Farsightedness and Cautiousness in Coalition Formation*

Ana Mauleon
LABORES (URA 362, CNRS),
Université catholique de Lille,
France.

Vincent Vannetelbosch[†]
FNRS, IRES and CORE,
Université catholique de Louvain,
Belgium.

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Abstract

We adopt the largest consistent set defined by Chwe [J. of Econ. Theory 63 (1994), 299-235] to predict which coalition structures are possibly stable in games with positive spillovers. We also introduce a refinement, the largest cautious consistent set. For games with positive spillovers, many coalition structures may belong to the largest consistent set. The grand coalition, which is the efficient coalition structure, always belongs to the largest consistent set and is the unique one to belong to the largest cautious consistent set.

Keywords: coalition formation, farsightedness, cautiousness, positive spillovers, largest consistent set.
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†Corresponding author address: Department of Economics, Université catholique de Louvain, Place Montesquieu 3, B-1348 Louvain-la-Neuve, Belgium. E-mail: vannetelbosch@ires.ucl.ac.be

1 Introduction

Many social, economic and political activities are conducted by groups or coalitions of individuals. For example, consumption takes place within households or families; production is carried out by firms which are large coalitions of owners of different factors of production; workers are organized in trade unions or professional associations; public goods are produced within a complex coalition structure of federal, state, and local jurisdictions; political life is conducted through political parties and interest groups; and individuals belong to networks of formal and informal social clubs.

The formation of coalitions has been a major topic in game theory, and has been studied mainly using the framework of cooperative games in coalitional form (see Aumann and Drèze, 1974). Unfortunately, externalities among coalitions cannot be considered within such framework (see Bloch, 1997). As a consequence, the formation of coalitions has been described in the recent years as noncooperative simultaneous or sequential games, which are usually solved using the Nash equilibrium concept or one of its refinements. The most disturbing feature of simultaneous coalition formation games is that the agents cannot be farsighted in the sense that individual deviations cannot be countered by subsequent deviations (see Hart and Kurz, 1983). In order to remedy this weakness sequential coalition formation games have been proposed (see Bloch, 1996). Nevertheless, these sequential games are quite sensitive to the exact coalition formation process and rely on the commitment assumption. Once some agents have agreed to form a coalition they are committed to remain in that coalition. They can neither leave the coalition nor propose to change the coalition at subsequent stages.

Coalition formation games in effectivity form as in Chwe (1994) specify what each coalition can do if and when it forms. This representation of games allows us to study economic and social activities where the rules of the game are rather amorphous or the procedures are rarely pinned down (e.g. in sequential bargaining or coalition formation without a rigid protocol), and for which classical game theory could lead to a solution which relies heavily on an arbitrarily chosen procedure or rule. For games in effectivity form where coalitions can form through binding or non-binding agreements and actions are public, Chwe (1994) has proposed an interesting solution concept, the largest consistent set. This solution concept predicts which coalitions structures are possibly stable and could emerge. Chwe's approach has a number of nice features. Firstly, it does not rely on a very detailed description of the coalition formation process as noncooperative sequential games do. No commitment assumption is imposed. Secondly, it incorporates the farsightedness of the coalitions. A coalition considers the possibility that, once it acts, another coalition might react, a third coalition might in turn react, and so on without limit.

However, the largest consistent set suffers from a number of drawbacks, some of them pointed out by Chwe himself. For instance, the largest consistent set may fail to satisfy the requirement of individual rationality. An individual that is given the choice between two moves, where one yields with certainty a higher payoff than the other, might choose the move leading to the lower payoff according to the largest consistent set. This is perhaps somewhat less disturbing than it seems at first sight, since the largest consistent set aims to be a weak concept, a concept that rules out with confidence, but is not so good at picking out. The largest consistent set may also fail to satisfy the requirement of cautiousness. Hence, we introduce a refinement, called the largest cautious consistent set.

Two different notions of a coalitional deviation or move can be found in the game-theoretic literature. Strict deviation: a group of players or a coalition can deviate only if each of its members can be made better off. Weak deviation: a group of players or a coalition can deviate only if at least one of its members is better off while all other members are at least as well off. A weak deviation or move requires only one player to be better off as long as all other members of the group are not worse off, whereas under a strict deviation or move, all deviating players must be better off. We shall distinguish between the indirect strict dominance relation and the indirect weak dominance relation in the definition of the largest (cautious) consistent set. The indirect strict (weak) dominance relation captures the fact that farsighted coalitions consider the end coalition structure that their move(s) may lead to, and that only strict (weak) deviations or moves will be engaged.

We find that the largest (cautious) consistent set is sensitive to the exact definition of the indirect dominance relation. In general there is no relationship between the largest (cautious) consistent set based on the indirect strict dominance and the largest (cautious) consistent set based on the indirect weak dominance. The largest consistent set is never empty whenever the set of coalition structures is finite. Unfortunately, the largest cautious consistent set might be empty in some situations.

However, we show that the largest cautious consistent set refines considerably the largest consistent set in coalition formation games satisfying the properties of positive spillovers, negative association, individual free-riding incentives and efficiency of the grand coalition. Positive spillovers restrict the analysis to games where the formation of a coalition by other players increases the payoff of a player. Negative association imposes that, in any coalition structure, small coalitions have higher per-member payoffs than big coalitions. Individual free-riding incentives assume that a player becomes better off leaving any coalition to be alone. An economic situation satisfying these properties is a cartel formation game under Cournot competition. Public goods coalitions satisfy these properties under some conditions.

We show that many coalition structures may belong to the largest consistent set in coalition formation games satisfying the four properties imposed on the payoffs. The grand coalition always belongs to the largest consistent set. The stand-alone coalition structure (where all players are singletons) is never stable under the largest consistent set based on the indirect weak dominance relation. However, the largest cautious consistent set singles out the grand coalition, which is the efficient coalition structure.

The paper has been organized as follows. In Section 2 we introduce some notations, primitives and definitions of indirect dominance. We present the solution concepts of Chwe (1994), and we propose a refinement, the largest cautious consistent set. In Section 3 we use the above mentioned concepts to predict which coalition structures are stable in coalition formation with positive spillovers. In Section 4 we analyze and characterize the stable outcomes in the cartel formation game. We also introduce a congestion or monitoring cost and we discuss the role of monitoring costs in the determination of largest consistent sets. Finally, Section 5 concludes.

2 Farsighted Coalitional Stability

The players are forming coalitions and inside each coalition formed the members share the coalition gains from cooperation. Let $\mathcal{P}$ be the finite set of coalition structures. A coalition structure $P = \{S_1, S_2, \ldots, S_m\}$ is a partition of the player set $N = \{1, 2, \ldots, n\}$, $S_i \cap S_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{m} S_i = N$. Let $|S_i|$ be the cardinality of coalition S_i. Gains from cooperation are described by a valuation V which maps the set of coalition structures $\mathcal{P}$ into vectors of payoffs in $\mathbb{R}^n$. The component $V_i(P)$ denotes the payoff obtained by player i if the coalition structure P is formed.

How the coalition formation proceeds? What coalitions can do if and when they form is specified by $\{\rightarrow_S\}_{S \subseteq N, S \neq \emptyset}$, where $\{\rightarrow_S\}$, $S \subseteq N$, is an effectiveness relation on $\mathcal{P}$. For any $P, P' \in \mathcal{P}$, $P \rightarrow_S P'$ means that if the coalition structure P is the status-quo, coalition S can make the coalition structure P' the new status-quo. After S deviates to P' from P, coalition S' might move to P'' where $P' \rightarrow_{S'} P''$, etc. All actions are public. If a status-quo P is reached and no coalition decides to move from P, then P is a stable coalition structure. A coalition formation game in effectivity form G is $(N; \mathcal{P}; V; \{\rightarrow_S\}_{S \subseteq N, S \neq \emptyset})$.

2.1 Indirect Strict or Weak Dominance

As Konishi et al. (1999) mention, the game-theoretic literature uses two different notions of a coalitional deviation or move.

- Strict Deviation: A group of players or a coalition can deviate only if each of its members can be made better off; and
- Weak Deviation: A group of players or a coalition can deviate only if at least one of its members is better off while all other members are at least as well off.

A weak deviation or move requires only one player to be better off as long as all other members of the group are not worse off, whereas under a strict deviation or move, all deviating players must be better off. Hence, we shall distinguish between the indirect strict dominance relation and the indirect weak dominance relation.

The indirect strict dominance relation captures the fact that farsighted coalitions consider the end coalition structure that their move(s) may lead to, and that only strict deviations or moves will be engaged. A coalition structure P' indirectly strictly dominates P if P' can replace P in a sequence of moves, such that at each move all deviators are better off at the end coalition structure P' compared to the status-quo they face. Formally, indirect strict dominance is defined as follows.

Definition 1 A coalition structure P is indirectly strictly dominated by P', or $P \ll P'$, if there exists a sequence $P^0, P^1, \ldots, P^m$ (where $P^0 = P$ and $P^m = P'$) and a sequence $S_0, S_1, \ldots, S_{m-1}$ such that $P^j \to_{S_j} P^{j+1}$, $V_i(P') > V_i(P^j)$ for all $i \in S_j$, for $j = 0, 1, \ldots, m-1$.

Direct strict dominance is obtained by setting $m = 1$ in Definition 1. A coalition structure P is directly strictly dominated by P', or $P < P'$, if there exists a coalition S such that $P \to_S P'$ and $V_i(P') > V_i(P)$ for all $i \in S$. Obviously, if $P < P'$, then $P \ll P'$. The definition of the indirect strict dominance relation $\ll$ is traditional: it is customary to require that a coalition will deviate or move only if all of its members are made better off at the end coalition structure, since changing the status-quo is costly, and players have to be compensated for doing so.

But sometimes some players may be indifferent between the status-quo they face and a possible end coalition structure, while others are better off at this end coalition structure. Then, it should not be too difficult for the players who are better off at this end coalition structure to convince the indifferent players to join them to move towards this end coalition structure. The indirect weak dominance relation captures this idea. A coalition structure P' indirectly weakly dominates P if P' can replace P in a sequence of moves, such that at each move all deviators are at least as well off at the end coalition structure P' compared to the status-quo they face, and at least one deviator is better off at P'. Formally, indirect weak dominance is defined as follows.

Definition 2 A coalition structure P is indirectly weakly dominated by P', or $P \underline{\ll} P'$, if there exists a sequence $P^0, P^1, \ldots, P^m$ (where $P^0 = P$ and $P^m = P'$) and a sequence $S_0, S_1, \ldots, S_{m-1}$ such that $P^j \to_{S_j} P^{j+1}$, $V_i(P') \geq V_i(P^j)$ for all $i \in S_j$, and $V_i(P') > V_i(P^j)$ for some $i \in S_j$, for $j = 0, 1, \ldots, m-1$.

Direct weak dominance is obtained by setting $m = 1$ in Definition 2. A coalition structure P is directly weakly dominated by P', or $P \leq P'$, if there exists a coalition S such that $P \to_S P'$, $V_i(P') \geq V_i(P)$ for all $i \in S$ and $V_i(P') > V_i(P)$ for some $i \in S$. Obviously, if $P \leq P'$ then $P \underline{\ll} P'$. Also, if P is indirectly strictly dominated by P', then P is indirectly weakly dominated by P'. Of course the reverse is not true. To summarize, we have

$$P < P' \Rightarrow P \leq P' \Rightarrow P \underline{\ll} P'$$
$$P < P' \Rightarrow P \ll P' \Rightarrow P \underline{\ll} P'$$

2.2 The Largest Consistent Set

Based on the indirect strict dominance relation, the largest consistent set due to Chwe (1994) is defined as follows. A set $Y \subseteq \mathcal{P}$ is consistent if $P \in Y$ if and only if $\forall P', S$ such that $P \to_S P'$, $\exists P'' \in Y$, where $P' = P''$ or $P' \ll P''$, such that we do not have $V_i(P) < V_i(P'')$ for all $i \in S$. The largest consistent set $LCS(G, \ll)$ is the consistent set such that if $Y \subseteq \mathcal{P}$ is consistent then $Y \subseteq LCS(G, \ll)$. Although there can be many consistent sets, Chwe (1994) has shown that there uniquely exists a largest consistent set $LCS(G, \ll)$, which contains all others. The largest consistent set $LCS(G, \ll)$ can also be defined in an iterative way as in Chwe (1994).

Definition 3 Let $Y^0 \equiv \mathcal{P}$. Then, Y^k $(k = 1, 2, \ldots)$ is inductively defined as follows: $P \in Y^{k-1}$ belongs to Y^k if and only if $\forall P', S$ such that $P \to_S P'$, $\exists P'' \in Y^{k-1}$, where $P' = P''$ or $P' \ll P''$, such that we do not have $V_i(P) < V_i(P'')$ for all $i \in S$. The largest consistent set $LCS(G, \ll)$ is $\bigcap_{k \geq 1} Y^k$.

That is, a coalition structure $P \in Y^{k-1}$ is stable (at step k) and belongs to Y^k, if all possible deviations are deterred. Consider a deviation from P to P' by coalition S. There might be further deviations which end up at P'', where $P' \ll P''$. There might not be any further deviations, in which case the end coalition structure $P'' = P'$. In any case, the end coalition structure P'' should itself be stable (at step $k-1$), and so, should belong to Y^{k-1}. If some member of coalition S is worse off or equal off at P'' compared to the original coalition structure P, then the deviation is deterred. Since $\mathcal{P}$ is finite, there exists $m \in \mathbb{N}$ such that $Y^k = Y^{k+1}$ for all $k \geq m$, and Y^m is the largest consistent set $LCS(G, \ll)$. If a coalition structure is not in the largest consistent set, it cannot be stable. The largest consistent set is the set of all coalition structures which can be possibly stable.

We define in a similar way the largest consistent set based on the indirect weak dominance relation. A set $Y \subseteq P$ is consistent if $P \in Y$ if and only if $\forall P', S$ such that $P \to_S P'$, $\exists P'' \in Y$, where $P' = P''$ or $P' \underline{\ll} P''$, such that we do not have $V_i(P) \leq V_i(P'')$ for all $i \in S$ and $V_i(P) < V_i(P'')$ for some $i \in S$. The largest consistent set $LCS(G, \underline{\ll})$ is the consistent set such that if $Y \subseteq P$ is consistent then $Y \subseteq LCS(G, \underline{\ll})$. The proof of Chwe (1994) can be easily adapted to show that there uniquely exists a largest consistent set $LCS(G, \underline{\ll})$. The largest consistent set $LCS(G, \underline{\ll})$ can also be redefined in an iterative way.

Definition 4 Let $Y^0 \equiv P$. Then, Y^k $(k = 1, 2, \ldots)$ is inductively defined as follows: $P \in Y^{k-1}$ belongs to Y^k if and only if $\forall P', S$ such that $P \to_S P'$, $\exists P'' \in Y^{k-1}$, where $P' = P''$ or $P' \underline{\ll} P''$; such that we do not have $V_i(P) \leq V_i(P'')$ for all $i \in S$ and $V_i(P) < V_i(P'')$ for some $i \in S$. The largest consistent set $LCS(G, \underline{\ll})$ is $\bigcap_{k \geq 1} Y^k$.

That is, a coalition structure $P \in Y^{k-1}$ is stable (at step k) and belongs to Y^k, if all possible deviations are deterred. Consider a deviation from P to P' by coalition S. There might be further deviations which end up at P'', where $P' \underline{\ll} P''$. There might not be any further deviations, in which case the end coalition structure $P'' = P'$. In any case, the end coalition structure P'' should itself be stable (at step $k - 1$), and so, should belong to Y^{k-1}. If some member of coalition S is worse off or all members of S are equal off at P'' compared to the original coalition structure P, then the deviation is deterred. Since P is finite, there exists $m \in \mathbb{N}$ such that $Y^k = Y^{k+1}$ for all $k \geq m$, and Y^m is the largest consistent set $LCS(G, \underline{\ll})$.

The following example shows that the largest consistent set is sensitive to the exact definition of the indirect dominance relation. Figure 1 depicts a three-player coalition formation game in effectivity form, where only three coalition structures are feasible: $\{12, 3\}$, $\{1, 2, 3\}$ and $\{1, 23\}$. The payoff vectors associated with those three partitions are given in Figure 1 as well as the possible moves from each partition. For instance, player 1 can move from $\{12, 3\}$ where he gets 1 to $\{1, 2, 3\}$ where he gets 2. We have $\{12, 3\} < \{1, 2, 3\}$ (hence $\{12, 3\} \underline{<} \{1, 2, 3\}$) and $\{1, 2, 3\} \underline{<} \{1, 23\}$. It follows that $LCS(G, \ll) = \{\{1, 2, 3\}, \{1, 23\}\}$ and $LCS(G, \underline{\ll}) = \{\{12, 3\}, \{1, 23\}\}$. In general, these two indirect dominance relations (weak or strict) might yield two very different largest consistent sets.

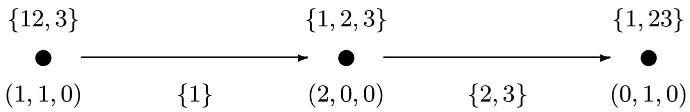

Figure 1. The largest consistent set is sensitive to the indirect dominance relation.

2.3 The Largest Cautious Consistent Set

Similarly to the rationalizability concepts,[1] the largest consistent set does not determine what will happen but what can possibly happen. The following example shows that the largest consistent set is not consistent with cautiousness. Figure 2 depicts a three-player coalition formation game in effectivity form, where the feasible coalition structures are: $\{123\}$, $\{1,23\}$ and $\{1,2,3\}$. The payoff vectors associated with those three partitions are given in Figure 2 as well as the possible moves from each partition. For instance, player 1 can move from $\{123\}$ where he gets 1 to $\{1,23\}$ where he gets 2. We have $\{123\} < \{1,23\}$ and $\{1,23\} < \{1,23\}$. It follows that $LCS(\Gamma,\ll) = LCS(\Gamma,\ll) = \{\{123\},\{1,2,3\}\}$. But player 1 cannot end worse off by engaging a move from $\{123\}$ compared to what he gets in $\{123\}$.

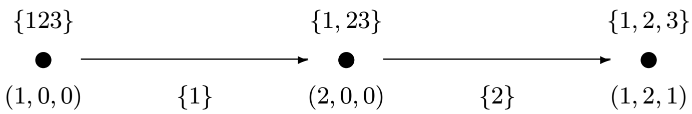

Figure 2. The largest consistent set is not consistent with cautiousness.

To give sharper predictions we propose to refine the largest consistent set. Applying the spirit of some refinements of the rationalizability concept to the largest consistent set leads to the following definition of the largest cautious consistent set derived from either the indirect strict dominance relation or the indirect weak dominance relation. Formally, the largest cautious consistent set $LCCS(\Gamma,\ll)$ based on the indirect strict dominance is defined in an iterative way.

Definition 5 Let $Z^0 \equiv \mathcal{P}$. Then, Z^k $(k = 1,2,\ldots)$ is inductively defined as follows: $P \in \mathcal{P}$ belongs to Z^k if and only if $\forall P', S$ such that $P \rightarrow_S P'$, $\exists \alpha = \left(\alpha(P^1),\ldots,\alpha(P^m)\right)$ satisfying $\sum_{j=1}^{m}\alpha(P^j) = 1$, $\alpha(P^j) \in (0,1]$, that gives positive weight to each $P^j \in Z^{k-1}$, where $P' = P^j$ or $P' \ll P^j$, such that we do not have

$$V_i(P) < \sum_{\substack{P^j \in Z^{k-1} \\ P'=P^j \text{ or } P' \ll P^j}} \alpha(P^j)\cdot V_i(P^j) \text{ for all } i \in S.$$

[1] See Bernheim (1984), Herings and Vannetelbosch (1999), Pearce (1984).

The largest cautious consistent set $LCCS(G, \ll)$ is $\bigcap_{k\geq 1} Z^k$.

The idea behind the largest cautious consistent set $LCCS(G, \ll)$ is that once a coalition S deviates from P to P', this coalition S should contemplate the possibility to end with positive probability at any coalition structure P'' not ruled out[2] and such that $P' = P''$ or $P' \ll P''$. Hence, a coalition structure P is never stable if a coalition S can engage a deviation from P to P' and doing so there is no risk that some coalition members will end worse off or equal off.

The definition, based on the indirect weak dominance, of the largest cautious consistent set $LCCS(G, \underline{\ll})$ is as follows.

Definition 6 Let $Z^0 \equiv \mathcal{P}$. Then, Z^k $(k = 1, 2, \ldots)$ is inductively defined as follows: $P \in Z^{k-1}$ belongs to Z^k if and only if $\forall P', S$ such that $P \to_S P'$, $\exists \alpha = (\alpha(P^1), \ldots, \alpha(P^m))$ satisfying $\sum_{j=1}^{m} \alpha(P^j) = 1$, $\alpha(P^j) \in (0, 1)$, that gives positive weight to each $P^j \in Z^{k-1}$, where $P' = P^j$ or $P' \underline{\ll} P^j$, such that we do not have

$$V_i(P) \leq \sum_{\substack{P^j \in Z^{k-1} \\ P' = P^j \text{ or } P' \underline{\ll} P^j}} \alpha(P^j) \cdot V_i(P^j) \text{ for all } i \in S, \text{ and}$$

$$V_i(P) < \sum_{\substack{P^j \in Z^{k-1} \\ P' = P^j \text{ or } P' \underline{\ll} P^j}} \alpha(P^j) \cdot V_i(P^j) \text{ for some } i \in S.$$

The largest cautious consistent set $LCCS(G, \underline{\ll})$ is $\bigcap_{k\geq 1} Z^k$.

Once a coalition S deviates from P to P', this coalition S should contemplate the possibility to end with positive probability at any coalition structure P'' not ruled out and such that $P' = P''$ or $P' \underline{\ll} P''$. Hence, a coalition structure P is never stable if a coalition S can engage a deviation from P to P' and doing so some coalition members will be better off but there is no risk that some coalition members will end worse off.

Obviously, the largest cautious consistent set is a refinement of the largest consistent set.

Theorem 1 $LCCS(G, \ll) \subseteq LCS(G, \ll)$ and $LCCS(G, \underline{\ll}) \subseteq LCS(G, \underline{\ll})$.

Proof. It suffices to show that $Z^k \subseteq Y^k$ for all k. We prove this by induction on k. For $k = 0$, this is true since $Z^0 = Y^0$. Now, let $Z^{k-1} \subseteq Y^{k-1}$ and let $P \in Z^k$. Then it is straightforward that $P \in Y^k$. ■

[2] On the contrary, in the largest consistent set once a coalition S deviates from P to P', this coalition S only contemplates the possibility to end with probability one at a coalition structure P'' not ruled out and such that $P' = P''$ or $P' \ll P''$.

Unfortunately, the largest cautious consistent set $LCCS(G;\prec)$ or $LCCS(G;\preceq)$ might be empty in some situations. In general there is no relationship between $LCCS(G;\prec)$ and $LCCS(G;\preceq)$. In the example of Figure 1, we have $LCCS(G;\prec) = \{\{1,2,3\},\{1,23\}\}$ and $LCCS(G;\preceq) = \{\{12,3\},\{1,23\}\}$. Nevertheless, we will show that the largest cautious consistent set refines considerably the largest consistent set in coalition formation games with positive spillovers (and that, both sets $LCS(G;\prec)$ and $LCS(G;\preceq)$ coincide).

The following example (see Figure 3) illustrates that the largest cautious consistent set $LCCS(G;\prec)$ or $LCCS(G;\preceq)$ might be empty, while the largest consistent set $LCS(G;\prec)$ or $LCS(G;\preceq)$ is not. Figure 3 depicts a three-player coalition formation game in effectivity form. The payoff vectors associated with the partitions are given in Figure 3 as well as the possible moves from each partition. For instance, the coalition of players 2 and 3 can move from $\{13,2\}$ where they get respectively $(1,1)$ to $\{1,23\}$ where they get $(0,0)$. We have $\{123\} \ll \{12,3\}$, $\{1,2,3\} \ll \{13,2\}$, $\{1,23\} \ll \{12,3\}$, $\{123\} \prec \{13,2\}$, but also $\{12,3\} \prec \{13,2\}$ and $\{13,2\} \prec \{12,3\}$. It follows that $LCCS(G;\prec) = LCCS(G;\preceq) = \emptyset$ but $LCS(G;\prec) = LCS(G;\preceq) = \{\{12,3\},\{13,2\}\}$. Indeed, it is intuitively reasonable that no outcome can be possibly cautiously stable in this example. Player 1 or the coalition formed by players 2 and 3 cannot end worse off by engaging a move from $\{12,3\}$ and $\{13,2\}$, respectively.

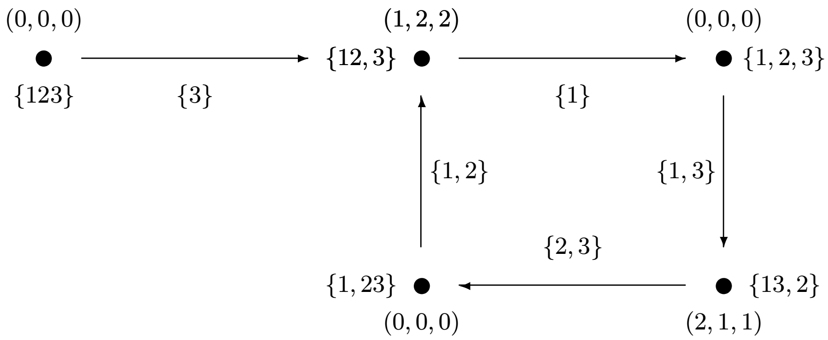

Figure 3. The largest cautious consistent set might be empty.

One condition on the game G in effectivity form which guarantees that the largest cautious consistent set is nonempty is that the coalition formation game in effectivity form is acyclic.

Definition 7 A coalition formation game in effectivity form G is acyclic if the effectiveness relation, $\{\to_S\}_{S\subseteq N}$, is such that there does not exist a sequence $P^0, P^1, \ldots, P^m$ (where $P^0 = P$ and $P^m = P$) and a sequence $S_0, S_1, \ldots, S_{m-1}$ such that $P^j \to_{S_j} P^{j+1}$, for $j = 0, 1, \ldots, m-1$.

Theorem 2 *If the coalition formation game in effectivity form G is acyclic, then the sets $LCCS(G;\succ)$ and $LCCS(G;\succeq)$ are nonempty.*

Proof. Since $\mathcal{P}$ is finite and G is acyclic, there exists $P\in\mathcal{P}$ such that there does not exist $P'\in\mathcal{P}$ and $S\subseteq N$ such that $P\rightarrow_S P'$. In other words P is an end coalition structure from which no move is possible. Hence, P belongs to $LCCS(G;\succ)$ and $LCCS(G;\succeq)$. ■

3 Coalition Formation with Positive Spillovers

3.1 Conditions on the Payoffs

Gains are assumed to be positive, $V_i(P)>0$ for all $i\in N$, for all $P\in\mathcal{P}$. We assume symmetric or identical players and equal sharing of the coalition gains among coalition members.[3] That is, in any coalition S_i belonging to P, $V_j(P)=V_l(P)$ for all $j,l\in S_i$, $i=1,\ldots,m$. So, let $V(S_i,P)$ denote the payoff obtained by any player belonging to S_i in the coalition structure P. We focus on coalition formation games satisfying the following conditions on the per-member payoffs.

(P.1) *Positive Spillovers.* $V(S_i, P\setminus\{S_1,S_2\}\cup\{S_1\cup S_2\})>V(S_i,P)$ for all players belonging to S_i, $S_i\neq S_1,S_2$.

Condition (P.1) restricts our analysis to games with positive spillovers, where the formation of a coalition by other players increases the payoff of a player.

(P.2) *Negative Association.* $V(S_i,P)<V(S_j,P)$ if and only if $|S_i|>|S_j|$.

Condition (P.2) imposes that, in any coalition structure, small coalitions have higher per-member payoffs than big coalitions.

(P.3) *Individual Free-Riding.* $V(\{j\},P\setminus\{S_i\}\cup\{S_i\setminus\{j\},\{j\}\})>V(S_i,P)$ for all $j\in S_i$, $S_i\in P$.

Condition (P.3) is related to the existence of individual free-riding incentives. That is, if a player leaves any coalition to be alone, then he is better off.

(P.4) *Efficiency.* $\nexists P=\{S_1,S_2,\ldots,S_m\}\in\mathcal{P}$ such that $P\neq\{N\}$ and $\sum_{i=1}^{m}V(S_i,P)\cdot|S_i|\geq V(N)\cdot|N|$.

[3] Ray and Vohra (1999) have provided a justification for the assumption of equal sharing rule. In an infinite-horizon model of coalition formation among symmetric players with endogenous bargaining, they have shown that in any equilibrium without delay there is equal sharing. See also Bloch (1996).

Finally, condition (P.4) assumes that the grand coalition is the only efficient coalition structure with respect to payoffs, where $V(N)$ denotes the payoff of any player belonging to the grand coalition $\{N\}$.

An economic situation satisfying these four conditions is a cartel formation game with Cournot competition as in Bloch (1997) and Yi (1997). Let $p(q) = a - q$ be the inverse demand (q is the industry output). The industry consists of $|N|$ identical firms. Inside each cartel, we assume equal sharing of the benefits obtained from the cartel's production. Once stable agreements on cartel formation have been reached, we observe a Cournot competition among the cartels. The payoff for each firm in each possible coalition structure is well defined. Firm i's cost function is given by $d \cdot q_i$, where q_i is firm i's output and d ($a > d$) is the common constant marginal cost. As a result, the per-member payoff in a cartel of size $|S|$ is, for all firms belonging to S,

$$V(S;P) = \frac{(a-d)^2}{|S| \cdot (|P|+1)^2}, \tag{1}$$

where $|P|$ is the number of cartels within P.

Lemma 1 Output cartels in a Cournot oligopoly with the inverse demand function $p(q) = a - q$ and the cost function $d(q_i) = d \cdot q_i$ satisfy (P.1)-(P.4):

Yi (1997) asserted that conditions (P.1) and (P.2) are satisfied. It is straightforward to show that (P.3) and (P.4) are also satisfied.

A second economic application of games with positive spillovers are economies with pure public goods. The model we study is inspired from Bloch (1997) and Yi (1997) wherein we introduce congestion. The economy consists of $|N|$ agents. At cost $d_i(q_i)$, agent i can provide q_i units of the public good. Let $q = \sum_i q_i$ be the total amount of public good. The utility each agent obtains from the public good depends positively on the total amount of public good provided, but negatively on the number of coalition partners: $U_i(q) = (|S|)^{-\alpha} \cdot q$ for all $i \in S$, where parameter $\alpha > 0$ measures the degree of congestion. Each agent owns a technology to produce the public good, and the cost of producing the amount q_i of the public good is given by $d_i(q_i) = \frac{1}{2}(q_i)^2$. Since individual cost functions are convex and exhibit decreasing returns to scale, it is cheaper to produce an amount q of public goods using all technologies than using a single technology. In stage one the coalition formation takes place. Inside each coalition, we assume equal sharing of the production. Once stable agreements on coalition formation have been reached, each coalition of agents acts noncooperatively. On the contrary, inside every coalition, agents act cooperatively and the level of public good is chosen to maximize the sum of utilities of the coalition members. That is, for any coalition structure $P = \{S_1; S_2; \ldots; S_m\}$, the level of public good q_i chosen by the coalition S_i solves

$$\max_{q_i} |S_i| \cdot \left[(|S_i|)^{-\alpha} \left(q_i + \sum_{j \neq i} q_j \right) - \frac{1}{2} \left(\frac{q_i}{|S_i|} \right)^2 \right]$$

yielding a total level of public good provision for the coalition S_i equal to $q_i = (|S_i|)^{2-\alpha}$, $i = 1, \ldots, m$. The per-member payoff in a coalition of size $|S_i|$ is given by

$$V(S_i; P) = (|S_i|)^{-\alpha} \cdot \sum_{j=1}^{n} (|S_j|)^2 - \frac{1}{2} (|S_i|)^{2-2\alpha}, \tag{2}$$

for all agents belonging to S_i, $i = 1, \ldots, m$.

Contrary to the cartel formation game with Cournot competition, it depends on the number of agents $|N|$ and the degree of congestion α whether public goods coalitions satisfy conditions (P1)-(P4). For instance, public goods coalitions with utility function $U_i(q) = (|S|)^{-.35} \cdot q$ for all $i \in S$ and cost function $c_i(q_i) = \frac{1}{2}(q_i)^2$ satisfy (P.1)-(P.4) if $|N| \in [4; 10]$. Notice that, for $|N| < 4$ the condition (P.3) is violated, while for $|N| > 10$ it is (P.4) which is violated.

3.2 The Effectiveness Relation

Remember that what coalitions can do if and when they form is specified by $\{\rightarrow_S\}_{S \subseteq N, S \neq \emptyset}$, where $\{\rightarrow_S\}$, $S \subseteq N$, is an effectiveness relation on P. Restrictions are imposed on the coalition formation process in G through the effectiveness relation $\{\rightarrow_S\}$ which is defined as follows: $P \rightarrow_S P'$ if and only if (i) $\{S_i \setminus (S_i \cap S) : S_i \in P\} = \{S'_i \in P' : S'_i \subseteq N \setminus S\}$ and (ii) $\exists \{S'_1, \ldots, S'_l\} \subseteq P'$ such that $\bigcup_{j=1}^{l} S'_j = S$. Condition (i) simply means that no simultaneous deviations are possible. If the players in S deviate leaving their coalition(s) in P, the non-deviating players do not move. Nevertheless, once S has moved, the players not in S can react to the deviation of S. Condition (ii) simply allows the deviating players in S to form one or several coalitions in the new status-quo P'. Non-deviating players do not belong to those new coalitions.

3.3 Stable Coalition Structures

Before stating the results, we introduce some definitions or notations. A coalition structure P is symmetric if and only if $|S_i| = |S_j|$ for all $S_i, S_j \in P$. We denote by $P^* = \{N\}$ the grand coalition (P^* is a symmetric coalition structure) and by $\overline{P}$ the stand-alone coalition structure: $\overline{P} = \{S_1, \ldots, S_n\}$ with $|S_i| = 1$ for all $S_i \in \overline{P}$.

The following two lemmas partially characterize the largest consistent set for the coalition formation game in effectivity form G under conditions (P.1)-(P.4). Lemma 2 states that any coalition structure, wherein some coalition members would receive less than in the stand-alone coalition structure, is never stable.

Lemma 2 Under (P.1)-(P.4), if there exists $S \in P$ such that $V(S,P) < V(S',\overline{P})$, then $P \notin LCS(\Gamma, \prec)$ and $P \notin LCS(\Gamma, \preceq)$.

Proof. Condition (P.2) implies that in any coalition structure P, $V(S_i,P) < V(S_j,P)$ if and only if $|S_i| > |S_j|$. To prove Lemma 2, we proceed by steps.

Step one. Firstly, we show that all coalition structures $P \in \mathcal{P}$ containing only one coalition S with $|S| > 1$ and $V(S,P) < V(S',\overline{P})$, do not belong to $LCS(\Gamma, \prec)$ and $LCS(\Gamma, \preceq)$. Obviously, $P \prec \overline{P}$ and the deviation $P \to_S \overline{P}$ cannot be deterred. Indeed, any deviation from $\overline{P}$ of players that did not belong to S in P will improve, by (P.1), the payoff of players that were in S (in P) and are singletons in $\overline{P}$. Therefore, $P \notin LCS(\Gamma, \prec)$ and $P \notin LCS(\Gamma, \preceq)$.

Step two. Secondly, we show that all coalition structures $P \in \mathcal{P}$ containing only two coalitions S_1, S_2 with $|S_1| \geq |S_2| > 1$ and $V(S_1,P) < V(S',\overline{P})$, do not belong to $LCS(\Gamma, \prec)$ and $LCS(\Gamma, \preceq)$. Condition (P.1) implies that the coalition S_1 has incentives to split into singletons. Indeed, $V(S_1,P') > V(S',\overline{P})$ and $P \prec P'$ where $P' = P \setminus S_1 \cup \{j\}_{j \in S_1}$. The deviation $P \to_{S_1} P'$ cannot be deterred. Indeed,

- if $V(S_2,P') < V(S',\overline{P})$, then using the argumentation of step one, the deviation $P' \to_{S_2} \overline{P}$ is not deterred and $\overline{P} \gg P$. Therefore, $P \notin LCS(\Gamma, \prec)$ and $P \notin LCS(\Gamma, \preceq)$.

- if $V(S_2,P') > V(S',\overline{P})$, we have to show that any deviation from P' of players in $N \setminus S_1$ will never make players in S_1 worse off than in P. Two kinds of deviations are possible. First, the players in S_2 form a bigger coalition with players not in S_1. Then, by condition (P.1), the players in S_1 that now are singletons obtain a payoff even greater than in P'. Second, some player(s) leave(s) S_2 to form singleton(s). Then, the players that were in S_1 are worse than in P' but, by (P.1), they are better off or at least not worse off than in $\overline{P}$, and $V(S_1,P) < V(S',\overline{P})$. Therefore, there is no other coalition structure P'' such that $P'' \gg P'$ and $V(\{j\},P'') < V(S_1,P)$ for some $j \in S_1$. Hence, $P \notin LCS(\Gamma, \prec)$ and $P \notin LCS(\Gamma, \preceq)$.

Step three. Thirdly, proceeding as above, we can show that all coalition structures $P \in \mathcal{P}$ containing only three coalitions S_1, S_2, S_3 with $|S_1| \geq |S_2| \geq |S_3| > 1$ and $V(S_1,P) < V(S',\overline{P})$, do not belong to $LCS(\Gamma, \prec)$ and $LCS(\Gamma, \preceq)$. And so on. ■

The grand coalition structure which is the efficient one always belongs to the largest consistent set, and is possibly stable.

Lemma 3 Under (P.1)-(P.4), $P^* \in LCS(\Gamma, \prec)$ and $P^* \in LCS(\Gamma, \preceq)$.

Proof. To prove that $P^* \in Y^k$ $(k \geq 1)$ we have to show that for all $P \neq P^*$ we have $P \succ P^*$. That is, we show that P^* could be stable since any deviation $P^* \rightarrow_S P$ can be deterred by the threat of ending in P^*. The proof is done in two steps.

Step A. By (P.2) and (P.4) the players belonging to the biggest coalition (in size) in any $P \neq P^*$ are worse than in P^*. Also, all players prefer P^* to $\overline{P}$, and $P^* > \overline{P}$.

Step B. Take the sequence of moves where at each move one player belonging to the biggest coalition (in the current coalition structure) deviates to form a singleton, until the coalition structure $\overline{P}$ is reached. From $\overline{P}$ occurs the deviation $\overline{P} \rightarrow_N P^*$.

Therefore, (A)-(B) imply that $P^* \gg P$ for all $P \neq P^*$. ■

From these two lemmas, we obtain a sufficient condition such that the largest consistent set singles out the grand coalition.

Proposition 1 Under (P.1)-(P.4), if each non-symmetric coalition structure $P \in \mathcal{P}$ is such that there exists $S \in P$ satisfying $V(S;P) < V(S';\overline{P})$, then $LCS(G;\succ) = \{P^*\}$ and $LCS(G;\succeq) = \{P^*\}$.

Proof. Lemma 2 tells us that coalition structures $P \in \mathcal{P}$, where $\exists S \in P$ such that $V(S;P) < V(S';\overline{P})$, do not belong to $LCS(G;\succ)$ and $LCS(G;\succeq)$. So, $Y^1 \subseteq \mathcal{P} \backslash \{P \in \mathcal{P} : \exists S \in P$ for which $V(S;P) < V(S';\overline{P})\}$. The conditions (P.2) and (P.4) imply that all symmetric coalition structures P $(\neq P^*)$ are such that $V(S_i;P) = V(S_j;P)$ and $V(N) > V(S_i;P)$ for all $S_i, S_j \in P$ (it implies that $P^* > P$ for all P $(\neq P^*)$ symmetric). So, the deviation $P \rightarrow_N P^*$ (where P symmetric) cannot be deterred since $\nexists P'$ such that $P' \gg P^*$ and $P' \in Y^1$. Therefore, $LCS(G;\succ) = LCS(G;\succeq) = \{P^*\}$. ■

We now show that the stand-alone coalition structure, i.e. the coalition structure consisting only of singletons, is never stable under the largest consistent set based on the indirect weak dominance relation.

Proposition 2 Under (P.1)-(P.4), $\overline{P} \notin LCS(G;\succeq)$.

Proof. From Definition 4 and Lemma 2, we have that $Y^0 \equiv \mathcal{P}$ and $Y^1 = \{P \in \mathcal{P} : \forall P', S$ such that $P \rightarrow_S P'$, $\exists P'' \in Y^0$, where $P' = P''$ or $P' \succeq P''$, we do not have $V_i(\cdot P) \leq V_i(\cdot P'')$ for all $i \in S$ and $V_i(\cdot P) < V_i(\cdot P'')$ for some $i \in S\} \subseteq \mathcal{P} \backslash \{P \in \mathcal{P} : \exists S \in P$ for which $V(S;P) < V(S';\overline{P})\}$. Next we show that $\overline{P} \notin Y^2 = \{P \in \mathcal{P} : \forall P', S$ such that $P \rightarrow_S P'$, $\exists P'' \in Y^1$, where $P' = P''$ or $P' \succeq P''$, we do not have $V_i(\cdot P) \leq V_i(\cdot P'')$ for all $i \in S$ and $V_i(\cdot P) < V_i(\cdot P'')$ for some $i \in S\}$. Any coalition structure $P \in Y^1$ is such that $\forall S \in P : V(S;P) \geq V(S';\overline{P})$. By (P.2) and (P.4), the coalition structure $P^* = \{N\}$

is efficient and $V(N) > V(S'; \overline{P})$ for all $i \in N$. Therefore, $\overline{P} \notin Y^2$ because the deviation $\overline{P} \rightarrow_N P^*$ cannot be deterred. Indeed, for all $P'' \in Y^1$, where $P^* = P''$ or $P^* \succeq P''$, we have $V_i(\cdot \overline{P}) \leq V_i(\cdot P'')$ for all $i \in N$ and $V_i(\cdot \overline{P}) < V_i(\cdot P'')$ for some $i \in N$, by (P.1). ■

However, this results does not hold when we consider the definition of the largest consistent set based on the indirect strict dominance relation. The stand-alone coalition structure, $\overline{P}$, may belong to $LCS(G; \succ)$.

Example 1. $|N| = 4$.

Partitions	Payoffs
{4}	(8;8;8;8)
{3;1}	(4;4;4;12)
{2;2}	(5;5;5;5)
{2;1;1}	(3;3;8;8)
{1;1;1;1}	(4;4;4;4)

Consider Example 1 with four players. Throughout all the examples, we make a slight abuse of notation. For instance, {3;1} should not be interpreted as a single coalition structure but as the four coalition structures, composed by two coalitions of size 3 and 1, that can be formed by four players. Example 1 shows how the use of the indirect strict or weak dominance matters. Firstly, we characterize $LCS(G; \succeq)$. In the first round of the iterative procedure to compute $LCS(G; \succeq)$, we eliminate the coalition structures {2;1;1} and {1;1;1;1}. Indeed, the deviations {2;1;1} → {1;1;1;1} and {1;1;1;1} → {4} are not deterred. In the second round, we cannot eliminate other coalition structures since any possible deviations from {4} or {3;1} or {2;2} are deterred. Then, $LCS(G; \succeq) = \{\{4\}; \{3;1\}; \{2;2\}\}$. Secondly, we characterize $LCS(G; \succ)$. We can only eliminate the coalition structure {2;1;1}. The deviation {1;1;1;1} → {4} is deterred by the move from {4} to {3;1}. The deviation {1;1;1;1} → {2;2} is deterred by the sequence of moves from {2;2} to {2;1;1} to {1;1;1;1} to {4} to {3;1}. Then, $LCS(G; \succ) = \{\{4\}; \{3;1\}; \{2;2\}; \{1;1;1;1\}\}$.

3.4 Cautiously Stable Coalition Structures

In most economic situations satisfying the conditions (P.1)-(P.4), many coalition structures belong to the largest consistent set. Indeed, the largest consistent set aims to be a weak concept which rules out with confidence. On the contrary, the largest cautious consistent set aims to be better at picking out. The largest cautious consistent set singles out the grand coalition.

Proposition 3 *Under (P.1)-(P.4), $LCCS(G; \succ) = \{P^*\}$ and $LCCS(G; \succeq) = \{P^*\}$.*

Proof. From Definition 6 we have $Z^0 = \mathcal{P}$.

Step one. From Lemma 2 and Definition 6, it is straightforward that the set of coalition structures $\{P \in \mathcal{P} : \exists S \in P \text{ such that } V(S;P) < V(S';\overline{P})\}$ does not belong to Z^1. On the contrary, we can see that $P^* \in Z^1$. Consider first any possible deviation of any coalition S to any coalition structure P containing only coalitions S with $V(S;P) > V(S';\overline{P})$, and such that $P > P^*$. By (P.2) and (P.4) the players belonging to the biggest coalition (in size) in any $P \in \mathcal{P}\setminus\{P^*\}$ are worse than in P^*. From P, take the sequence of moves where, at each move, one of the players of the biggest coalition in size deviates to form a singleton, until we arrive to $\overline{P}$. From $\overline{P}$ occurs the deviation to some coalition structure P' which is a permutation of players in P (that is, $|P| = |P'|$ and $\forall S \in P$, there exists a coalition $S' \in P'$ such that $|S| = |S'|$), and such that the initial player who has deviated from P is occupying now in P' the position of some player i belonging to the coalition S that, initially, has moved from P^* to P. This means that $P' \gg P$ and at least one of the initial deviating players of coalition S from P^* (player i) is worse off in P' compared to P^*. Therefore, every possible deviation from P^* to some coalition structure P with all $S \in P$ such that $V(S;P) > V(S';\overline{P})$, is deterred because there always exists a coalition structure P', with $P' \gg P$ and such that $V(\cdot P') < V(\cdot P^*)$ for some player $i \in S$ and $P^* \rightarrow_S P$. Finally, we have to consider any possible deviation of some coalition S from P^* to P with $P > P^*$ and such that for some $S'' \in P$ we have $V(S'';P) < V(S';\overline{P})$. If such a deviation does exist, it will be deterred because $\overline{P} \gg P$ and all $i \in N$ get a payoff $V(\cdot\overline{P}) < V(\cdot P^*)$. Then, $P^* \in Z^1$, and $Z^1 \subseteq \mathcal{P}\setminus\{P \in \mathcal{P} : \exists S \in P \text{ such that } V(S;P) < V(S';\overline{P})\}$.

Step two. Take the coalition structure $\overline{P}$ and any other coalition structure $P \in Z^1 \subseteq \mathcal{P}\setminus\{P \in \mathcal{P} : \exists S \in P \text{ such that } V(S;P) < V(S';\overline{P})\}$ containing some coalition S that obtains a payoff $V(S;P) = V(S';\overline{P})$. Obviously, P and $\overline{P}$ do not belong to Z^2 since for all $P';S$ such that $P \rightarrow_S P'$ or $\overline{P} \rightarrow_S P'$, the expected payoff obtained by assigning positive probabilities to all coalition structures $P'' \in Z^1$, with at least $P' = P''$ or $P' \ll P''$, is strictly preferred to $V(S;P)$ for all players in S, given that $P^* \gg P'$ for all $P' \in \mathcal{P}$ and $V(\cdot\overline{P}) < V(\cdot P^*)$. Using the same reasoning as in step one, one can show that $P^* \in Z^2$, with $Z^2 \subseteq \mathcal{P}\setminus\overline{P}\} \cup \{P \in \mathcal{P} : \exists S \in P \text{ such that } V(S;P) \leq V(S';\overline{P})\}$.

Step three. Take the coalition structure(s) $P \in Z^2$ containing the coalition S that obtains the smallest payoff. Obviously, P does not belong to Z^3 since for all $P';S$ such that $P \rightarrow_S P'$, the expected payoff obtained by assigning positive probabilities to all coalition structures $P'' \in Z^2$, with $P' = P''$ or $P' \ll P''$, is strictly preferred to $V(S;P)$ for all players in S, given that $P^* \gg P'$ for all $P' \in \mathcal{P}$, and $V(S;P) < V(\cdot P^*)$ for all $i \in S$ (the deviating coalition). One can use the same reasoning used in step one to show that $P^* \in Z^3$. And so on, until we have eliminated all $P \in \mathcal{P} \backslash \{P^*\}$ (given that, by (P.2) and (P.4), the players belonging to the biggest coalition (in size) in any $P \in \mathcal{P} \backslash \{P^*\}$ are worse than in P^*).

Now, consider P^*. From Lemma 3, we know that for all $P \neq P^*$, P ¿ P^*. Then, $LCCS(G; ¿) = \{P^*\}$ and $LCCS(G; \underline{¿}) = \{P^*\}$, since for all P', S such that $P^* \to_S P'$, the expected payoff obtained by assigning positive probability to P^* (the only coalition structure not yet eliminated in the iterative procedure described above) and with P' ¿ P^*, is equally preferred to the payoff obtained in P^* for all $i \in S$ (the initial deviating coalition) ■

This result is due to the basic idea behind the largest cautious consistent set. Intuitively, at each iteration in the definition of the largest cautious consistent set, we rule out the coalition structure wherein some players receive less or equal than what they could obtain in all candidates to be stable (i.e. all coalition structures not ruled out yet) since these players cannot end worse off by engaging a move.

Example 2. Public goods coalitions with $|N| = 4$ and $\alpha = .35$.

Partitions	Payoffs
$\{4\}$	$(6.82, 6.82, 6.82, 6.82)$
$\{3, 1\}$	$(4.72, 4.72, 4.72, 9.5)$
$\{2, 2\}$	$(5.05, 5.05, 5.05, 5.05)$
$\{2, 1, 1\}$	$(3.48, 3.48, 5.5, 5.5)$
$\{1, 1, 1, 1\}$	$(3.5, 3.5, 3.5, 3.5)$

In the first round of the iterative procedure to compute the largest consistent set, we eliminate the coalition structures $\{1, 1, 1, 1\}$ and $\{2, 1, 1\}$. Indeed, the deviations $\{1, 1, 1, 1\} \to \{4\}$ and $\{2, 1, 1\} \to \{1, 1, 1, 1\}$ are not deterred. In the second round, we cannot eliminate other coalition structures since any possible deviations from $\{4\}$ or $\{3, 1\}$ or $\{2, 2\}$ are deterred. For example, the deviation $\{3, 1\} \to \{2, 1, 1\}$ by one of the player who obtains 4.72 as payoff is deterred since there exists a sequence of moves $\{2, 1, 1\} \to \{4\} \to \{3, 1\}$ ending at $\{3, 1\}$ such that at each move the deviating players prefer the ending coalition structure to the status-quo they face and the original deviating player is not better off (he obtains still 4.72 as payoff). Therefore, the largest consistent set is $LCS(G; \underline{¿}) = \{\{4\}, \{3, 1\}, \{2, 2\}\}$.

But $\{3, 1\}$ and $\{2, 2\}$ do not belong to the largest cautious consistent set. Indeed, the deviation $\{3, 1\} \to \{2, 1, 1\}$ by one of the player who obtains 4.72 as payoff is not deterred since all coalition structures that indirectly dominate $\{2, 1, 1\}$ and not yet eliminated are $\{4\}$, $\{3, 1\}$ and $\{2, 2\}$. Hence, the expected payoff of the original deviating player, obtained by assigning positive probabilities to $\{4\}$, $\{3, 1\}$ and $\{2, 2\}$, is greater than 4.72.

Once $\{3,1\}$ is eliminated, the deviation $\{2,2\} \to \{4\}$ is not deterred. Therefore, $\{3,1\}$ and $\{2,2\}$ do not belong to the largest cautious consistent set which singles out $\{4\}$.

4 Cartel Formation with Quantity Competition

In the cartel formation game with Cournot competition, the largest consistent set based on the indirect weak dominance relation singles out for $|N| \leq 4$ the grand coalition $P^* = \{N\}$. But as $|N|$ grows, many coalition structures may belong to $LCS(G; \underline{\tau})$.

Example 3. $|N| = 6$, $d = 0$, $a = 1$.

Partitions	Payoffs
$\{6\}$	(.0417; .0417; .0417; .0417; .0417; .0417)
$\{5,1\}$	(.0222; .0222; .0222; .0222; .0222; .111)
$\{4,2\}$	(.0278; .0278; .0278; .0278; .0556; .0556)
$\{3,3\}$	(.0370; .0370; .0370; .0370; .0370; .0370)
$\{4,1,1\}$	(.0156; .0156; .0156; .0156; .0625; .0625)
$\{3,2,1\}$	(.0208; .0208; .0208; .0312; .0312; .0625)
$\{2,2,2\}$	(.0312.0312; .0312; .0312; .0312; .0312)
$\{3,1,1,1\}$	(.0133; .0133; .0133; .0400; .0400; .0400)
$\{2,2,1,1\}$	(.0200; .0200; .0200; .0200; .0400; .0400)
$\{2,1,1,1,1\}$	(.0139; .0139; .0278; .0278; .0278; .0278)
$\{1,1,1,1,1,1\}$	(.0204; .0204; .0204; .0204; .0204; .0204)

In the first round of the iterative procedure to compute the largest consistent set, we eliminate the coalition structures $\{2,1,1,1,1\}$, $\{3,1,1,1\}$, $\{4,1,1\}$. The deviations $\{2,1,1,1,1\} \to \{1,1,1,1,1,1\}$, $\{3,1,1,1\} \to \{1,1,1,1,1,1\}$, $\{4,1,1\} \to \{1,1,1,1,1,1\}$ are not deterred. Also, we can eliminate $\{2,2,1,1\}$: the deviation $\{2,2,1,1\} \to \{2,1,1,1,1\}$ is not deterred. In the second round, we delete the coalition structure $\{1,1,1,1,1,1\}$: the deviation $\{1,1,1,1,1,1\} \to \{3,3\}$ is not deterred. No more coalition structures can be eliminated at the next rounds. For example, the deviation from $\{2,2,2\}$ to $\{6\}$ is deterred by the further deviation to $\{5,1\}$. Therefore, $\{\{6\}, \{5,1\}, \{4,2\}, \{3,3\}, \{3,2,1\}, \{2,2,2\}\}$ is the largest consistent set $LCS(G; \underline{\tau})$. The sum of the payoffs associated to coalition structures $\{6\}$, $\{5,1\}$, $\{2,2,2\}$ are .2502, .222, .1872, respectively.

We now turn to the characterization of the largest consistent set for $|N| \leq 10$.

Proposition 4 *In the cartel formation game under Cournot competition, $LCS(G; \underline{\tau}) = \{P^*\}$ for $|N| \leq 4$, and $LCS(G; \underline{\tau}) = \mathcal{P} \setminus \{\{\overline{P}\} \cup \{P \in \mathcal{P} : \exists S \in P \text{ such that } V(S; P) < V(S'; \overline{P})\}\}$ for $5 \leq |N| \leq 10$.*

The proof of this proposition can be found in the appendix. Some remarks can be made. Firstly, P^* always belongs to the largest consistent set $LCS(G;\ll)$ (see Lemma 3), while $\overline{P}$ never belongs to $LCS(G;\ll)$ (see Proposition 2). Secondly, for $10 \geq |N| \geq 5$, all symmetric coalition structures, except $\overline{P}$, belong to $LCS(G;\ll)$. Finally, all non-symmetric coalition structures P such that $\nexists S \in P$ with $V(S;P) < V(S';\overline{P})$ belong to $LCS(G;\ll)$.

We compare now the outcomes obtained under the largest consistent set (and the largest cautious consistent set) with those obtained under a sequential game of coalition formation with fixed payoff division proposed by Bloch (1996). A fixed protocol is assumed and the sequential game proceeds as follows. Player 1 proposes the formation of a coalition S_1 to which he belongs. Each prospective player answers the proposal in the order fixed by the protocol. If one prospective player rejects the proposal, then he makes a counter-proposal to which he belongs. If all prospective players accept, then the coalition S_1 is formed. All players in S_1 withdraw from the game, and the game proceeds among the players belonging to $N \setminus S_1$. This sequential game has an infinite horizon, but the players do not discount the future. The players who do not reach an agreement in finite time receive a payoff of zero. Contrary to the largest consistent set, this sequential game relies on the commitment assumption. Once some players have agreed to form a coalition they are committed to remain in that coalition.

Consider the following finite procedure to form coalitions. First, player 1 starts the game and chooses an integer s_1 in the interval $[1;|N|]$. Second, player s_1+1 chooses an integer s_2 in $[1;|N|-s_1]$. Third, player s_1+s_2+1 chooses an integer s_3 in $[1;|N|-s_1-s_2]$. The game goes on until the sequence $(s_1;s_2;s_3;\ldots)$ satisfies $\sum_j s_j = |N|$. For symmetric valuations, if the finite procedure yields as subgame perfect equilibrium a coalition structure with the property that payoffs are decreasing in the order in which coalitions are formed, then this coalition structure is supported by the (generically) unique symmetric stationary perfect equilibrium (SSPE) of the sequential game (see Bloch, 1996). This result makes easy the characterization of the SSPE outcome of the cartel formation game.

Lemma 4 (Bloch, 1996) In Bloch's sequential coalition formation game under Cournot competition, any symmetric stationary perfect equilibria (SSPE) is characterized by $P = \{S^*;\{i\}_{i\notin S_1}\}$ where $|S^*|$ is the first integer following $(2n+3-y)\frac{1}{2}$, where $y = \sqrt{4n+5}$. If y is an integer, $|S^*|$ can take on the two values $(2n+3-y)\frac{1}{2}$ and $(2n+5-y)\frac{1}{2}$.

Intuitively, in the sequential game, firms commit to stay out of the cartel until the number of remaining firms equals the minimal profitable cartel size (this is the smallest coalition size for which a coalition member obtains a higher payoff than if all coalitions are singletons, and is equal to $|S^*|$). From Proposition 4 and Lemma 4, the relationship between the largest consistent set $LCS(G;\tau)$ and SSPE follows straightforwardly.

Proposition 5 *In the cartel formation game under Cournot competition with $|N| \leq 10$, the coalition structures supported by any symmetric stationary perfect equilibria (SSPE) of Bloch's sequential game always belong to the largest consistent set $LCS(G;\tau)$.*

Assume now that each player belonging to a coalition S have to support a monitoring or congestion cost $c(S)$ which is increasing with the coalition size and has the following functional form.[4] For all $S \subseteq N$, $c(S) = c \cdot (|S|-1)^{\phi}$ for $|S| > 1$ and $c(S) = 0$ for $|S| = 1$, with $c, \phi > 0$. For $c = 0$, the monitoring is said to be costless. For $c > 0$, the monitoring is said to be costly. As a result, the per-member expected payoff in a cartel of size $|S|$ becomes for all firms belonging to S,

$$V(S;P) = \frac{(a-d)^2}{|S| \cdot (|P|+1)^2} - c \cdot (|S|-1)^{\phi}. \tag{3}$$

It should be noted that, once a monitoring cost is introduced, the valuation still satisfies the properties of positive spillovers, negative association and individual free-riding. However, the grand coalition may be inefficient. Example 4 illustrates that a monitoring cost may refine the largest consistent set and single out the grand coalition.

Example 4. $|N| = 6$, $d = 0$, $a = 1$, $c = .00433$ and $\phi = .5$.

Partitions	Payoffs
$\{6\}$	(.0320; .0320; .0320; .0320; .0320; .0320)
$\{5,1\}$	(.0135; .0135; .0135; .0135; .0135; .1110)
$\{4,2\}$	(.0203; .0203; .0203; .0203; .0513; .0513)
$\{3,3\}$	(.0309; .0309; .0309; .0309; .0309; .0309)
$\{4,1,1\}$	(.0081; .0081; .0081; .0081; .0625; .0625)
$\{3,2,1\}$	(.0147; .0147; .0147; .0269; .0269; .0625)
$\{2,2,2\}$	(.0269. .0269; .0269; .0269; .0269; .0269)
$\{3,1,1,1\}$	(.0072; .0072; .0072; .0400; .0400; .0400)
$\{2,2,1,1\}$	(.0157; .0157; .0157; .0157; .0400; .0400)
$\{2,1,1,1,1\}$	(.0096; .0096; .0278; .0278; .0278; .0278)
$\{1,1,1,1,1,1\}$	(.0204; .0204; .0204; .0204; .0204; .0204)

Applying the iterative procedure to Example 4, we obtain that the largest consistent set $LCS(G;\tau)$ which is $\{\{6\}\}$. The sum of the payoffs associated to coalition structure

[4] Monitoring or congestion costs may emerge because larger coalitions face higher organizational costs, or moral hazard problems as in Espinosa and Macho-Stadler (2002)'s study of cartel formation in a Cournot oligopoly with teams.

f6g is 192. We observe that the welfare (de…ned as the sum of the payo¤s) is greater than the one associated to some stable coalition structures when monitoring is costless. That is, costly monitoring might re…ne the largest consistent set as well as improving the welfare.

In Table 1, we report the coalition structures supported by SSPE, LCS(G; ¿) or LCCS(G; ¿) in both examples of the cartel formation game. Except the earlier mentioned relationships between these solution concepts, these two examples show that there are no other relationships.[5]

	SSPE	LCS(G; ¿)	LCCS(G; ¿)
Example 3	ff5;1gg	(f5;1g; f4;2g; f3;3g; f6g; f3;2;1g; f2;2;2g)	ff6gg
Example 4	ff6gg	ff6gg	ff6gg

Table 1: The largest consistent set, the largest cautious consistent set and SSPE in the examples of the cartel formation game.

5 Conclusion

We have adopted the largest consistent set due to Chwe (1994) to predict which coalition structures are possibly stable in coalition formation games with positive spillovers. We have also introduced a re…nement, the largest cautious consistent set. For games satisfying the properties of positive spillovers, negative association, individual free-riding incentives and e¢ciency of the grand coalition, many coalition structures may belong to the largest consistent set. The grand coalition, which is the e¢cient coalition structure, always belongs to the largest consistent set and is the unique one to belong to the largest cautious consistent set.

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A Appendix

Proof of Proposition 4.

Part 1: $|N| \leq 4$. Simple computations show that each non-symmetric coalition structure $P \in \mathcal{P}$ is such that there exists $S \in P$ with $V(S;P) < V(S';\overline{P})$. From Proposition 1, we have $LCS(G;\gg) = \{P^*\}$ for $|N| \leq 4$.

Part 2: $5 \leq |N| \leq 10$. From Lemmas 2 and 3 and Proposition 2, we have $\{P \in \mathcal{P} : \exists S \in P$ such that $V(S;P) < V(S';\overline{P})\} \notin LCS(G;\gg)$, $\overline{P} \notin LCS(G;\gg)$ and $P^* \in LCS(G;\gg)$, respectively.

To prove that $\{P \in \mathcal{P} : P \neq \overline{P}, P^*$ and $V(S;P) \geq V(S';\overline{P})$ for all $S \in P\} \subseteq LCS(G;\gg)$, we have to show that all possible deviations from P can be deterred. Two kinds of possible deviations that benefit the deviating players have to be considered.

Firstly, we consider the splitting deviations $P \to_S P'$ such that $|P'| > |P|$. The condition (P.1) implies that the players in $N \setminus S$ are worse off in P'. Then, conditions (P.1) and (P.3) imply that further splitting deviations of players in $N \setminus S$ can occur and lead to some P'' where $P'' \to_N P$ and $V(\cdot P'') < V(\cdot P)$ for all $i \in N$. Therefore, $P' \gg P$ and the deviation $P \to_S P'$ is deterred.

Secondly, we consider the enlarging deviations $P \to_S P'$ such that $|P'| < |P|$ and $V(S;P') > V(\cdot P)$ for all $i \in S$. Then, $P' \succeq P$. Notice that by (P.1)-(P.4) and the payoff structure in the cartel formation game (Expression 1) we have $P' > P$ if and only if $|P'| < |P|$ and both coalition structures P and P' are symmetric. Then, the coalition S which moves from P to P' is $S = N$. Two cases should be distinguished:

(i) $P' = P^*$. Take the deviation $P^* \to_{\{i\}} P''$ where player i deviates to form a singleton with $P'' > P'$. It can be shown that $V(\cdot P'') < V(\cdot P)$ for some $i \in S$ (the initial deviating coalition) and $V(\cdot P'') \geq V(S';\overline{P})$ for all $S'' \in P''$. Then, the deviation $P \to_N P^*$ is deterred. From Expression 1 we get, for $S'' \in P''$ such that $|S''| = |N| - 1$,

$$V(S'';P'') = \frac{(a-d)^2}{9(|N|-1)} \geq \frac{(a-d)^2}{(|N|+1)^2} = V(S';\overline{P}) \text{ iff } (|N|)^2 - 7(|N|) + 10 \geq 0,$$

condition which is satisfied for $5 \leq |N| \leq 10$. Moreover, we have to show that $V(S'';P'') < V(\cdot P)$ for some $i \in S = N$ (the initial deviating coalition). Since P is symmetric, we have to compare $V(S'';P'')$ with the payoff obtained in the symmetric coalition structures. Given that $5 \leq |N| \leq 10$ the only symmetric coalition structures we could have are such that their payoffs will be

$$\frac{(a-d)^2}{2\left(\frac{|N|}{2}+1\right)^2}, \quad \frac{(a-d)^2}{9\frac{|N|}{2}} \quad \text{or} \quad \frac{(a-d)^2}{\frac{|N|}{3}\left(\frac{|N|}{3}+1\right)^2}.$$

So,

$$V(S'';P'') = \frac{(a-d)^2}{9(|N|-1)} < \frac{(a-d)^2}{2(\frac{|N|}{2}+1)^2} \Leftrightarrow \frac{(|N|)^2}{2} - 7(|N|) + 11 < 0,$$

$$V(S'';P'') = \frac{(a-d)^2}{9(|N|-1)} < \frac{(a-d)^2}{9\frac{|N|}{2}} \Leftrightarrow |N| > 2,$$

$$V(S'';P'') = \frac{(a-d)^2}{9(|N|-1)} < \frac{(a-d)^2}{\frac{|N|}{3}(\frac{|N|}{3}+1)^2} \Leftrightarrow \frac{(|N|)^2}{9} + \frac{2(|N|)}{3} + 1 < 27 - \frac{27}{|N|},$$

and all these conditions are satisfied for $5 \le |N| \le 10$. Hence, the deviation $P \to_N P^*$ is deterred (with P symmetric).

(ii) $P' \neq P^*$ (i.e. all players deviate from P to another symmetric coalition structure P', with $P' \neq P^*$). From P' a player i deviates to form a singleton. That is, $P' \to_{\{i\}} P''$. From P'', take the sequence of moves where, at each move, one of the players belonging to the biggest coalition in size, deviates to form a singleton until we arrive to $\overline{P}$. From $\overline{P}$ occurs the deviation of coalition $N \setminus \{i\}$ to the coalition structure P''' with $P''' = \{N \setminus \{i\}; \{i\}\}$ and such that $P''' \succeq P'$. As before, it is immediate to see that $V(\{N \setminus \{i\}\}; P''') < V(\cdot; P)$ for some player $i \in N$ (the initial deviating coalition) whenever P is symmetric. So, the deviation $P \to_N P'$ (with P and P' symmetric) is deterred.

Therefore, the enlarging deviations $P \to_S P'$ are deterred. ■