A New Family of Allocation Rules for Network Games

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Abstract

Previous allocation rules for network games, such as the Myerson Value, implicitly or explicitly take the network structure as fixed. In many situations, however, the network structure can be altered by players. This means that the value of alternative network structures (not just sub-networks) can and should influence the allocation of value among players on any given network structure. I present a family of allocation rules that incorporate information about alternative network structures when allocating value.

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1 Introduction

This paper examines the allocation of value among players connected by a network. Behind this is the idea that the productivity of the players in some economic or social situation depends on the ways in which they are "connected." The applications are quite wide and varied, ranging from communicating information about job openings to modeling international trade agreements or political alliances. How the fruits of the total productive value might end up being allocated or transferred among players turns out to be important, not only in terms of fairness considerations, but also because it determines players incentives to form various networks.

Myerson's (1977) value for communication games views a network as a fixed primitive that defines the opportunities for communication among the players in a cooperative game. As such, one the network is fixed, a communication game may be viewed as a cooperative game where the role of the network is to define which coalitions can function. Those feasible coalitions are the ones who can freely communicate via the given network (so that any two players in the coalition are path connected in the network via players in the coalition).

Jackson and Wolinsky (1996) showed that the Myerson value has a direct extension to network games. Network games are richer than both cooperative games and communication games, as the value that accrues to a given set of players depends on how those players are connected in terms of network connections, rather than just the fact that they are connected.[1]

While network games are richer than communication games and readily apply to a wide class of situations, the extension of the Myerson value to network games still inherits most of the perspective it has in communication games. Most importantly, the network is implicitly viewed as fixed when value is being allocated. In particular, while

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[1] Let me be explicit in terms of the terminology I follow. A *cooperative game* is the well-known concept where a characteristic function specifies how much value any given coalition of players can produce. A *communication game* is a combination of a cooperative game plus a network (i.e., a nondirected graph). The network determines which coalitions can function, as only players who are path connected via the network can generate value together. This class of games was introduced by Myerson (1977), although he referred to the graphs as cooperation structures. The literature that followed Myerson has used the term communication structures, and so I will call those communication games. The third class of games are *network games* where a value function specifies how much value is produced by a particular network of players. These games were introduced by Jackson and Wolinsky (1996) and allow the value to depend not only on who is connected, but how they are connected. Thus it allows for costs, benefits, and externalities across links in a network.

the Myerson Value involves Shapley Value style calculations that account for how the network is built up, it does not account for the value of alternative network structures that might have formed.

Here, I take the view that the allocation of value is something that is occurring together with or even before the formation of a network, or at least that allocation rules should sometimes allow and account for the possibility that alternative networks could have been formed. From this point of view an allocation rule should depend on the value that might accrue to a series of potential networks and not just the network that forms and its subnetworks. In particular, evaluating the value of a given link or player involves how that link or player might contribute to various networks and how others might potentially serve as a substitute.

To understand why this issue arises in network games, but not in cooperative games, it is important to recognize that in a cooperative game we generally assume that it is the grand coalition that forms. Value operators such as the Shapley value decompose the grand coalition in evaluating players' contributions, but the issue of what other coalitions might have formed, or what other players might have played the role of a given player never even arise. In contrast, in the context of a network game, it is often (perhaps even generally) the case given any implicit or explicit costs to links that the set of efficient networks will not include the complete network. This means that we must care about how to allocate value to some networks that are not complete networks. In such cases, the allocation of value may depend on information about the roles of given players that involve calculations based on networks that are not subnetworks of a given network. For instance, if we consider a player at the center of a star network (where all players are connected to this center player but not directly to each other), how much value accrues to this player might depend on whether this player is the only one who can serve this function, or whether it is the case than any player could equally well have served as the center in a productive way.

The paper proceeds as follows. Section 2 provides definitions for network games. Section 3 provides an example illustrating some of the issues. Section 4 provides a menu of properties that we might want an allocation rule to satisfy. Section 5 introduces two new allocation rules and characterizes them. Section 6 provides a definition of the core for network games and two other new allocation rules. Section 7 concludes with a comparison of the properties satisfied by various allocation rules.

2 Network Games and Allocation Functions

Players

$N = \{1, \ldots, n\}$ is a set of players who are connected in some network relationship.

Networks

A network is a list of which pairs of players are linked to each other and is modeled as a non-directed graph. A network is thus a list of *unordered* pairs of players $\{i, j\}$, where $\{i, j\} \in g$ indicates that i and j are linked under the network g.

For simplicity, write ij to represent the link $\{i, j\}$, and so $ij \in g$ indicates that i and j are linked under the network g.

More formally, let g^N be the set of all subsets of N of size 2. $G = \{g \subset g^N\}$ denotes the set of all possible networks or graphs on N.

For instance, if $N = \{1, 2, 3\}$ then $g = \{12, 23\}$ is the network where there is a link between players 1 and 2, a link between players 2 and 3, but no link between players 1 and 3.

Given any $S \subset N$, let g^S be the set of all subsets of S of size 2, so that g^S is the complete network among the players in S.

Let

$$g|_S = \{ij \ : \ ij \in g \text{ and } i \in S, j \in S\}.$$

Thus $g|_S$ is the network found deleting all links except those that are between players in S.

The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network obtained by deleting link ij from an existing network g is denoted $g - ij$.

Let $N(g)$ be the set of players who have at least one link in g. That is,

$$N(g) = \{i | \exists j \text{ s.t. } ij \in g\}.$$

Let $n(g) = \#N(g)$ be the number of players involved in g.

Let $L_i(g)$ be the set of links that player i is involved in, so that

$$L_i(g) = \{ij | \exists j \text{ s.t. } ij \in g\},$$

and let $\ell_i(g) = \#L_i(g)$.

Let $\ell(g) = \sum_i \ell_i(g)$ be the number of links in g.

Paths and Components

A *path* in a network $g \in G$ between players i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.

Looking at the path relationships in a network naturally partitions a network into different connected subgraphs that are commonly referred to as components.

A *component* of a network g, is a nonempty subnetwork $g' \subset g$, such that

- if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and
- if $i \in N(g')$ and $j \notin N(g')$ then there does not exist a path in g between i and j.

Thus, the components of a network are the distinct connected subgraphs of a network.

The set of components of g is denoted $C(g)$. Note that $g = \cup_{g' \in C(g)} g'$.

Under this definition of component, a completely isolated player who has no links is not considered a component.

Value Functions

A *value function* is a function $v : G \to I\!R$.

The set of all possible value functions is denoted V.

A value function specifies the total value that is generated by a given network structure. The calculation of value may involve both costs and benefits and is a richer object than a characteristic function of a cooperative game, or that induced in a communication game, as it allows the value that accrues to depend on the network structure and not only on the coalition of players involved (or who can communicate).

When we begin to think about the possibilities available to a set of links at an *ex ante* stage, before any network is fixed in place, it will generally be natural to think of the value as being generated by the maximum over possible networks that could be formed using those links. This is captured by its monotonic cover.[2]

Given a value function v, its *monotonic cover* $\widehat{v}$ is defined by

$$\widehat{v}(g) = \max_{g' \subset g} v(g').$$

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[2] In a situation where v is component additive, the monotonic cover is also the superadditive cover. However, if v is not component additive so that there are externalities across components, then v can be monotonic without being superadditive.

Monotonicity

A value function v is *monotonic* if $v(g') \geq v(g)$ whenever $g \subset g'$.

Note that a value function is monotonic if and only if $v = \widehat{v}$.

In general the value functions that are natural in network games will *not* be monotonic. Nevertheless, information from their monotonic covers will be useful in designing allocation rules.

A Basis for Value Functions

It will be useful to define a basis for the set of value functions.

Let v_g denote the value function that satisfies

$$v_g(g') = \begin{cases} 1 & \text{if } g \subset g' \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Call such a v_g a *basic* value function.

Note that any v can be written as a linear combination of v_g's. That is, for any v we can write $v = \sum_g c_g v_g$ for some collection of scalars c_g. This follows from viewing the v's as vectors in $I\!R^{2^{n(n-1)/2}}$ and noting that the $2^{n(n-1)/2}$ different v_g's are linearly independent and so form a basis for $I\!R^{2^{n(n-1)/2}}$.

Network Games

A *network game* is a pair, (N, v), of a set of players and a value function.

Allocation Rules

How the value generated by a network is allocated among the players, either through their decisions or perhaps even by some outside intervention, is described by an allocation rule.

An *allocation rule* is a function $Y : G \times V \to I\!R^N$ such that $\sum_i Y_i(g, v) = v(g)$ for all v and g.

Note that balance ($\sum_i Y_i(g, v) = v(g)$) is built into the definition of an allocation rule.

It is important to note that an allocation rule depends on both g and v. This allows an allocation rule to take full account of an player i's role in the network. This includes not only what the network configuration is, but also and how the value generated depends on the overall network structure. For instance, consider a network $g = \{12, 23\}$ in a situation where $v(g) = 1$. Player 2's allocation might be very different on what the value of other networks are. For instance, if $v(\{12, 23, 13\}) = 0 = v(\{13\})$, then in a sense 2 is essential to the network and may receive a large allocation. If on the other hand $v(g') = 1$ for all networks, then 2's role is not particularly special. This information might turn out to be relevant, which is why the allocation rule is allowed (but not required) to depend on it.

3 The Myerson Value and some Examples

Myerson (1977) developed a variation of the Shapley value for communication games. This rule was subsequently referred to as the Myerson value (e.g., see Aumann and Myerson (1988)). The Myerson value also has a corresponding allocation rule in the context of network games which is a direct generalization of the version defined by Myerson for communication games, as shown by Jackson and Wolinsky (1996). That allocation rule is expressed as follows.

$$Y_i^{MV}(g,v) = \sum_{S \subset N \setminus \{i\}} (v(g|_{S \cup i}) - v(g|_S)) \left(\frac{\#S!(n - \#S - 1)!}{n!} \right) \tag{1}$$

Component Balance

A value function v is *component additive* if $v(g) = \sum_{g' \in C(g)} v(g')$ for any $g \in G$ and $g' \in C(g)$.

An allocation rule Y is *component balanced* if for any component additive v

$$\sum_{i \in N(g')} Y_i(g', v) = v(g').$$

Note that component balance only makes requirements on Y for v's that are component additive, and Y can be arbitrary otherwise. If v is not component additive, then requiring component balance of an allocation rule $Y(\cdot, v)$ would necessarily violate balance.

Component balance requires that if a value function is component additive, we might wish to have the value generated by any component to be allocated to the players among that component.

Equal Bargaining Power

An allocation rule satisfies *equal bargaining power*[3] if for any component additive v and $g \in G$

$$Y_i(g,v) - Y_i(g - ij, v) = Y_j(g,v) - Y_j(g - ij, v).$$

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[3] This was called "fairness" by Myerson (1977).

Note that equal bargaining power does *not* require that players split the marginal value of a link. It just requires that they equally benefit or suffer from its addition. It is possible (and generally the case) that $Y_i(g) - Y_i(g - ij) + Y_j(g) - Y_j(g - ij) \neq v(g) - v(g - ij)$.

A Characterization of the Myerson Value

The following characterization of the Myerson value is from Jackson and Wolinsky (1996) and is an easy extension of a theorem of Myerson (1977).[4]

THEOREM **1** *Y satisfies component balance and equal bargaining power if and only if $Y(g, v) = Y^{MV}(g, v)$ for all $g \in G$ and any component additive v.*

Note that the above characterization only holds on component additive value functions.[5]

Examples and Criticisms of Component Balance and Equal Bargaining Power

EXAMPLE **1** *Insensitivity of the Myerson Value to Alternative Networks*

Let $v(\{12\}) = v(\{23\}) = 1$, $v(\{12, 23\}) = 1$, and $v(g) = 0$ for all other networks. Let $v'(g) = 1$ for all $g \neq \emptyset$. That is, under v' the value any non-empty network is 1. Note that

$$Y^{MV}(\{12, 23\}, v) = Y^{MV}(\{12, 23\}, v') = \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right).$$

[FIGURE 1 Here.]

Here, player 2 gets a bigger allocation in the network $\{12, 23\}$ than the other players. This reflects player 2's status in two links in the network, and comes about through the Shapley value style calculations underlying the Myerson value, where we can think of building up the network $\{12, 23\}$ by adding players one at a time.

While player 2's status is special in the network $\{12, 23\}$, player 2's status is not at all special in the value function v'. If the allocation of value is decided upon based at

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[4] Dutta and Mutuswami (1997) extend the characterization to allow for weighted bargaining power, and show that one obtains a version of a weighted Myerson Value.

[5] Myerson (1977) built component balance into his definition of an allocation rule, as naturally the value that accrued to the grand coalition in a communication game is determined in a component additive way. However, once one turns to network games one admits the possibility of externalities across components and has to be explicit about the component additivity of the value function in order to extend Myerson's result.

an ex ante stage, or based on the symmetry of value it is not clear that player 2 should enjoy any special treatment. In fact, it seems natural that under v' all players should receive equal payments.

The insensitivity of the Myerson value to such changes in v is the subject of this paper, and I propose alternative allocation rules that will account for such variations in value functions.

Before proceeding, let us examine some examples that provide a detailed look at some of the shortcomings of the axioms underlying the Myerson value, which will help us in understanding the issues more deeply and in developing alternative allocation rules.

EXAMPLE **2** *A Criticism of Equal Bargaining Power*

Let $v(\{12\}) = v(\{23\}) = 1$ and $v(g) = 0$ for all other networks.

Any allocation rule, including the Myerson value, that satisfies equal bargaining power (and allocates 0 to the players on the empty network) will have $Y_1(\{12\}, v) = Y_2(\{12\}, v)$.

[Figure 2 here]

Here, there is a real asymmetry among the players and player 2 is more a critical player than the others. It is not at all clear why we should require that the allocation to players 1 and 2 be the same in the network $\{12\}$, as player 2 has a viable outside option while player 1 does not.

If we do want to impose some sort of fairness or equality in bargaining power, it should only apply when the roles of the players in question are really comparable. This underlies the anonymity, strong anonymity, and equal treatment of equals properties defined in the next section.

EXAMPLE **3** *A Look at Component Balance*

Let $v(g) = \frac{5}{4}$ if $\ell(g) = 1$, $v(g) = 2$ if $\ell(g) = 2$, and $v(g) = 0$ otherwise.

In this case, the Myerson Value allocates $\frac{5}{8}$ to each player involved in the link in a one link network; and $\frac{1}{2}$ to players having one link in a two link network and 1 to the player having two links in a two link network.

[Figure 3 here]

Why might one criticize component balance here? It is based on noting that component balance sits in a sort of "no-man's-land" here, not quite strong enough to capture its normative arguments and at the same time strong enough to rule out other natural conditions. To be more explicit, a natural and reasonable argument behind component balance is that it makes sure that components of a network receive their due value which prevents their members from wanting to walk away and reallocate their value among themselves. However, if one uses that sort of motivation to justify component balance, then it is logical to worry about all coalitional deviations, not just those of components. Thus, the natural strengthening of component balance is a core property. Note however, that component balance does not guarantee that an allocation lies in the core (for a formal definition see Section 6) when the core is nonempty. For example, under the Myerson value above, if the network is $\{12, 23\}$, then players 1 and 3 receive $\frac{1}{2}$ each, and would benefit by deviating and forming their own one link network where they could receive $\frac{5}{8}$ each.

On the other hand, while component balance stops short of dealing with all relevant coalitional possibilities, it is already strong enough to be in conflict with some fairness and anonymity properties. In the above example, the v is completely anonymous and all players are completely interchangeable. If we then think of the value being bargained upon at an ex ante stage, it is not clear why value is not split equally among the players *regardless* of which of which network is formed. This strong form of anonymity is precluded by component balance.

The criticisms of the two properties lie on different levels. The criticism of equal bargaining power is that it is a fundamentally flawed property. The criticism of component balance is that it is not strong enough to reflect its real normative grounding, and yet strong enough to conflict with other properties. Thus, one might prefer to either strengthen component balance if one really is concerned with what value is generated by various coalitions and components, or abandon the property if one is more concerned with other normative properties.

I will suggest alternatives below, which are to weaken equal bargaining power to only apply in situations where players are equal in some broader sense, and to replace component balance either with a core requirement or with some anonymity and equity properties. Depending on the variation, one will end up with very different allocation rules.

4 Properties of Value Functions and Allocation Rules

I now list a series of properties of allocation rules. These will be used in characterizing different allocation rules. As such, some combinations of these properties will be inconsistent with each other. I will provide the full list of properties and a brief discussion of some of the motivation behind each before returning to provide characterizations of the various rules.

Anonymity

Given a permutation of players π (a bijection from N to N) and any $g \in G$, let $g^\pi = \{\pi(i)\pi(j) | ij \in g\}$. Thus, g^π is a network that shares the same architecture as g but with the specific players permuted.

Given a permutation π, let v^π be defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in G$.

An allocation rule Y is *anonymous* if for any v, $g \in G$, and permutation π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$.

Anonymity of an allocation rule requires that if all that has changed is the labels of the players and the value generated by networks has changed in an exactly corresponding fashion, then the allocation only change according to the relabeling. Of course, anonymity is a type of fairness condition that has a rich axiomatic history, and also naturally arises situations where Y represents the utility or productive value coming directly from some social network.

Note that anonymity allows for asymmetries in the ways that allocation rules operate even in completely symmetric networks. For instance, anonymity does *not* require that each player in a complete network get the same allocation. That would be true only in the case where v was in fact anonymous. Generally, an allocation rule can respond to different roles or powers of players and still be anonymous.

Most allocation rules that one can think of satisfy this version of anonymity.

Equal Treatment of Equals

Given any two players i and j define a permutation π^{ij} such that $\pi^{ij}(i) = j$ and $\pi^{ij}(j) = i$ and $\pi(k) = k$ for all $k \notin \{i, j\}$. Say that i and j are *equals under* v if $v(g^{\pi^{ij}}) = v(g)$ for all g.

An allocation rule Y satisfies *equal treatment of equals* if for any $v \in V$, players i and j who are equals under v, and $g \in G$ such that $g^{\pi^{ij}} = g$, $Y_j(g, v) = Y_i(g, v)$.

Equal treatment of equals says that all allocation rule should give the same payoff to players who play exactly the same role in terms of symmetric position in a network under a value function that depends on them in exactly the same way.[6]

Equal treatment of equals is implied by anonymity and might be thought of as more of a symmetry condition that anonymity, and is also a condition that has a rich background in the axiomatic literature that will be satisfied by most any allocation rule.

Strong Anonymity

A value function v is *anonymous* if $v(g^\pi) = v(g)$ for any $g \in G$ and permutation π.

Anonymous value functions are those such that the architecture of a network matters, but not the labels of players.

An allocation rule Y is *strongly anonymous* if for any anonymous v, $g \in G$, and players i and j $Y_j(g, v) = Y_i(g, v)$.

Strong anonymity states that if v is anonymous, then the allocation be independent of players' positions in the network.

If the allocation is agreed upon at the same time (or before) the network is being formed, then the strong version of anonymity makes sense. From an ex ante point of view, when a given network g is formed and the value function is anonymous, then any permutation of g would have resulted in the same value and so in a sense the particular positions of the players become irrelevant. While players' ex post positions in the network may be asymmetric, their ex ante potential roles and contributions are completely identical.

Note that strong anonymity is in conflict with component balance. Note also that it implies that the allocation is completely egalitarian whenever v is anonymous.

Ex Ante Rules

An allocation rule Y is an *ex-ante* rule if $Y(g, v) = Y(g, \widehat{v})$ for all v and efficient g (relative to v).

The idea that the allocation only depends on the super-additive cover of the value function is one property that is implied from the perspective that the allocation is being decided upon when the network is being formed - thus at an ex ante time. With the idea that sub-optimal networks would not be formed, the allocation of value would

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[6] There are various versions of this property. One might slightly strengthen the condition to require that $Y_j(g^{\pi^{ij}}, v) = Y_i(g, v)$ for any g. This condition would also be satisfied by almost any natural allocation rule.

then only depend on the value of optimal networks given some set of available links, and so the monotonic cover is all that enters into the calculations.

Note that this equivalence is only required on efficient networks, as the value on other networks might not even be the same (i.e., $v(g) \neq \widehat{v}(g)$) and so the condition would be impossible to impose except on efficient networks.

Proportionality

An allocation rule Y is *proportional* if for each i and v either $Y_i(g, v) = 0$ for all g, or for any g and g' such that $v(g') \neq = 0$

$$\frac{Y_i(g, v)}{Y_i(g', v)} = \frac{v(g)}{v(g')}.$$

When one takes an ex ante perspective, the question of how value is allocated on inefficient networks becomes a bit tricky, as it is not clear why they should ever form. Nonetheless, as we know that there may exist some conflicts between stability and efficiency (e.g., see Jackson and Wolinsky (1996)), it is important to have something to say about these allocations, both because such networks might end up forming, and also because their allocations might end up being important "off-the-equilibrium-path" considerations.

The proportionality condition takes an obvious (but certainly not the only natural) path to allocation of value on inefficient networks. Given the ex ante perspective, players allocations can be determined on efficient networks and then rescaled for inefficient ones. One might think of this as a way of saying that the efficient allocations have taken into account all of the relevant bargaining and decision making that would go on in terms of determining players' relative contributions or power due to a given value function.

Additivity

An allocation rule Y is *additive* if for any v and v', and scalars $a \geq 0$ and $b \geq 0$

$$Y(g^N, av + bv') = aY(g^N, v) + bY(g^N, v').$$

Additivity is a well-known foundation for Shapley-style calculations. Note that in the context of network games it may be in conflict with making decisions from an ex ante perspective. In particular, the super-additive cover of $av + bv'$ is not necessarily the same as the sum of the super-additive covers.

This suggests a more limited version of additivity.

Weak Additivity

An allocation rule Y is *weakly additive* if for any monotonic v and v', and scalars $a \geq 0$ and $b \geq 0$

$$Y(g^N, av + bv') = aY(g^N, v) + bY(g^N, v').$$

and if $av - bv'$ is monotonic, then

$$Y(g^N, v - v') = aY(g^N, v) - bY(g^N, v').$$

The weaker version of additivity is one that is adapted to hold together with the ex ante perspective as it only applies in monotonic cases. While the value functions that we are interested in will rarely be monotonic, this condition will still be useful as in allocating value a value functions monotonic cover will be an important ingredient in the calculations and we can apply the condition to monotonic covers.

The reasoning behind (weak) additivity is that if we are making calculations regarding allocations based on what players or links contribute to various network possibilities, and we increase or decrease those contributions, we should treat those increases or decreases in the same way that we treat the original contributions.

Link-Based Rules

An allocation rule Y is *link-based* if there exists $\phi : V \times G \to I\!R^{n(n-1)/2}$ such that $\sum_{ij \in g^N} \phi_{ij}(g, v) = v(g)$, and

$$Y_i(g, v) = \sum_{j \neq i} \frac{\phi_{ij}(g, v)}{2}$$

When thinking about how we measure players' contributions or how they bargain over their worth, this may be done directly in terms of adding or removing the players, or it may be done in terms of the links that they control. That is, we may think of assigning allocations to *links* and then this indirectly determines how value accrues to players.

The perspective of assigning values to links rather than players was first taken by Meeson (1988) (see also Borm, Owen, and Tijs (1992)) in the context of communication games and resulted in a variation on the Myerson value called the position value. More generally, we may think of the alternatives of doing player based or link based calculations when deriving any allocation rule.

Equal Treatment of Vital Players

An allocation rule Y satisfies *equal treatment of vital players* if v_g is a basic value function for some g, then

$$Y_i(g, v_g) = \begin{cases} \frac{1}{n(g)} & \text{if } i \in N(g) \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Basic value functions are ones where the players involved in g are all vital to the functioning of any network, in the sense that no value is generated without all the players in g being present and that no other players contribute in any way.

In such a setting, the players in g are all in some sense "equals" and all others are worthless. In this case, this equal treatment property allocates value equally to each player.

Equal Treatment of Vital Links

An allocation rule Y satisfies *equal treatment of vital links* if v_g is a basic value function for some $g \neq \emptyset$, then

$$Y_i(g, v_g) = \frac{\ell_i(g)}{2\ell(g)}$$

This property is a variation on equal treatment of vital players, where it is links who are viewed as being vital rather than players, and will correspond to a link-based version of an allocation rule.

This also might be thought of as reflecting the idea that it is not really the players that are vital, but certain collections of links (i.e., those in g). As such from one perspective it is the links which are all equal.

5 Two New Allocation Rules Based on Ex Ante Calculations

Let me now propose two variations on an allocation rule that while still based on Shapley Value style calculations take an ex ante perspective.

One of the variations is a link-based rule while the other is player based. It is not clear to me that either one is clearly more pertinent than the other. In one sense the link-based rule is richer in that it considers variations on a link-by-link basis, and presumably players could choose to withhold certain links and not others. On the other hand, it is really the players who control the links and not the links themselves who accumulate value (as becomes clearer when dealing with a core definition). I first present the link-based rule and then the player-based version.

The Link-Based Ex Ante Bargaining Rule

$$Y_i^{LBEAB}(g,v) = \frac{v(g)}{\widehat{v}(g^N)} \sum_{j \neq i} \left[\sum_{g \subset g^N - ij} \frac{1}{2} \left(\widehat{v}(g+ij) - \widehat{v}(g) \right) \left(\frac{\#g!([n(n-1)/2] - \#g - 1)!}{[n(n-1)/2]!} \right) \right].$$

On a superficial level this rule bears some similarities to the Myerson Value because we see Shapley Value style calculations. However, it is a quite different allocation rule. In fact, it violates both equal bargaining power and component balance, and is characterized by some conditions that are violated by the Myerson value. Most importantly it is an ex ante rule and thus accounts for the value of all prospective networks.

The calculations are as follows. First, one may think of value being allocated to links, and so the allocation to a given player i is simply that value summed across links. Note that value is allocated to a given link whether or not it is present in a given network - which reflects the ex ante perspective. The allocation on a given network is then rescaled in proportion to the value that would be allocated on an efficient network. The $\frac{1}{2}$ reflects the fact that the value of a given link is controlled by two players. The remaining part of the calculation is then based on a Shapley Value allocation of value to links as to what they contribute to the overall value possible under v. This calculation is from an ex ante perspective, so it is relative to the monotonic cover $\widehat{v}$.

Theorem 2 *An allocation rule satisfies equal treatment of vital links, weak additivity, and is an ex ante and proportional rule if and only if it is Y^{LBEAB}.*

Note that the above characterization holds for all value functions, not just component additive ones.

Proof of Theorem 2: It is easily checked that Y^{LBEAB} satisfies equal treatment of vital links and is an ex ante and proportional rule. Let us show that it satisfies weak additivity.

Consider any monotonic v and v', and scalars $a \geq 0$ and $b \geq 0$. Then $av + bv'$ is monotonic and the same as its monotonic cover. So,

$$Y_i^{LBEAB}(g^N, av + bv')$$

$$= \sum_{j \neq i} \left[\sum_{g \subset g^N - ij} \frac{1}{2} \left(av(g+ij) + bv'(g+ij) - av(g) - bv'(g) \right) \left(\frac{\#g!([n(n-1)/2] - \#g - 1)!}{[n(n-1)/2]!} \right) \right].$$

Given the monotonicity of v and v', the right hand side can be rewritten as $aY_i^{LBEAB}(g^N, v) + bY_i^{LBEAB}(g^N, v')$ which is the desired conclusion. Next, suppose that $av - bv'$ is monotonic. Then we can write

$$Y_i^{LBEAB}(g^N, av - bv')$$

$$= \sum_{j \neq i} \left[\sum_{g \subset g^N - ij} \frac{1}{2} \left(av(g+ij) - bv'(g+ij) - av(g) + bv'(g) \right) \left(\frac{\#g!([n(n-1)/2] - \#g - 1)!}{[n(n-1)/2]!} \right) \right].$$

Given the monotonicity of v and v', the right hand side can be rewritten as $aY_i^{LBEAB}(g^N, v) - bY_i^{LBEAB}(g^N, v')$ which is again the desired conclusion.

Next, let us verify that any allocation rule satisfying the claimed properties must be Y^{LBEAB}. Consider an allocation rule satisfying the given properties. It is enough to show that it is uniquely determined.

First, under proportionality, if we know $Y(v, g)$ on some efficient g, then we know $Y(v, g')$ on any network.

Second, the fact that Y is an ex ante allocation rule implies that $Y(v, g) = Y(\widehat{v}, g)$ on efficient networks, and so we need only show that $Y(\widehat{v}, g)$ is uniquely determined on an efficient network. In particular, since Y is an ex ante rule and $\widehat{v}$ is monotonic, it is enough to show that $Y(\widehat{v}, g^N)$ is uniquely determined.

Third, by the monotonicity of $\widehat{v}$, we can write

$$\widehat{v} = \sum_{g'} c_{g'} v_{g'},$$

for a set of scalars $c_{g'}$'s.

Fourth, Let $G^- = \{g' : c_{g'} < 0\}$ and let $G^+ = g^N \setminus G^-$. So we can write

$$\widehat{v} = \sum_{g' \in G^+} c_{g'} v_{g'} - \sum_{g' \in G^-} |c_{g'}| v_{g'}.$$

Thus, by weak additivity

$$Y(g^N, \widehat{v}) = Y(g^N, \sum_{g' \in G^+} c_{g'} v_{g'}, g^N) - Y(g^N, \sum_{g' \in G^-} |c_{g'}| v_{g'}, g^N).$$

Applying weak additivity again allows us to write

$$Y(g^N, \widehat{v}) = \sum_{g'} c_{g'} Y(g^N, v_{g'}).$$

Since Y is a proportional rule and both g^N and g' are efficient for $v_{g'}$, $Y(g^N, v_{g'}) = Y(g', v_{g'})$. By equal treatment of vital links, this is uniquely determined. ∎

The Player-Based Ex Ante Bargaining Rule

The following is the player-based version of the ex ante bargaining allocation rule.

$$Y_i^{PBEAB}(g,v) = \frac{v(g)}{\widehat{v}(g^N)} \sum_{S \subset N \setminus \{i\}} (\widehat{v}(g^{S \cup i}) - \widehat{v}(g^S)) \left(\frac{\#S!(n - \#S - 1)!}{n!} \right) \qquad (2)$$

Theorem **3** *An allocation rule satisfies equal treatment of vital links, weak additivity, and is an ex ante and proportional rule if and only if it is* Y^{PBEAB}.

The proof of Theorem 3 is analogous to that for Theorem 2.

Let us now compare the allocation of value by the ex ante bargaining rules to that of the Myerson Value.

Example **4** *Example 1 Revisited*

This is a slight generalization of Example 1.

Consider value functions v and v' are defined as follows.

$v(\{12\}) = v(\{23\}) = 1$, $v(\{12, 23\}) = w \geq 1$, and $v(g) = 0$ for all other networks.

$v'(g) = w$ for all g with at least two links and $v'(g) = 1$ on g with one link.

The Myerson Value allocates as follows.

$$Y^{MV}(\{12, 23\}, v) = Y^{MV}(\{12, 23\}, v') = \left(\frac{w}{3} - \frac{1}{6}, \frac{w}{3} + \frac{1}{3}, \frac{w}{3} - \frac{1}{6} \right).$$

Here the link- and player-based ex ante allocation rules provide different allocations depending on values of networks that are not subnetworks of $\{12, 23\}$.

$$Y^{LBEAB}(\{12, 23\}, v) = \left(\frac{w}{4}, \frac{w}{2}, \frac{w}{4} \right).$$

$$Y^{PBEAB}(\{12, 23\}, v) = \left(\frac{w}{3} - \frac{1}{6}, \frac{w}{3} + \frac{1}{3}, \frac{w}{3} - \frac{1}{6} \right).$$

$$Y^{LBEAB}(\{12, 23\}, v') = \left(\frac{w}{3}, \frac{w}{3}, \frac{w}{3} \right).$$

$$Y^{PBEAB}(\{12, 23\}, v') = \left(\frac{w}{3}, \frac{w}{3}, \frac{w}{3} \right).$$

[Figure 4 here]

There are several things to note. Both allocation rules split values equally under v' which is consistent with the ex ante perspective and the strong anonymity that they both satisfy.

The player-based allocation rule coincides with the Myerson Value on v since under v only subnetworks of $\{12, 23\}$ generate any value. This differs from the link-based version. The link-based ex ante rule is easy to see here as only the links $\{12\}$ and $\{23\}$ generate value, but not the link $\{13\}$, and so they split the value equally on $\{12, 23\}$. Player 2 being involved in twice as many links as the others ends up with twice the value.

6 The Core of a Network Game and the Networkolus

While the ex ante allocation rules proposed in the previous section have nice properties in terms of their ex ante perspective, there is a shortcoming that they inherit from their Shapley Value origins. Namely, they are not always in the core of a network game. In situations where players are deciding on the allocation of the value of a network at the same time that they are forming the network there is a natural core definition that captures some constraints on the allocation of value that would be required to avoid certain forms of instability.

The Core of a Network Game

A network-allocation pair $g \subset g^N$ and $y \in I\!R^n$ is in the *core* of the network game (N, v) if $\sum_i y_i \leq v(g)$ and $\sum_{i \in S} y_i \geq \widehat{v}(g^S)$ for all $S \subset N$.

The core of a network game provides a natural look at how the allocation of value of a network can be taken together with the formation of a network, especially regarding accounting for coalitional incentives.

Most of the literature on network games has takes the allocation rule as given and then examines some non-cooperative formation procedure.[7] There are some papers that

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[7] For instance, see Aumann and Myerson (1988) for an extensive form game of network formation, Jackson and Wolinsky (1996) for notions of pairwise stability, Dutta and Mutuswami (1997) for Nash and coalition-proof Nash equilibria of network formation games, Watts (2001) and Jackson and Watts (2002) for dynamic processes of network formation, and Jackson and van den Nouweland (2000) for a strong stability notion, and Bala and Goyal (2000) for a one-sided (directed) network formation

have analyzed the allocation of value at the same time as the formation of a network.[8] Those papers examine specific demand-based noncooperative network formation games. The core provides a protocol-free way analyzing the simultaneous allocation of value and formation of a network.

Core Consistency

An allocation rule Y is *core consistent*, if for any v such that the core is nonempty, there exists at least one g such that $(g, Y(g,v))$ is in the core.

Example **5** *The Core*

Consider the value function v from Example 4 except allowing w to be any number, defined by $v(\{12\}) = v(\{23\}) = 1$, $v(\{12,23\}) = w$, and $v(g) = 0$ for all other networks.

Note that the core is always non-empty, regardless of the value of w. If $w \leq 1$, then the networks $\{12\}$ and $\{23\}$ together with an allocation of $(0,1,0)$ are in the core. If $w > 1$, then the network $\{12,23\}$ together with any allocations such that $y_1 + y_2 \geq 1$ and $y_2 + y_3 \geq 1$ are in the core. For instance, the allocation $(\frac{w-1}{2}, 1, \frac{w-1}{2})$ is in the core together with the network $\{12,23\}$.

Note, however, that when $w = \frac{7}{6}$, then the allocations of the Myerson, Player-Based Ex Ante, and Link-Based Ex Ante allocation rules are not in the core and thus fail core consistency.

[Figure 5 here]

The failure of Shapley Value style calculations to turn up core allocations is well-known from the cooperative game theory literature and it appears here too.

The ex ante approach that I have proposed here can also be combined with other ways of dividing up value among a group of players (or links). Thus, there is a natural variation of the Nucleolus that can be defined for network games.

The Nucleolus defined by Schmeidler (1969) has the very desirable property that it always lies in the core of a cooperative game whenever that object is non- empty, among some other nice properties. There is a natural analog of the nucleolus that can

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process.

[8] For instance, see Slikker and van den Nouweland (2000), Currarini and Morelli (2000), and Mutuswami and Winter (2000). See also, Navarro and Perea (2002), which provides for an implementation of the Myerson Value. It would be interesting to see if their work could be extended to cover the Ex Ante Bargaining allocation rules defined here.

be defined from an ex ante perspective in network games. I will refer to that allocation rule as the networkolus. There are two versions, depending on whether the definition is link-based or player-based.

I will begin with the player-based version as that always lies in the core of a network game when it is non-empty.

The Player-Based Networkolus

Let $B(g,v) = \{x \in I\!R^n | \sum_i x_i = v(g)\}$ be the balanced allocations for g under v.

Let $e_S(x) = \sum_{i \in S} x_i - \widehat{v}(g^S)$ be the excess allocated to coalition S relative to their monotonic value under v, and let $e(x)$ denote the vector with entries for each nonempty $S \subset N$.

Let $Y^{PN}(g^N, \widehat{v}) = y$ be the unique allocation such that $e(y)$ leximin dominates $e(x)$ for all $x \in B(g^N, \widehat{v})$. Then define $Y^{PN}(g,v) = \frac{v(g)}{\widehat{v}(g)} Y^{PN}(g^N, \widehat{v})$.

We can also consider a link-based version of the networkolus.

The Link-Based Networkolus

Let $B^\ell(g,v) = \{x \in I\!R^{n(n-1)/2} | \sum_i x_i = v(g)\}$ be the balanced allocations to links for g under v.

Let $e^\ell_g(x) = \sum_{ij \in g} x_{ij} - \widehat{v}(g)$ be the excess allocated to set of links g relative to their monotonic value under v, and let $e^\ell(x)$ denote the vector with entries for each nonempty $g \subset g \subset N$.

Let $Y_i^{LN}(g^N, \widehat{v}) = \sum_{j \neq i} y_{ij}$ be the unique allocation such that $e^\ell(y)$ leximin dominates $e^\ell(x)$ for all $x \in B^\ell(g^N, \widehat{v})$. Then define $Y^{LN}(g,v) = \frac{v(g)}{\widehat{v}(g)} Y^{LN}(g^N, \widehat{v})$.

Example **6** *The Networkolus and the Core*

Consider the value function v from Example 5 defined by $v(\{12\}) = v(\{23\}) = 1$, $v(\{12,23\}) = w$, and $v(g) = 0$ for all other networks.

Recall that the Myerson Value, Link-Based Ex Ante Bargaining Allocation Rule and the Player-Based Ex Ante Bargaining Allocation Rule failed to be core consistent in this example.

Let us examine the networkolus in this example, and consider the case where $3 \geq w \geq 1$.[9]

$Y^{PN}(\{12,23\}, v) = (\frac{w-1}{2}, 1, \frac{w-1}{2})$ which indeed is in the core.

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[9] The same conclusions hold in other cases, but involve different calculations.

Note however, that the link-based version of the networkolus is not in the core. In this particular example it actually coincides with Y^{LBEAB}. That is,

$Y^{LN}(\{12, 23\}, v) = (\frac{w}{4}, \frac{w}{2}, \frac{w}{4})$ which is not in the core.

[Figure 6 here]

The difficulty with the link-based networkolus is that it is making allocations based on links rather than players, while it is the players who are critical in core calculations. This actually suggests a refinement of the core, which is a link-based core.

The Link-Core of a Network Game

A network-allocation pair $g \subset g^N$ and $y \in I\!R^{n(n-1)/2}$ is in the *link-core* of the network game (N, v) if $\sum_{ij} y_{ij} \leq v(g)$ and $\sum_{ij \in g'} y_{ij} \geq \widehat{v}(g')$ for all $g' \subset g^N$.

The link-core of a network game provides a refinement of the core where it is links that are being allocated value. When the link-core is nonempty, it is straightforward to see that the core is nonempty.[10] However, there are many situations (such as under the v in Example 6) where the link-core is empty.

An allocation rule Y is *link-core consistent*, if for any v such that the link-core is nonempty, there exists at least one g and $y \in I\!R^{n(n-1)/2}$ such that (g, y) is in the link-core and $\sum_{j \neq i} y_{ij} = Y_i(g, v)$ for each i.

Interestingly, the link-based networkolus is link-core consistent but not core consistent, while the player-based networkolus is core consistent but not link-core consistent.

Other Ex Ante Allocation Rules

As should be clear by now, there is a general approach to defining allocation rules for network games that take an ex ante perspective. For instance, one might want to define per-capita versions of the networkolus. To do this, let ϕ be your favorite imputation rule from cooperative game theory. Noting that $\widehat{v}(g^S)$ defines a characteristic function as we vary S, we can apply ϕ directly and denote the resulting allocations by $\phi(\widehat{v})$.

To define a player-based rule, set $Y^{\phi}(g, v) = \frac{v(g)}{\widehat{v}(g)} \phi(\widehat{v})$.

To define a link-based rule, we can apply ϕ to groups of links rather than players (so on $\widehat{v}(g)$, viewing g as a coalition of links) and denote the resulting allocations to links by ϕ^{ℓ}. Next, set $Y_i^{\phi^{\ell}}(g, v) = \frac{v(g)}{\widehat{v}(g)} \frac{1}{2} \sum_j \phi_{ij}^{\ell}(\widehat{v})$.

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[10] Let $g, \widehat{y}$ be in the link-core. Let $y \in I\!R^n$ be defined by $y_i = \sum_{j \neq i} \widehat{y}_{ij}$. Then (g, y) lies in the core.

7 A Comparison of Allocation Rules

Before comparing a few allocation rules in terms of some of the conditions I have discussed here,[11] a let us recall two other allocation rules from Jackson and Wolinsky (1996).

The Egalitarian Allocation Rule

The *egalitarian* allocation rule Y^e is defined by

$$Y_i^E(g,v) = \frac{v(g)}{n}$$

for all i and g.

The egalitarian allocation rule splits the value of a network equally among all members of a society regardless of what their role in the network is. It is clear that the egalitarian allocation rule will have very nice properties in terms of aligning player incentives with efficiency.

The Component-Wise Egalitarian Allocation Rule

The egalitarian rule violates component balance. The following modification of the egalitarian rule respects component balance.

The *component-wise egalitarian* allocation rule Y^{ce} is defined as follows for component additive v's and any g.

$$Y_i^{CE}(g,v) = \begin{cases} \frac{v(h)}{|N(h)|} & \text{if there exists } h \in C(g) \text{ such that } i \in h, \\ 0 & \text{otherwise.} \end{cases}$$

For any v that is not component additive, set $Y^{ce}(\cdot,v) = Y^e(\cdot,v)$.

The component-wise egalitarian splits the value of a component network equally among all members of that component, but makes no transfers across components.

The component-wise egalitarian rule has some nice properties in terms of aligning player incentives with efficiency, although not quite to the extent that the egalitarian rule does.[12]

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[11] There are many other conditions one might consider, as many conditions from cooperative games and communication games have natural analogs in the context of network games. For instance, although I have not provided a characterization of the networkolus, but there should be natural variations on Peleg's (1986) reduced game property to derive an analog of Sobolev's (1975) characterization of the nucleolus.

[12] See Jackson and Wolinsky (1996) Section 4 for some detailed analysis of the properties of the egalitarian and component-wise egalitarian rules.

	Y^{MV}	Y^{LBEAB}	Y^{LBEAB}	Y^{LN}	Y^{PN}	Y^{E}	Y^{CE}
anonymity	+	+	+	+	+	+	+
strong anonymity	-	+	+	+	+	+	-
equal treatment of vital links	-	+	-	+	-	-	-
equal treatment of vital players	+	-	+	-	+	-	-
additivity	+	-	-	-	-	+	+
weak additivity	+	+	+	-	-	+	+
ex ante	-	+	+	+	+	+	-
proportional	-	+	+	+	+	+	-
equal bargaining power	+	-	-	-	-	+	-
component balance	+	-	-	-	-	-	+
core consistency	-	-	-	-	+	-	-
link-core consistency	-	-	-	+	+	-	-

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