-1, \eta-1) \geq f$. Any player i in g_{-l} should have no incentive to form a link with the isolated player l, i.e. $\psi(\eta, 0) < f$. Now consider any $\eta' \neq \eta$ and assume that g_{-l} is symmetric of degree η'. If $\eta' > \eta > 0$, then $\psi(\eta'-1, \eta'-1) \leq \psi(\eta, \eta'-1) < \psi(\eta, 0) < f$ and players in g_{-l} have an incentive to delete their links. Similarly, let $0 \leq \eta' < \eta$. Then $\psi(\eta', 0) \geq \psi(\eta-1, 0) \geq \psi(\eta-1, \eta-1) \geq f$ and a player in g_{-l} has an incentive to form a link with the isolated player for generic values of f. Further, $\psi(0, \eta') \geq \psi(0, \eta-1) \geq \psi(\eta-1, \eta-1) \geq f$ and the isolated player has an incentive to reciprocate the link for generic values of f. This establishes the uniqueness result.

We now take up the case of pws-equilibrium g in which there is no singleton component. Suppose $g_{i,j} = 1$ for some $i, j \in N_q(g)$, $q \in \{2, ..., m\}$. Consider some $k, l \in N_1(g) \cup \ldots \cup N_{q-1}(g)$ and suppose that $g_{k,l} = 0$. Since $\eta_k(g) \leq \eta_i(g) - 1$ and $\eta_l(g) \leq \eta_j(g) - 1$:

$$\begin{aligned} \psi(\eta_k(g), \eta_l(g)) \ &\geq \psi(\eta_i(g)-1, \eta_j(g)-1) \ \geq\ f \\ \psi(\eta_l(g), \eta_k(g)) \ &\geq \psi(\eta_j(g)-1, \eta_i(g)-1) \ \geq\ f \end{aligned} \tag{55}$$

Therefore, k and l have an incentive to form a link for generic values of f, a contradiction.

Now suppose that $g_{i,j} = 0$ for some $i, j \in N_q(g)$, $q \in \{1, ..., m-1\}$ in a pws-equilibrium network. Consider any $p, r \in N_{q+1} \cup \ldots \cup N_m(g)$ such that $g_{p,r} = 1$. Since $\eta_i(g) \leq \eta_p(g) - 1$ and $\eta_j(g) \leq \eta_r(g) - 1$:

$$\begin{aligned} \psi(\eta_p(g)-1, \eta_r(g)-1) \ &\leq \psi(\eta_i(g), \eta_j(g)) \ \leq\ f \\ \psi(\eta_r(g)-1, \eta_p(g)-1) \ &\leq \psi(\eta_j(g), \eta_i(g)) \ \leq\ f \end{aligned} \tag{56}$$

Therefore p and r have an incentive to delete their link for generic values of f, a contradiction that establishes the result.

△

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