THE MINIMAL DOMINANT SET IS A NON-EMPTY CORE-EXTENSION

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ABSTRACT. A set of outcomes for a TU-game in characteristic function form is dominant if it is, with respect to an outsider-independent dominance relation, accessible (or admissible) and closed. This outsider-independent dominance relation is restrictive in the sense that a deviating coalition cannot determine the payoffs of those coalitions that are not involved in the deviation. The minimal (for inclusion) dominant set is non-empty and for a game with a non-empty coalition structure core, the minimal dominant set returns this core.

1. Introduction

For a TU-game in coalitional form, there are two fundamental and strongly linked problems: (*i*) what coalitions will form, and (*ii*) how will the members of these coalitions distribute their total coalitional worth. We attempt to answer these questions. Following Harsányi (1974), we presuppose some bargaining process among the players. At first, one of the players proposes some outcome (a payoff vector augmented with a coalition structure). In case some coalition could gain by acting for themselves, it can reject this initial outcome and propose a second outcome. Of course, in order to be able to make a counter-proposal, the deviating coalition is a member of the new coalition structure and none of the players in the deviating coalition looses and some win when moving towards the new outcome. We impose an additional condition that we call *outsider-independence*: a coalition C that belongs to the initial coalition structure and that does not contain a deviating player survives the deviation; the players in C stay together and keep their pre-deviation payoffs. This contrasts with, for example, the approach by Sengupta and Sengupta (1994) and Shenoy (1980, Section 5). They also study coalition formation in a TU-framework, but their domination relation does not incorporate such an outsider-independence condition: the deviating coalition is allowed to determine the payoffs and the structure of *all* players. This seems unrealistic to us. In contrast, our approach is based on the observation that outsiders' payoffs are unaffected by the formation of the deviating coalition and hence outsiders do not necessarily notice the deviation until the new coalition structure is announced.

Once such a counter-proposal has popped up, another coalition may reject this counter-proposal in favor of a third outcome, and so forth. This bargaining process generates a dominating chain of outcomes. In case the game has a non-empty coalition structure core (Greenberg, 1994, Section 6), the bargaining process enters this core after a finite number

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of steps (this is shown in Kóczy and Lauwers, 2001). Conclusion: the coalition structure core, if non-empty, is accessible.

Similarly to the core, the coalition structure core has an important shortcoming: non-emptiness is far from being guaranteed. The present paper tackles games with an empty set of undominated outcomes.

We impose three conditions upon a solution concept. *First*, we insist on accessibility: from each outcome there is a dominating chain that enters the solution. *Second*, the solution is closed for domination: each outcome that dominates an outcome in the solution also belongs to the solution. The intuition behind this axiom is straightforward. In case there are no "undominated outcomes", there might exist "undominated sets" of outcomes. Such a set must be closed for outsider-independent domination. A collection of outcomes that combines accessibility and closedness is said to be a dominant set. And, *third*, from all the dominant sets, we only retain the minimal (with respect to inclusion) ones.
The following observation provides a further argument in favor of these three conditions: in case the game generates undominated outcomes, then the accessibility of the coalition structure core implies that this core is the unique minimal dominant set. Uniqueness and non-emptiness extends to arbitrary games:

> THEOREM A. *Let (N, v) be a TU-game. Then, there is exactly one minimal dominant set. Moreover, this minimal dominant set is non-empty.*

In other words, the minimal dominant set is a non-empty coalition structure core extension. On the one hand, the three conditions we impose upon a solution concept are strong enough to filter out the coalition structure core (in case it is non-empty), and on the other hand these conditions are weak enough to return a non-empty set of outcomes in case the game has an empty coalition structure core. As a matter of fact, the minimal dominant set meets Zhou's (1994) minimal qualifications for a solution concept: non-imposition with respect to the coalition structure[1] and non-emptiness.

We close the discussion on Theorem A with an example. Consider a three player game with an empty core: singletons have a zero value, pairs have a value equal to 8, and the grand coalition has a value 9. The payoff vector $(4, 4, 0)$ supported by the coalition structure $(\{1, 2\}, \{3\})$ belongs to the minimal dominant set. However, this outcome is not efficient: the total payoff in this vector amounts to 8, where the value 9 is obtainable. On the other hand, the efficient outcome $(3, 3, 3; \{1, 2, 3\})$ does not belong to the minimal dominant set. Hence, the minimal dominant set might contain inefficient outcomes and at the same time there might be efficient outcomes outside the minimal dominant set. While the core selects those outcomes that satisfy efficiency and stability, these two properties are not so well linked as soon the core is empty (Section 5 returns to this issue).

[1]In the framework of endogenous coalition formation, a solution concept "is not a priori defined for payoff vectors of a particular coalition structure, and it does not always contain payoff vectors of every coalition structure," (Zhou, 1994, p513).

Along the proof of Theorem A we come across the following properties of the outsider-independent domination relation. First, the set of outcomes that indirectly dominate an (initial) outcome is closed in the Euclidean topology. And, second:

> Theorem B. *Let (N, v) be a game. Then there exists a natural number $\tau = \tau(N, v)$ such that for all outcomes a and b we have that a indirectly dominates b if and only if there exists a dominating chain from b to a of length at most τ.*

As a consequence, the accessibility axiom can be sharpened: for each game (N, v) the minimal dominant set can be reached via $\tau = \tau(N, v)$ subsequent counter-proposals. This number τ can be imposed as a time-limit for the completion of the bargaining process.

Theorem B dramatically improves previous results on the accessibility of the core. We mention two of them. First, Wu (1977) showed the existence of a bargaining scheme that converges to the core and rephrased this result as "the core is globally stable". Second, Sengupta and Sengupta (1996) construct for each imputation a sequence of dominating imputations that enters the core in finitely many steps. We extend these results to the coalition structure core (and to the minimal dominant set); in addition, here we provide an upper bound for the length of the dominating chains.

The next section collects notation and definitions. Section 3 considers dominating chains, the length of such chains, and proves Theorem B. Section 4 defines the minimal dominant set and proves Theorem A. Section 5 lists some deficiencies and some properties of the minimal dominant set. An example indicates that the outsider-independency-condition rightly prevents some outcomes (that belong to the solution of Sengupta and Sengupta, 1994) from entering the minimal dominant set.

2. Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be a set of n players. Non-empty subsets of N are called coalitions. A *coalition structure* is a set of pairwise disjoint coalitions so that their union is N and represents the breaking up of the grand coalition N. Let $\mathcal{P}$ and $\mathcal{Q}$ be two coalition structures such that for each coalition C in $\mathcal{Q}$ we have that either C belongs to $\mathcal{P}$ or there exists a coalition in $\mathcal{P}$ that includes C, then $\mathcal{Q}$ is *finer* than $\mathcal{P}$ (and $\mathcal{P}$ is *coarser* than $\mathcal{Q}$). For a coalition structure $\mathcal{P} = \{C_1, C_2, \ldots, C_m\}$ and a coalition C, the *partners' set* $P(C, \mathcal{P})$ of C in $\mathcal{P}$ is defined as the union of those coalitions in $\mathcal{P}$ that have a non-empty intersection with C:

$$P(C,\mathcal{P}) \;=\; \{i\,|\,i \in C_j \text{ with } j \text{ such that } C_j \cap C \neq \varnothing\} \;=\; \bigcup_{C_j \cap C \neq \emptyset} C_j.$$

Its complement $O(C, \mathcal{P}) = N \setminus P(C, \mathcal{P})$ is said to be the set of *outsiders*.

A characteristic function $v : 2^N \setminus \{\varnothing\} \to \mathbb{R}$ assigns a real value to each coalition. The pair (N, v) is said to be a transferable utility game in characteristic function form, in short, a game.

An *outcome* of a game (N, v) is a pair $(x, \mathcal{P})$ with x in $\mathbb{R}^n$ and $\mathcal{P}$ a coalition structure of N. The vector $x = (x_1, x_2, \ldots, x_n)$ lists the payoffs of each player and satisfies

$$\forall i \in N : x_i \geq v(\{i\}) \qquad \text{and} \qquad \forall C \in \mathcal{P} : x(C) = v(C),$$

with $x(C) = \sum_{j \in C} x_j$. The first condition is known as individual rationality: player i will cooperate to form a coalition only if his payoff x_i exceeds the amount he obtains on his own. The second condition combines feasibility and the myopic behavior of the players, it states that each coalition in the coalition structure $\mathcal{P}$ allocates its value among its members. Outcomes with the same payoff vector are said to be *payoff equivalent*.

The set of all outcomes is denoted by $\Omega(N, v)$. The set $\Omega(N, v)$ is non-empty: it contains the outcome in which the grand coalition is split up in singletons.

In case the grand coalition forms, then an outcome is a pair $(x, \mathcal{P})$ with $\mathcal{P} = \{N\}$, $x_i \geq v(\{i\})$, and $x(N) = \sum_{i \in N} x_i = v(N)$. As such, outcomes generalize imputations.

Now, we define the *outsider-independent* dominance relation. An interpretation and a discussion follows. Later on, we use the shorthand *o-i-domination*. We keep *domination* as a reference to the concept of Sengupta and Sengupta (1994).[2]

Definition 2.1. Let $x, y \in \mathbb{R}^n$ and let C be a coalition. Then $x \geq_C y$ if $x_i \geq y_i$ for each player i in C. And, $x >_C y$ (vector x dominates y by C) if $x \geq_C y$ and $x(C) > y(C)$.

Let (N, v) be a game and let $a = (x, \mathcal{P})$ and $b = (y, \mathcal{Q})$ be two outcomes. Then, outcome *a outsider-independent dominates b by C*, denoted by $b \xrightarrow{C} a$, if

- $\mathcal{P}$ contains C,
- $\mathcal{P}$ contains all coalitions in $\mathcal{Q}$ that do not intersect C,
- in $\mathcal{P}$ the players in $P(C, \mathcal{Q}) \setminus C$ form singletons,[3]
- $x >_C y$, and
- the restrictions of x and y to $O(C, \mathcal{Q})$ coincide.

Furthermore, we hold on the next terminology:

- C is called the deviating coalition, its members are *deviators*.

Finally, outcome *a outsider-independent dominates b*, denoted by $b \longrightarrow a$, if $\mathcal{P}$ contains a coalition C such that $b \xrightarrow{C} a$.

This o-i-domination relation can be interpreted in a dynamic way. Let $(y, \mathcal{Q}) \xrightarrow{C} (x, \mathcal{P})$ and consider $(y, \mathcal{Q})$ as the initial outcome. Note that the initial partition $\mathcal{Q}$ and the deviating coalition C completely determine the new partition $\mathcal{P}$. Also, the deviating coalition C enforces the new outcome $(x, \mathcal{P})$. Indeed, in order to obtain a higher total payoff, coalition C separates from its partners (and at least one member of C is strictly better off). The players in $P(C, \mathcal{Q}) \setminus C$ become ex-partners of C and fall apart in singletons. Finally, the outsiders, i.e. the players not in $P(C, \mathcal{Q})$, are left untouched.

The new outcome is achieved independently of the outsiders. This is in strong contrast

[2]Outcome $(x, \mathcal{P})$ dominates outcome $(y, \mathcal{Q})$, if $\mathcal{P}$ contains a coalition C such that $x(C) = v(C) > y(C)$ and for each j in C one has $x_j \geq y_j$ (Sengupta and Sengupta, 1994, p349).

[3]This condition can be relaxed.

to the "classical" domination relation where the deviating coalition dictates the payoffs and the coalition structure for the whole set of players. Hence, in employing this classical domination relation one implicitly assumes the cooperation of the outsiders even in case the proposed outcome is less favorable for them.

Definition 2.1 also models a merger: the deviating coalition is the union of some of the coalitions in the initial coalition structure.

In case one is concerned with coalition formation processes, o-i-dominance seems to be a natural and a straightforward extension of the domination relation at the level of payoff vectors. On the other hand, if outcome b is dominated by a at the level of payoff vectors, then there exists an outcome a' that o-i-dominates b. Therefore, the set of o-i-*un*dominated outcomes coincides with the set of undominated outcomes.

Definition 2.2. Let (N, v) be a game. The *coalition structure core* $C(N, v)$ is the set of outcomes that are not o-i-dominated.
Equivalently, the pair $(x, \mathcal{P})$ is in the coalition structure core if and only if it satisfies feasibility and coalitional rationality:

- for each coalition C in $\mathcal{P}$ we have $x(C) \leq v(C)$, and
- for each coalition S we have $x(S) \geq v(S)$.

The coalition structure core might contain payoff equivalent outcomes; and in case "the" core is non-empty (i.e. in case the grand coalition forms), then the coalition structure core includes the core.

3. Dominating chains

We introduce *sequential o-i-domination* and we show that in order to check for this, one can concentrate on chains the length of which does not exceed some upper bound.

Definition 3.1. Let $a, b \in \Omega$. Outcome a is said to be *accessible* from b (denoted by $b \stackrel{\wedge}{\rightarrow} a$ or $a \stackrel{\wedge}{\leftarrow} b$), if one of the following conditions holds

- a and b are payoff equivalent, or
- a *sequentially o-i-dominates* b, i.e. there exists a natural number T and a sequence of outcomes
$$a_0 = b,\ a_1,\ \ldots,\ a_{T-1},\ a_T = a$$
such that a_t o-i-dominates a_{t-1} for $t = 1, 2, \ldots, T$. The sequence
$$a_0 = b \longrightarrow a_1 \longrightarrow \ldots \longrightarrow a_{T-1} \longrightarrow a_T = a$$
is called an o-i-dominating chain of length T.

This accessibility relation $\stackrel{\wedge}{\longrightarrow}$ is the transitive and reflexive closure of the o-i-domination relation $\longrightarrow$.
Two different outcomes might be accessible from each other. E.g. payoff equivalent outcomes are accessible from each other; this boils down to the implicit assumption that repartitioning involves no costs in case the payoff vector does not change.

The accessibility relation describes a possible succession of transitions from one outcome to another. An initial outcome is proposed and the players are allowed to deviate from it. We are interested in the outcomes that will appear at the *end* of a sequence of transitions. Some of the outcomes will definitely disappear, while others show up again and again. As such, the game is absorbed in (hopefully) a small set of outcomes. The following result gives a precise content to the expression "end of an o-i-dominating chain".

Theorem 3.2. *Let (N, v) be a game. Then there exists a natural number τ such that for all outcomes a and b in $\Omega(N, v)$ we have that a is accessible from b if and only if there exists an o-i-dominating chain from a to b of length at most τ.*

The *if*-part in the above statement (accessibility if there is a chain) is immediate. In order to prove the *only-if*-part, we need some additional preparations.

- First, the set $\Omega(N, v)$ of outcomes is partitioned such that two outcomes of the same class induce similar deviations,
- Second, the set N is partitioned according to the behavior of the players in an o-i-dominating chain.

The finiteness of these operations is crucial in the proof of Theorem 3.2. We start the discussion with the partitioning of the set of outcomes.

Definition 3.3. Let (N, v) be a game. Two outcomes $(x, \mathcal{P})$ and $(y, \mathcal{Q})$ are similar if they satisfy the following list of conditions:

- $\mathcal{P} = \mathcal{Q}$,
- for each coalition C we have, $x(C) \geq v(C)$ if and only if $y(C) \geq v(C)$, and
- for each coalition C, for each coalition structure $\mathcal{C}$ of C, and for each D in $\mathcal{C}$, we have

$$x(D) - v(D) \geq v(C) - v(\mathcal{C}) \quad \text{iff} \quad y(D) - v(D) \geq v(C) - v(\mathcal{C}), \qquad (*)$$

  where $v(\mathcal{C}) = \Sigma_{E \in \mathcal{C}}\, v(E)$.

In this way the set $\Omega(N, v)$ of outcomes is partitioned into a finite number of classes. The number of classes in this partition depends upon the cardinality of N.

Definition 3.4. Let (N, v) be a game and let

$$b = (x_0, \mathcal{P}_0) \xrightarrow{D_1} (x_1, \mathcal{P}_1) \xrightarrow{D_2} \dots \xrightarrow{D_t} (x_t, \mathcal{P}_t) \xrightarrow{D_{t+1}} \dots \xrightarrow{D_T} (x_T, \mathcal{P}_T) = a,$$

be a o-i-dominating chain from b to a. We interpret t as a time index.
For each $t = 0, 1, \dots, T - 1$ we divide the set of players into two subsets:

- The set W_t of winning players collects those players who, from t onwards, are either outsiders or deviators. Formally: i belongs to W_t if

$$i \in O(D_s, \mathcal{P}_{s-1}) \cup D_s, \text{ for all } s = t + 1, \dots, T.$$

  From t onwards the payoff of a winning player cannot decrease.

- The set L_t of losing players collects those players who, at a certain point in time, are left behind as singletons. Formally: i belongs to L_t if
$$\text{there exists } \ s \geq t+1 \text{ such that } i \in P(D_s, \mathcal{P}_{s-1}) \setminus D_s.$$
Let $\ell(t,i) \geq t+1$ denote the first time (after t) that player i is standing alone, i.e. $\{i\} \in \mathcal{Q}_{\ell(t,i)}$.

Obviously, along the o-i-dominating chain we have
$$W_0 \subseteq W_1 \subseteq \ldots \subseteq W_{T-1} = O(D_T, \mathcal{P}_{T-1}) \cup D_T.$$
Indeed, once a player is winning, his status cannot change. As a consequence we obtain
$$L_0 \supseteq L_1 \supseteq \ldots \supseteq L_{T-1} = P(D_T, \mathcal{P}_{T-1}) \setminus D_T.$$
Furthermore, at each moment t a losing player i with $\ell(t-1,i) = t$ *might* move up to the class W_t of winning players.
Since winners and losers are completely determined by the coalition structures and the deviating coalitions, this division of N into winning and losing players does not depend upon the individual payoffs.

Proof of Theorem 3.2 (Only-if part).

The key idea is that any chain from b to a longer than τ can be made shorter. We construct such a shorter chain. First, we locate two compatible outcomes c and c'. Next, we trisect the chain ($b \hat{\rightarrow} c$, $c \hat{\rightarrow} c'$, $c' \hat{\rightarrow} a$), we remove the middle part, and we reattach the head and the tail. Since the outcomes c and c' are not likely to be identical the tail of the chain must be modified; we keep the deviating coalitions and we adjust the outcomes along the tail.

We proceed in four steps. The first step is the surgical one: we locate two compatible outcomes and we make the cuts; here we implicitly define the value of τ. In Step 2 we show that the first deviation in the tail of the original chain can be attached to the head. Then, the second deviation is attached (Step 3) and so forth (Step 4).

Step 1. Starting up the proof.
If the length of the o-i-dominating chain from b to a is large enough (larger than τ), then there exist two outcomes $c = (y_0, \mathcal{Q}_0)$ and $c' = (z_0, \mathcal{Q})$ in the o-i-dominating chain that (i) are similar and (ii) partition the players (winning versus losing) in the same way. Indeed, there are only a finite number of different classes of similar outcomes and there are only a finite number of ways to split up the finite set N of players into two subsets. We write $\mathcal{Q}_0$ instead of $\mathcal{Q}$ and we assume that $(y_0, \mathcal{Q}_0)$ comes later than $(z_0, \mathcal{Q}_0)$. Denote the sets of winning and losing players for the outcomes $(y_0, \mathcal{Q}_0)$ and $(z_0, \mathcal{Q}_0)$ by W_0 and L_0. In sum, we have the following o-i-dominating chain
$$b = (x_0, \mathcal{P}_0) \longrightarrow \ldots \longrightarrow \overbrace{(z_0, \mathcal{Q}_0)}^{W_0, L_0} \longrightarrow \ldots \longrightarrow \overbrace{(y_0, \mathcal{Q}_0)}^{W_0, L_0} \longrightarrow \ldots \longrightarrow (x_m, \mathcal{P}_m) = a.$$
We rename the last part in this original o-i-dominating chain and we indicate the deviating coalitions:

$$(x_0,\mathcal{P}_0)\longrightarrow\ldots\longrightarrow\overbrace{(z_0,\mathcal{Q}_0)}^{W_0,L_0}\underbrace{\longrightarrow\ldots\longrightarrow\overbrace{(y_0,\mathcal{Q}_0)}^{W_0,L_0}}_{\text{middle part}}\underbrace{\xrightarrow{C_1}(y_1,\mathcal{Q}_1)\xrightarrow{C_2}\ldots\xrightarrow{C_T}(y_T,\mathcal{Q}_T)}_{y-\text{chain}}.$$

We show the existence of payoff vectors $z_1, z_2, \ldots, z_T$ such that this initial chain from b to a (of length m) can be shortened to

$$(x_0,\mathcal{P}_0)\longrightarrow\ldots\longrightarrow\overbrace{(z_0,\mathcal{Q}_0)}^{W_0,L_0}\underbrace{\xrightarrow{C_1}(z_1,\mathcal{Q}_1)\xrightarrow{C_2}\ldots\xrightarrow{C_T}(z_T,\mathcal{Q}_T)=a}_{z-\text{chain}}.$$

Since the coalition structure $\mathcal{Q}_0$ and the deviating coalitions $C_1, C_2, \ldots, C_T$ coincide along the initial y-chain and the new z-chain, both chains generate the same sets W_s and L_s of winning and of losing players, $s = 1, 2, \ldots, T-1$.

Along the z-chain, the payoffs of certain players are straightforward. Indeed, in the step $\mathcal{Q}_s \xrightarrow{C_{s+1}} \mathcal{Q}_{s+1}$, each player i in $P(C_{s+1}, \mathcal{Q}_s) \setminus C_{s+1}$ drops off as a singleton and obtains his stand alone value. Furthermore, the post-deviation payoff of an outsider (i.e. a player in $O(C_{s+1}, \mathcal{Q}_s)$) is equal to his pre-deviation payoff. Hence, it is sufficient to concentrate on the payoffs of the deviators.

Step 2. The first deviation: $\mathcal{Q}_0 \xrightarrow{C_1} \mathcal{Q}_1$.
The similarity of $(z_0, \mathcal{Q}_0)$ and $(y_0, \mathcal{Q}_0)$ implies that $z_0(C_1) < v(C_1)$. Hence coalition C_1 has an incentive to deviate. The payoff of a deviator depends upon the status of the deviating coalition:

1. C_1 is a subset of W_0.
   Then we define $z_{1,k} = y_{1,k}$ for each k in C_1. This can be done because (i) player k in C_1 is winning (from $(z_0, \mathcal{Q}_0)$ onwards) such that $z_{0,k} \le y_{1,k}$ and (ii) coalition C_1 is deviating such that $y_1(C_1) = v(C_1)$.
   Also, the inclusion $C_1 \subset W_0$ implies that the players in C_1 glue together and will not be separated in subsequent steps.
2. C_1 intersects L_0.
   Then we allocate the surplus $v(C_1) - z_0(C_1)$ to those players who are the first to drop off as singletons in subsequent deviations (i.e. losing players k in C_1 with the smallest $\ell(1, k)$-value). In other words, the payoff of such a player is *temporarily* increased and will fall back on his stand alone value later on.
   The payoffs of the remaining players in C_1 stay at the pre-deviation level.

We close this step with the following observations. If player i moves up from L_0 to W_1, then the singleton coalition $\{i\}$ belongs to $\mathcal{Q}_1$ and $z_{1,i} = y_{1,i} = v(\{i\})$. Players in W_0 either have their initial z_0-payoff or obtained a y_1-payoff.

Step 3. The second deviation: $\mathcal{Q}_1 \xrightarrow{C_2} \mathcal{Q}_2$.
Let us investigate the composition of the deviating coalition C_2. We regard this deviation as a merger of a set $\mathcal{C}$ of (possibly singleton) coalitions in $\mathcal{Q}_1$ that pick up further players from other coalitions. Let D denote the set of these picked-up players.

We have to check whether coalition C_2 can improve upon $(z_1, \mathcal{Q}_1)$ by standing alone, i.e. $v(C_2) > z_1(C_2)$. In the above notation we have $\mathcal{C} \subset \mathcal{Q}_1$, and hence

$$z_1(C_2) \;=\; \Sigma_{\mathcal{C}}\, v(C) \;+\; z_1(D).$$

We investigate the nature of a player in D. Such a player in D cannot have a *temporarily* high payoff. We show this by contradiction and we assume that a player j in D has a temporarily high payoff. Player j is, by construction, a future loser that belonged to C_1. Since the surplus $v(C_1) - z_0(C_1)$ of the previous deviation was allocated to those losers that are the *first* to drop off, coalition C_2 can only contain player i in case C_2 includes C_1. Therefore, $j \in C_1 \in \mathcal{C}$ and j is not in D. A contradiction.
Conclude that each player in D was, in the previous step, either an outsider or a deviator. Now, we are able to specify the pre-deviation payoff $z_{1,i}$ of a player i in D:

- The payoff $z_{1,i}$ of an outsider is still at the z_0-level.
- The payoff $z_{1,i}$ of a deviator also is at the z_0-level. Indeed, in this case the deviating coalition C_1 is not included in C_2. Only the payoffs of those players that are the first to left behind as singletons were temporarily increased. Obviously, player i belongs to $C_1 \cap C_2$ and his payoff is equal to $z_{0,i}$.

Therefore, we can rewrite the previous equality:

$$z_1(C_2) \;=\; \Sigma_{\mathcal{C}}\, v(C) \;+\; z_0(D).$$

Next, we look at the y-chain. In the step $\mathcal{Q}_1 \xrightarrow{C_2} \mathcal{Q}_2$ the same decomposition of C_2 appears. Because C_2 improves upon y_1 and because players in D are either outsiders or deviators when moving from y_0 to y_1 we have

$$v(C_2) \;=\; y_2(C_2) \;>\; y_1(C_2) \;=\; \Sigma_{\mathcal{C}}\, v(C) \;+\; y_1(D) \;\geq\; \Sigma_{\mathcal{C}}\, v(C) \;+\; y_0(D);$$

Now use the similarity of the outcomes $(y_0, \mathcal{Q}_0)$ and $(z_0, \mathcal{Q}_0)$ (Condition $(*)$ in Def 3.3) and conclude that C_2 indeed has an incentive to deviate:

$$v(C_2) \;>\; z_1(C_2) \;=\; \Sigma_{\mathcal{C}}\, v(C) \;+\; z_0(D).$$

The payoff vector z_2 is defined in the same way as z_1. The payoff of a deviator depends upon the status of C_2.

1. C_2 is a subset of W_1.
   Then a deviator either already belonged to W_0 or obtained in the previous step his stand alone value; in both cases the payoff of the deviator can be lifted to the y_2-level.
2. C_2 intersects L_1.
   Then the payoff of a deviator is either equal to his pre-deviation payoff or is temporarily increased.

Step 4. The t-th deviation: $\mathcal{Q}_{t-1} \xrightarrow{C_t} \mathcal{Q}_t$.
The subsequent deviations by the coalitions $C_1, C_2, \ldots, C_{t-1}$ are all executed and the payoff vectors $z_1, z_2, \ldots, z_{t-1}$ are all defined. Again, we start with the decomposition of the

deviating coalition C_t. Since players now have a longer history, the decomposition of C_t is more complicated.

In the outcome $(z_{t-1}, \mathcal{Q}_{t-1})$ we distinguish four types of players: players with a temporarily high payoff, players (that do not form a singleton coalition) with a payoff at the y_k-level with $k \leq t-1$, players having their stand alone payoff, and untouched players with a payoff still at the z_0-level. By construction, these four types exhaust the set N of players. Indeed, when a player leaves his z_0-level, he either enters the y-level, or obtains a temporarily high payoff, or obtains his stand alone value.

Consider a player in C_t with a payoff at the y_k-level with $k \leq t-1$. By construction, a player can move up to the y_{t-1}-level only after joining a deviating coalition C_j that enters the set W_j of winners. Such a coalition C_j never breaks up. However, the coalition C_j can be picked up as a whole by a later deviating coalition. Let C_k be the latest deviating coalition that includes C_j and that is a subset of C_t (i.e. $C_j \subset C_k \subset C_t$). Let $\mathcal{C}_1$ collect these coalitions C_k. Note that two different coalitions in $\mathcal{C}_1$ must be disjoint.
Hence each player in C_t with a payoff at the y-level is sheltered in some coalition in $\mathcal{C}_1$.

Now, consider a player in C_t, not yet sheltered by $\mathcal{C}_1$, with a temporarily high payoff. Then C_t must include the entire deviating coalition C_j (with $j < t$) which was at the basis of this temporarily high payoff. Indeed, the surplus of a deviation was (in case C_j contains future losers) allocated to those players that are the first to drop off. Hence, if such a future loser is present in C_t, then the drop off has not yet happened. The coalition C_j is still together and is included in some deviating coalition C_k which is a subset of C_t (again let k be as large as possible, $j \leq k < t-1$)). Let $\mathcal{C}_2$ collect these coalitions C_k. Different coalitions in $\mathcal{C}_1 \cup \mathcal{C}_2$ are disjoint.
Now, each player with a payoff at the y-level or with a temporarily high payoff is sheltered in some coalition in $\mathcal{C}_1 \cup \mathcal{C}_2$.

Let S collect the remaining players in C_t with a payoff equal to their stand alone value. Such a player is been dropped off as a singleton coalition; later on such a player might become a winner in a deviating coalition that also contained losers.

Finally, let the coalition D collect the remaining players in C_t. They have a payoff at the z_0-level.

In contrast to Step 3, the coalitions in $\mathcal{C}_1, \mathcal{C}_2$ need not be present as coalitions in $\mathcal{Q}_{t-1}$, they are *included* in one of the coalitions in $\mathcal{Q}_{t-1}$.

In conclusion:

$$z_{t-1}(C_t) \;=\; \Sigma_{\mathcal{C}_1}\, v(C) \,+\, \Sigma_{\mathcal{C}_2}\, v(C) \,+\, \Sigma_S\, v(\{i\}) \,+\, z_0(D).$$

We have to check whether $v(C_t) > z_{t-1}(C_t)$.

Consider the same decomposition in the step $(y_{t-1}, \mathcal{Q}_{t-1}) \xrightarrow{C_t} (y_t, \mathcal{Q}_t)$. Since coalition C_t can improve upon y_{t-1}, we know

$$v(C_t) \;>\; \Sigma_{\mathcal{C}_1}\, v(C) \,+\, \Sigma_{\mathcal{C}_2}\, v(C) \,+\, y_{t-1}(S) \,+\, y_{t-1}(D).$$

For each player k in D we have $y_{t-1,k} \geq y_{0,k}$. For each player k in S we have $y_{t-1,k} \geq v(\{k\})$. Hence,

$$v(C_t) \;>\; \Sigma_{\mathcal{C}_1}\, v(C) \,+\, \Sigma_{\mathcal{C}_2}\, v(C) \,+\, \Sigma_S\, v(\{k\}) \,+\, y_0(D).$$

Use the similarity of the outcomes $(y_0, \mathcal{Q}_0)$ and $(z_0, \mathcal{Q}_0)$ (Condition $(*)$ in Def 3.3) and conclude that C_t indeed has an incentive to deviate.

The payoff $z_{t,k}$ with k in C_t depends upon the status of C_t and is lifted to the y_t-level ($C_t \subseteq W_{t-1}$), or is either equal to the pre-deviation payoff or is temporarily increased ($C_t \cap L_{t-1} \neq \varnothing$). □

4. The minimal dominant set

Here we introduce dominant sets and show that *the* minimal dominant set is non-empty. Let (N, v) be a game and let $\Omega = \Omega(N, v)$ be the set of all outcomes.

Definition 4.1. A set $\Delta \subseteq \Omega$ of outcomes is said to be dominant if it satisfies

accessibility: the set Δ is accessible from Ω, i.e. for each b in Ω there exists an a in Δ such that $b \hat{\rightharpoonup} a$, and

closure: the set Δ is closed for o-i-domination, i.e. for each a in Ω and each b in Δ, if $b \hat{\rightharpoonup} a$ then $a \in \Delta$.

For example, the set Ω of all outcomes is dominant. Furthermore, the complement $\Omega \setminus \Delta$ of a dominant set Δ is not dominant. The non-emptiness of the minimal dominant set will follow from the existence of outcomes that are maximal for the sequential o-i-dominance relation $\hat{\longrightarrow}$.

Definition 4.2. Outcome a is maximal for $\hat{\rightharpoonup}$ if for each outcome b in Ω that sequentially o-i-dominates a, we have that a sequentially o-i-dominates b.

In order to show the existence of a maximal outcome, we follow Kalai and Schmeidler (1977, Theorem 3) and use some standard arguments from topology. We embed the set Ω in the Euclidean space $\mathbb{R}^n$ by neglecting the coalition structures behind the outcomes. Formally, we study outcome vectors $x, y, \ldots$ instead of outcomes $(x, \mathcal{P}), (y, \mathcal{Q}), \ldots$. Observe that the set of all outcome vectors (i.e. the set Ω after neglecting the coalition structures) is compact. Furthermore, within the universe Ω we consider the relativization of the Euclidean topology to Ω. Theorem 3.2 implies the next *continuity* property.

Lemma 4.3. *Let $a, b \in \Omega$. The set $\hat{a} = \{c \in \Omega : a \hat{\rightharpoonup} c\}$ of outcomes that sequentially o-i-dominate a is closed (in the Euclidean topology). In addition, if $a \hat{\rightharpoonup} b$, then $\hat{a} \supset \hat{b}$.*

Proof. First, let $A \subset \Omega$ be a closed set of outcomes. Observe that the set A_1 of outcomes that outsider-independent o-i-dominate A (in one step) also is a closed set. According to Theorem 3.2 there exists a natural number τ such that

$$\hat{a} \;=\; \{c \in \Omega : \text{there is a chain from } a \text{ to } c \text{ of length smaller than } \tau\}.$$

Hence, $\hat{a}$ is the union of τ closed sets, and is therefore closed. The second statement (the finite intersection property along a chain) is obvious. □

Lemma 4.4. *The set Ω, equipped with the sequential o-i-dominance relation, has at least one maximal outcome.*

Proof. By Zorn's lemma it is sufficient to show that each chain in $(\Omega, \hat{\rightarrow})$ has an upper bound. Hence, let A be a chain in Ω. In case the chain contains an outcome a such that $\hat{a} = \{a\}$, then a is a maximal element. Otherwise, the intersection $\cap_{a \in A} \hat{a}$ of closed sets is non-empty (use the finite intersection property of closed sets in the compact set Ω). Each outcome in this intersection is an upper bound for the chain A. □

Now, we identify the minimal dominant set with the set of maximal outcomes.

Theorem 4.5. *Let (N, v) be a game and let Ω be the set of outcomes. Then, the minimal dominant set coincides with the set of maximal outcomes and is therefore non-empty.*

Proof. Let Δ be a minimal dominant set and let M collect the maximal outcomes.

First, let a be a maximal outcome. Because Δ satisfies accessibility, it contains an outcome b such that $a \hat{\rightarrow} b$. The maximality of a implies that $b \hat{\rightarrow} a$. Since Δ satisfies closure, a belongs to Δ. Conclusion: $M \subseteq \Delta$ and Δ is non-empty.

Next, suppose that a belongs to Δ and that b sequentially o-i-dominates a. Then, either a sequentially o-i-dominates b, or Δ is not a minimal o-i-dominant set: outcome a and each outcome that is sequentially o-i-dominated by a can be left out. Since Δ is assumed to be minimal, the outcome a must be maximal. Hence, $\Delta \subseteq M$. □

Finally, consider a game (N, v) with a non-empty coalition structure core $C(N, v)$. As the coalition structure core collects the o-i-undominated outcomes, it follows that the minimal dominant set Δ includes $C(N, v)$. As a matter of fact the equality $C(N, v) = \Delta$ holds:

Corollary 4.6. *Let (N, v) be a game. Then, the minimal dominant set is a non-empty coalition structure core extension.*

Proof. First, the minimal dominant set is non-empty (Theorem 4.5). Second, consider a game with a non-empty coalition structure core. The accessibility of the coalition structure core is proven in Kóczy and Lauwers (2001). Hence, the minimal dominant set coincides with the coalition structure core. □

5. Properties

We discuss some deficiencies and we list some properties of the minimal dominant set. Consider a game (N, v). Let Ω be the set of outcomes and let Δ be the minimal dominant set.

5.1. We start with the observation that an outcome in Δ might assign a positive payoff to a dummy player, i.e. a player i for which $v(\{i\}) = 0$ and $v(C \cup \{i\}) = v(C)$ for each coalition C. Indeed, consider a three player majority game augmented with two dummy players: $N = \{1, 2, 3, 4, 5\}$, $v(C) = 2$ if the intersection $C \cap \{1, 2, 3\}$ contains at least two players, all other coalitions have a value equal to 0.

The outcome $(1, 1, 0, 0, 0; \{1, 2\}, \{3\}, \{4\}, \{5\})$ belongs to Δ and is o-i-dominated by the outcome $a = (0, 1.2, 0.4, 0.4, 0; \{1\}, \{2, 3, 4\}, \{5\})$ which allocates a positive amount to player 4. Since Δ is closed for o-i-domination, outcome a belongs to Δ.

Sengupta and Sengupta (1994, Section 3.2) observe that this affliction is common to many solution concepts: the Aumann-Maschler set, the Mas-Collel bargaining set, the consistent bargaining set of Dutta et al., and the set of viable proposals by Sengupta and Sengupta all generate solutions for this game with a positive payoff for the dummy players.

An artificial way to circumvent this problem is to impose a stability condition upon the deviating coalitions. Call a coalition S *stable against splitting up* in case each proper partitioning $\mathcal{D}$ of S has a value that is strictly smaller than the worth of S, i.e. $v(\mathcal{D}) < v(S)$. In other words, a coalition will split up in case it can be partitioned without lowering its total worth. As such, a deviating coalition will never contain a dummy player and dummy players will end up in their stand alone position.

5.2. Next, we observe that the shortsightedness or myopia of the players may lead to inefficient coalition structures.

Definition 5.1. Let (N, v) be a game and let S be some coalition. A coalition structure $\mathcal{C}$ of S is said to be efficient if the total payoff $v(\mathcal{C}) = \Sigma_{E \in \mathcal{C}}\, v(E)$ decreases when the coalition structure $\mathcal{C}$ is made finer or coarser.

Efficiency combines stability against splitting up with stability against mergers, i.e. $\mathcal{C}$ does not contain coalitions A and B such that $v(A \cup B) > v(A) + v(B)$. The next example indicates that inefficient coalition structures might enter the minimal dominant set.

Example 5.2. *Repeat the three player game* (N, v) *with* $v(\{i\}) = 0, v(\{i, j\}) = 8$, *and* $v(N) = 9$. *The minimal dominant set is the union of two sets. The first one is the boundary of a triangle spanned by* $(8, 0, 0), (0, 8, 0), (0, 0, 8)$*:*

$$\Delta_1 = \{\ (x_1, x_2, x_3; \{i, j\}, \{k\}) \mid \{i, j, k\} = \{1, 2, 3\} \text{ and } x_i + x_j = 8, x_k = 0\ \}.$$

The second one is a part of a triangle spanned by $(9, 0, 0), (0, 9, 0), (0, 0, 9)$*:*

$$\Delta_2 = \{\ (x_1, x_2, x_3; N) \mid x_1 + x_2 + x_3 = 9 \text{ and } \exists\, k \in N : x_k \leq 1\ \}.$$

The outcomes in Δ_1 are inefficient. Coarsening the coalition structure $(\{i, j\}, \{k\})$ to N improves the value from 8 to 9. Furthermore, the efficient outcome $(3, 3, 3; N)$ does not belong to the minimal dominant set.

These observations raise a rather fundamental issue: the conflict between efficiency and undomination. Here we insisted on undomination. As a consequence, inefficient outcomes might enter and some efficient outcomes might leave the solution.

We do not regard this as a serious conceptual problem: we view the minimal dominant set as a *first* solution concept. In other words, outcomes outside the minimal dominant set certainly will not survive.[4] Hence, if one insists on efficiency, then one can select the

[4] The literature on tournaments provides an analogue (Laslier, 1997). The top-cycle gathers the maximal elements of a tournament, and the top-cycle is considered as a starting point for further investigations: most tournament solutions are top-cycle selections.

efficient outcomes out of the minimal dominant set. Since (i) each inefficient outcome is o-i-dominated by an efficient outcome and (ii) the minimal dominant set is closed for o-i-domination, this restriction is non-empty. In addition, this restricted set of efficient outcomes still satisfies accessibility. In the example, Δ_2 collects the efficient outcomes.

Analogously, if one insists on the dummy player axiom (i.e. dummy players obtain a zero payoff), then one can impose the above mentioned stability axiom on the deviating coalition.

5.3. Finally, we study the behavior of the minimal dominant set in composed games. Let (N_1, v_1) and (N_2, v_2) be two games, with N_1 and N_2 disjoint. The juxtaposition of these games is the game (N, v), with $N = N_1 \cup N_2$ and

$$v : 2^N \setminus \{\varnothing\} \longrightarrow \mathcal{R} : S \longmapsto v(S) = \begin{cases} v_1(S) & \text{if } S \subseteq N_1, \\ v_2(S) & \text{if } S \subseteq N_2, \\ 0 & \text{otherwise.} \end{cases}$$

In such a juxtaposition the restriction to one of the initial sets of players coincides with the corresponding initial game. On the other hand, cross-coalitions have a zero worth. Furthermore, in case $a_i = (x_i, \mathcal{P}_i)$ is an outcome of the game (N_i, v_i), $i = 1, 2$, then the juxtaposition $a_1 \times a_2 = (x_1, x_2\, ; \mathcal{P}_1 \cup \mathcal{P}_2)$ is an outcome of the game (N, v).
The next proposition indicates that the minimal dominant set behaves well with respect to such composed games.

Proposition 5.3. *The minimal dominant set of the juxtaposition of two games coincides with the juxtaposition of the two minimal dominant sets.*

Proof. Let (N, v) be the juxtaposition of the games (N_1, v_1) and (N_2, v_2). Let $(x_i, \mathcal{P}_i)$ be an outcome of the game (N_i, v_i) that is maximal for the sequential o-i-domination relation, $i = 1, 2$. In other words, let $(x_i, \mathcal{P}_i)$ belong to $\Delta(N_i, v_i)$.
Obviously, the juxtaposition $(x_1, x_2\, ; \mathcal{P}_1 \cup \mathcal{P}_2)$ is maximal. Hence, $\Delta(N, v)$ includes the juxtaposition of $\Delta(N_1, v_1)$ and $\Delta(N_2, v_2)$.
The inclusion $\Delta(N, v) \subseteq \Delta(N_1, v_1) \times \Delta(N_2, v_2)$ also is immediate. □

Although this property seems natural, it illuminates some advantages of the minimal dominant set above other solution concepts. Consider the juxtaposition of a small game with an empty and a large game with a non-empty core. As each outcome of this game is dominated, the coalition structure core is empty. Nevertheless, the composed game contains *almost* stable outcomes. The minimal dominant set is able to trace this locally stable behavior.

Furthermore, this property illustrates the implications of the *outsider-independence* assumption in the o-i-dominance relation. Consider the following juxtaposition.

Let $N = \{1, 2, 3, 4, 5\}$ and let

$$v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{4, 5\}) = 2,$$

all other coalitions have a zero value. The minimal dominant set of this game is equal to

$$\Delta = \{(x\,;\{i,j\}\,,\{k\}\,,\{4,5\}) \mid \{i,j,k\} = \{1,2,3\}\,,\ x_i + x_j = x_4 + x_5 = 2,\ x_k = 0\}\,.$$

When the deviating coalition is allowed to intervene in the structure of the outsiders, the set of maximal elements does contain outcomes that are not plausible. For example, the outcome $a = (1,1,0,0,0\,;\{1,2\}\,,\{3\}\,,\{4\}\,,\{5\})$ dominates in the sense of Sengupta and Sengupta (1994) the outcome $b = (1,1,0,1,1\,;\{1,2\}\,,\{3\}\,,\{4,5\})$. Indeed, start from b and consider a deviation by $\{2,3\}$ that separates players 4 and 5, next consider a deviation by $\{1,2\}$. This example shows that the set of viable proposals (i.e. the solution of Sengupta and Sengupta, 1994) does not satisfy the juxtaposition property.

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