

Sharing the surplus: A just and efficient proposal for environments with externalities*

Inés Macho-Stadler[†] David Pérez-Castrillo[‡] David Wettstein[§]

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Abstract

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1 Introduction

Achieving cooperation and sharing the resulting benefits in the presence of externalities has been the subject of several international agreements. The Kyoto protocol was drafted in 1997 to address climate control. It was further elaborated upon in the Buenos Aires plan of action put together in 1998. The Bonn agreements in 2001 resolved several outstanding issues and paved the way to the Marrakesh accords that contained a clearer picture of the Kyoto protocol. Of concern were the commitments undertaken by the various parties and the enforcement as well as compensation mechanisms set in place.

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[†]Dep. of Economics & CODE, Universitat Autònoma de Barcelona. Email: ines.macho@uab.es.

[‡]Corresponding author. Dep. of Economics & CODE, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain. Email: david.perez@uab.es.

[§]Department of Economics, Ben-Gurion University of the Negev. Email: wettstn@bgumail.bgu.ac.il.

The GATT (General Agreement on Tariffs and Trade) was signed in 1947 in Geneva and focused on trading arrangements. After eight rounds of multilateral trade negotiations, the Uruguay round concluded with the signing in 1994 of the Marrakesh agreement, closing down the GATT which was replaced in 1995 by the World Trade Organization (WTO). The WTO stated goals are to promote world trade. It examines among other things the effect of regional trade agreements on the world wide trading system and the ways to compensate (or reward) the parties adhering to its policy recommendations.

The Treaty on the Non-Proliferation of Nuclear Weapons was signed at Washington, London, and Moscow in 1968, and dealt with the nuclear arms race. The purpose was to prevent the spread of nuclear weapons and eventually lead to nuclear disarmament. The implementation of the treaty objectives had to overcome two major problems. The first one is assuring countries agreeing to it of adequate protection should a non-adhering country develop a nuclear arsenal. The second is the (economic) sanctions that should be imposed on violating countries, as well as the (economic) compensation for countries adhering to it.

On the micro scale, there are mergers and agreements between firms coordinating their market behavior. Two major issues are the payoffs expected by the parties involved, and the reorganization of activities which determines to a large degree the sharing of these payoffs.

A common denominator to all of the above scenarios is that they entail cooperation in the presence of externalities. Each one of those can be broadly described as a situation where, what a group of agents, taking a joint action, may expect to get, depends both on the action taken, as well as on the organization and actions of agents outside this group.

Recently there has been a surge of literature that deals with the question of what coalitions would arise in such cooperative environments and how would the gains of cooperation be shared among the agents. This “positive” approach has been taken by Bloch (1996) which studied the sequential formation of coalitions in environments with externalities through the analysis of extensive form games. Ray and Vohra (1999) allowed for more general environments while studying a similar problem. They defined an extensive form bargaining game, and studied its stationary subgame perfect equilibria outcomes with emphasis on the resulting coalition structure. In Ray and Vohra (2001) the provi-

sion of public goods was analyzed and the resulting equilibrium coalition structure was characterized. Bloch (2002) offered a survey of problems and results in the industrial organization literature.

These works, while discussing in part the efficiency of the outcome reached, did not address other normative properties of the resulting allocations. They refrained from taking the normative point of view asking what should the coalition structure and sharing of the surplus look like. This might in part be due to the fact that in contrast to cooperative environments with no externalities for which there exist focal solutions such as the Shapley value or the core, there is no “focal” solution for cooperative environments with externalities.

In this work we propose a general, yet simple, method of dividing the gains (costs) of cooperation in the presence of externalities. The solution offered can be applied to environments where the externalities are positive (that is, what a group of agents expects to get is larger the more grouped the rest of agent is) as well as to situations with negative externalities. It satisfies the desirable properties (axioms) of efficiency, anonymity (symmetry), linearity, and the further reasonable property that agents which have no effect whatsoever on the outcome (“dummy” agents) should not receive any part of the surplus. In contrast to the case of no externalities, where these conditions are sufficient to generate a unique sharing method (Shapley, 1953), there are several ways to satisfy them in the presence of externalities. We proceed to impose more stringent, yet reasonable, conditions leading to a unique way of surplus sharing.

First we study the implications of a stronger symmetry axiom, capturing the idea that all individuals with “identical power” should receive the same outcome. We prove that this leads to a natural method of constructing a solution, that is proceeding via averages. This method associates to each group of agents a value that is some average of what they can obtain in the different scenarios, and then it allocates to each agent the Shapley value of this average game. However, there are still several ways to form averages that satisfy all the requirements imposed so far. To reach a unique outcome, we add one more requirement, namely, that when a pair of agents has exactly the same power acting separately or together, the outcome received as a pair coincides with the outcome received as singletons. We then construct a sharing method that satisfies all the axioms and show

it is unique. We elaborate further on the average approach and present a “marginal view” of the method, similar to the popular marginal expression for the Shapley value for games with no externalities. We also prove the method is well-behaved with respect to the addition of “dummy” players, in that the addition of a “dummy” player leaves the outcomes of all other players intact.

Two previous attempts to provide sharing methods in the presence of externalities were Myerson (1977) and Bolger (1989). Our method is simpler and uses a more intuitive average approach. We also offer a more suitable definition of a “dummy” player than Myerson (1977) and avoid the Bolger (1989) problem where the addition of a “dummy” player may change the outcomes of other agents. The issue of coalition formation and value in environments with no externalities has recently been raised by Maskin (2003). He characterized another sharing method through the use of axioms inspired in part by a sequential process of coalition formation. He also showed it can be implemented by a non-cooperative game. We note that the value and coalitions structure predictions of Aumann and Dreze (1974) and Hart and Kurz (1983) dealt with environments with no externalities.

Finally, we emphasize the applicability of our proposal by applying our method to data provided by Eyckmans and Tulkens (2003) on international environmental agreements.¹ The environment consists of six agents: the U.S., Japan, European Union, China, Former Soviet Union, and Rest of the World. The data describes the coalitional payoffs for any partition of those agents. We calculate and comment on the outcomes prescribed by our value.

The paper proceeds as follows: Section 2 introduces the environment; Section 3 presents the four basic requirements of efficiency, symmetry, “dummy” player, and linearity and the class of sharing methods that satisfy them. Section 4 presents the new strong symmetry axiom as well as the average approach and shows the two are equivalent. Section 5 introduces the final similar influence axiom. It constructs a sharing method satisfying all axioms, shows it is unique, and discusses several properties of the new value. Section 6 offers a detailed comparison of our value with those of Myerson and Bolger.

¹See Eyckmans and Tulkens (2003) for a description of the computational model generating this data set.

Section 7 applies our value to data on environmental agreements, generating a proposal regarding the distribution of gains across the various countries. Section 8, concludes and offers further directions of research.

2 The environment

The economic environment we study can be described as follows. We denote by $N = \{1, \dots, n\}$ the set of players (or agents). A coalition S is a non-empty subset of N , $S \subseteq N$. An embedded coalition is a pair (S, P) , where S is a coalition and $P \ni S$ is a partition of N . An embedded coalition hence, specifies the coalition as well as the structure of coalitions formed by the other players. Let \mathcal{P} denote the set of all partitions of N . The set of embedded coalitions is denoted by ECL and defined by:

$$ECL = \{(S, P) \mid S \in P, P \in \mathcal{P}\}.$$

We denote by (N, v) a game in partition function form (or a partition function game (PF G)), where $v : ECL \rightarrow R$ is a characteristic function that associates a real number with each embedded coalition. Hence, $v(S, P)$ with $S \in P, P \in \mathcal{P}$, is the value of coalition S in a partition P . For technical purposes, we use the convention that the empty set \emptyset is in P for every $P \in \mathcal{P}$, and assume that the characteristic function satisfies $v(\emptyset, P) = 0$.

A game is *with no externalities* if and only if for any two partitions of the set of players $P, P' \in \mathcal{P}$ and any coalition S which belongs both to P and P' , the characteristic function satisfies $v(S, P) = v(S, P')$. This means that in a game with no externalities, the payoff that the players in a coalition S can jointly obtain if this coalition is formed is independent of the way the other agents are organized. Hence, the value of a coalition S can be written without reference to the organization of the remaining agents, $\hat{v}(S) \equiv v(S, P)$ for all $P \ni S, P \in \mathcal{P}$.

A game is *with externalities* if and only if there is at least one coalition $S \subseteq N$, and two partitions P and P' containing S , such that $v(S, P) \neq v(S, P')$. In this case, the value of some coalitions depend on the way the other players are organized, and it is necessary to specify not only the coalition whose value we are interested in but also the groups the other players form.

In this paper we make a proposal for the division of the surplus in such Partition Function Games. By a solution concept, or a *value*, we mean a mapping φ which associates with every game (N, v) a vector in R^n that satisfies $\sum_{i \in N} \varphi_i(N, v) = v(N, (N, \emptyset))$. A value determines the payoffs for every player. Note that by definition, we assume that all the agents end up together and that a value is always *efficient*.

We proceed in stages imposing a sequence of axioms that lead to a unique value. We first study the implications of the three basic axioms (in addition to efficiency) which are the natural analogues of the Shapley axioms. While imposing restrictions on the permissible values, the three requirements still leave us with a large family of values. We then strengthen the symmetry axiom in a natural way, addressing in a sense the “upsetting power” of the agents. This naturally narrows down the class of values and we show that in the presence of the other two axioms it is equivalent to defining a value through an averaging process. We then add a final assumption regarding the effect of a two agents merger or separation on the value and obtain a unique proposal. We provide two closed form expressions for the value, one corresponding to the average approach and the other identifying the simple expressions it assumes on a natural basis for the space of Partition Function Games.

3 The “basic” axioms

The most natural requirements to impose on a value are those underlying the construction of the Shapley value in games without externalities, namely the axioms of linearity, symmetry, and finally the “dummy” player axiom. These properties were the starting point of all previous attempts to construct values for Partition Function Games.

We first define the notions of a dummy player and the operation of addition, multiplication by a scalar, and permutation of games.

A player $i \in N$ is called a *dummy player* in the game (N, v) if and only if $v(S, P) = v(S', P')$ for any embedded coalition (S', P') that can be obtained from (S, P) by changing the affiliation of player i . Hence, for a player i to be a dummy player it must be the case that he alone receives zero for any organization of the other players. Also a dummy player has no effect on the payment any particular coalition S can obtain. In games in

partition function form, this also means that if player i is not a member of S , changing the organization of players outside S by moving player i around will not affect the value of S .²

The *addition* of two games (N, v) and (N, v') is defined as the game $(N, v + v')$ where $(v + v')(S, P) \equiv v(S, P) + v'(S, P)$ for all $(S, P) \in ECL$. Similarly, given the game (N, v) and the scalar $\lambda \in \mathbb{R}$, the game $(N, \lambda v)$ is defined by $(\lambda v)(S, P) \equiv \lambda v(S, P)$ for all $(S, P) \in ECL$.

Let σ be a permutation of N . Then the σ *permutation* of the game (N, v) denoted by $(\sigma N, \sigma v)$ is defined by $(\sigma v)(S, P) = v(\sigma S, \sigma P)$.

The three basic axioms a value φ should satisfy are immediately derived from the original Shapley (1953) value axioms and are:

1. **Linearity:** A value φ satisfies the linearity axiom if:
 - 1.1. For any two games (N, v) and (N, v') , $\varphi(N, v + v') = \varphi(N, v) + \varphi(N, v')$.
 - 1.2. For any game (N, v) and any scalar $\lambda \in \mathbb{R}$, $\varphi(N, \lambda v) = \lambda \varphi(N, v)$.³
2. **Symmetry:** A value φ satisfies the symmetry axiom if for any permutation σ of N , $\varphi(\sigma N, \sigma v) = \sigma \varphi(N, v)$.
3. **Dummy player:** A value φ satisfies the dummy player axiom if for any player i which is a dummy player in the game (N, v) , $\varphi_i(N, v) = 0$.

The axiom of linearity means that when a group of agents share the benefits (or the costs) steaming from two different issues, how much each agent obtains does not depend on whether they consider the two issues together or one by one. Hence, the agenda does not affect the final outcome. Also, the sharing does not depend on the unit used

²This definition of a dummy player agrees with the Bolger (1989) definition and it is different than the Myerson (1977) definition. A dummy player in Myerson is not necessarily a dummy player according to our definition. However a dummy player in our setup is a dummy player according to Myerson (1977) as well. See Section 6 for more details.

³Note that in games with no externalities, the axiom of Linearity can be reduced to part 1.1, since it (together with the other axioms) implies the property 1.2. In games with externalities this is not the case. Bolger (1989) uses the same version of the linearity axiom that we propose.

to measure the benefits. Symmetry is a property of anonymity: the payoff of an agent should not depend on his “name”. Finally, the dummy player axiom only makes sure that an agent with absolutely no influence on the gains that any coalition can obtain, should not receive nor pay anything.

Shapley (1953) proved that these three basic axioms characterize a unique value in the class of games with no externalities. Let us denote by (N, \widehat{v}) a game with no externalities, where $\widehat{v} : 2^N \rightarrow R$ is a function that gives the worth of each coalition (independently of the partition structure). The Shapley value ϕ can be written as:

$$\phi_i(N, \widehat{v}) = \sum_{S \subseteq N} \beta_i(S, n) \widehat{v}(S) = \sum_{\substack{S \subseteq N \\ S \ni i}} \beta_i(S, n) MC_i(S) \text{ for all } i \in N, \quad (1)$$

where $MC_i(S)$ is the marginal contribution of player $i \in S$ to the coalition S , $MC_i(S) \equiv \widehat{v}(S) - \widehat{v}(S \setminus \{i\})$, and we have denoted by $\beta_i(S, n)$ the following numbers:

$$\beta_i(S, n) = \begin{cases} \frac{(|S|-1)!(n-|S|)!}{n!} & \text{for all } S \subseteq N, \text{ for all } i \in S \\ -\frac{|S|!(n-|S|-1)!}{n!} & \text{for all } S \subseteq N, \text{ for all } i \in N \setminus S. \end{cases}$$

To see the implications of the three basic axioms on a value for Partition Function Games we start by considering a simple class of games a generic element of which is denoted by $(N, v_{S,P})$, with $(S, P) \in ECL$. The game $(N, v_{S,P})$ is defined by $v_{S,P}(S', P') = 1$ if $(S', P') = (S, P)$ and zero otherwise. It is easy to see that this set of games forms a *basis* for the set of Partition Function Games, since

$$v = \sum_{(S,P) \in ECL} v(S, P) \cdot v_{S,P}.$$

Let us denote by $\varphi_i(S, P) \equiv \varphi_i(N, v_{S,P})$ the outcome assigned by the value φ to player i in the game $(N, v_{S,P})$. The following proposition establish several properties of the value φ implied by the basic axioms.

Properties (a), (b), and (c) state immediate implications from, respectively, linearity, efficiency (implicitly included in the definition of a value), and symmetry. Property (d) highlights the implication of the fact that if a value satisfies the three basic axioms, then it must coincide with the Shapley value for games with no externalities.

Proposition 1 *If the value φ satisfies linearity, symmetry, and dummy player, then:*

- (a) $\varphi_i(N, v) = \sum_{(S,P) \in ECL} \varphi_i(S, P)v(S, P)$ for all $i \in N$.
- (b) $\sum_{i \in N} \varphi_i(S, P) = 0$ for all $(S, P) \in ECL$, $(S, P) \neq (N, (N, \emptyset))$.
- (c) $\varphi_i(S, P) = \varphi_j(S, P)$ for all $i, j \in S$, for all $(S, P) \in ECL$.
- (d) $\sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \varphi_i(S, P) = \beta_i(S, n)$ for all $S \subseteq N$, for all $i \in N$.

Proof. Properties (a), (b), and (c) are immediate. To prove (d), for any $S \subseteq N$ denote by (N, u_S) the game with no externalities defined by $u_S(S, P) = 1$ for any $P \ni S$ and zero otherwise. The Shapley value of player $i \in S$ in u_S is $\phi_i(N, u_S) = \beta_i(S, n)$ and it should coincide with $\varphi_i(N, u_S)$. Since $u_S = \sum_{P \ni S, P \in \mathcal{P}} v_{S,P}$, linearity of φ implies that $\varphi_i(N, u_S) = \sum_{P \ni S, P \in \mathcal{P}} \varphi_i(S, P)$, thus proving the proposition. ■

To derive further implications of the dummy player axiom (stemming from the structural properties of Partition Function Games) we introduce some additional notation. Consider a set of players N and a partition $P \in \mathcal{P}$. We denote $\mathcal{P}(i, P)$ the set of partitions that result from moving player $i \in N$ to any other element of P , including the empty set. For example, for $N = \{1, 2, 3, 4, 5\}$ and $P = (\{1, 2\}, \{3\}, \{4, 5\}, \emptyset)$, then

$$\mathcal{P}(1, (\{1, 2\}, \{3\}, \{4, 5\}, \emptyset)) = \{(\{2\}, \{1, 3\}, \{4, 5\}, \emptyset), (\{2\}, \{3\}, \{1, 4, 5\}, \emptyset), (\{1\}, \{2\}, \{3\}, \{4, 5\}, \emptyset)\}.$$

Proposition 2 *If the value φ satisfies linearity, symmetry, and dummy player, then*

$$\varphi_i(S, P) + \sum_{P' \in \mathcal{P}(i, P)} \varphi_i(S \setminus \{i\}, P') = 0$$

for all $i \in S$, for all $(S, P) \in ECL$ with $|S| > 1$. (2)

Proof. Consider $(S, P) \in ECL$ with $|S| > 1$ and $i \in S$. Define the game $(N, v_{S,P}^i)$ as $v_{S,P}^i(S', P') = 1$ for $(S', P') = (S, P)$ and for all (S', P') in $\mathcal{P}(i, P)$, otherwise $v_{S,P}^i(S', P') = 0$; that is, $v_{S,P}^i = v_{S,P} + \sum_{P' \in \mathcal{P}(i, P)} v_{S \setminus \{i\}, P'}$. The proposition follows immediately from the fact that player $i \in S$ is a dummy player in $v_{S,P}^i$, hence his value in $v_{S,P}^i$ must be zero. ■

While the basic axioms have yielded several restrictions on the behavior of values, they still leave a considerable amount of leeway as regarding the question of how one should distribute $v(N, (N, \emptyset))$ among the players. As it will become clear later, the two values of Myerson (1977) and Bolger (1989) indeed satisfy these basic axioms, as well as many

other possible values one could define. In the next section we describe an alternative (and stronger) symmetry axiom leading to a very natural method of constructing a value for Partition Function Games, namely “taking averages”.

4 The strong symmetry axiom and the average approach

In the same spirit as the axiom of symmetry, we find it natural to introduce a alternative symmetry axiom that further develops for games with externalities the idea that all individuals with “identical power” should receive the same outcome. To explain this axiom consider the basic game $(N, v_{S,P})$, where the only group of agents that produces worth is coalition S , when the rest of agents form precisely the coalition structure P . The symmetry axiom implies that all players in S receive the same outcome. The players outside S may conceivably receive different outcomes, depending on the coalition in P (other than S) they belong to. However, the situation of all the players outside S is, in some sense, similar: it is only when they form the partition P that the coalition S generates value. If any of them changes his position in P , he unilaterally brings about a zero outcome for S . This “equal distribution of power” leads to the following axiom which we call strong symmetry since it reinforces (and *it implies*) the symmetry axiom:

- 2'. Strong Symmetry: A value φ satisfies the strong symmetry axiom if $\varphi_i(S, P) = \varphi_j(S, P)$ for all $i, j \in S$, or $i, j \in N \setminus S$, for all $(S, P) \in ECL$.

The axiom of strong symmetry is intuitive from a normative point of view. Moreover, when we add it to the two basic axioms of linearity and dummy player, it allows us to look at the search for values for games with externalities in a different and very appealing way. We will refer to this way as the “average approach”.

The “*average approach*” associates with each Partition Function Game an “average game” with no externalities and assigns the players in the Partition Function Game their Shapley values in the proposed average game. The average game is constructed as follows: Given a Partition Function Game, (N, v) we define a game with no externalities (N, \tilde{v})

by assigning to each coalition $S \subseteq N$ the average value $\tilde{v}(S) \equiv \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P)v(S, P)$, with $\sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) = 1$. We refer to $\alpha(S, P)$ as the “weight” of partition $P \ni S$ in the computation of the value of coalition S . The idea of this average game is simple. The value of a coalition S does indeed depend of the organization of the other players; but we can think of such a value as some kind of average of the different values of the coalition S for all the possible partition structures it can belong to. Once the average game (N, \tilde{v}) is constructed, we compute the Shapley value of the game in the usual way.

If a value φ is constructed through the average approach, where the weights of the partitions for the different coalitions are given by the function $\alpha(S, P)$ for $(S, P) \in ECL$, then:

$$\varphi_i(N, v) = \sum_{S \subseteq N} \beta_i(S, n) \tilde{v}(S) = \sum_{S \subseteq N} \beta_i(S, n) \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P)v(S, P).$$

We say a value is constructed through the average approach, if it can be derived in the two stage procedure described above of constructing an average game and calculating its Shapley value. The following theorem shows the relationship between the average approach and the strong symmetry axiom:

Theorem 1 *A value φ satisfying linearity and dummy player, can be constructed through the average approach if and only if it satisfies the strong symmetry axiom.*

Proof. Linearity implies that the value φ is uniquely determined by the values it assumes on the games $(N, v_{S,P})$ for $(S, P) \in ECL$. If φ can be constructed through the average approach, then $\varphi_i(N, v_{S,P}) = \alpha(S, P)\beta_i(S, n)$ for all $i \in N$, for all $(S, P) \in ECL$. The expression $\beta_i(S, n)$ is the same for all players $i \in S$, and it is also the same for all players $i \in N \setminus S$. Therefore, the value φ satisfies the strong symmetry axiom.

Now assume φ satisfies the linearity, dummy player, strong symmetry axioms. We first show that the ratio $\varphi_i(N, v_{S,P})/\beta_i(S, n)$ is the same for any $i \in N$, $(S, P) \in ECL$. By strong symmetry, the property holds for $(S, P) = (N, \{N, \emptyset\})$. For $(S, P) \in ECL$, $(S, P) \neq (N, \{N, \emptyset\})$, both $\varphi_i(N, v_{S,P})$ and $\beta_i(S, n)$ are the same for all players in S , and they are also the same for all players in $N \setminus S$, because of the strong symmetry axiom. Moreover, by Proposition 1(b) (that holds because strong symmetry implies symmetry), $\sum_{i \in N} \varphi_i(N, v_{S,P}) = 0$, i.e., $|S| \varphi_i(N, v_{S,P}) + (n - |S|) \varphi_j(N, v_{S,P}) = 0$, for all $i \in S, j \in$

$N \setminus S$. Given that $|S| \beta_i(S, n) + (n - |S|) \beta_j(S, n) = 0$, for all $i \in S, j \in N \setminus S$, it also happens that $\varphi_i(N, v_{S,P}) / \beta_i(S, n) = \varphi_j(N, v_{S,P}) / \beta_j(S, n)$, for all $i \in S, j \in N \setminus S$.

Second, define the weights as follows: $\alpha(S, P) = \varphi_i(N, v_{S,P}) / \beta_i(S, n)$, for any $i \in N$. By Proposition 1(d), $\sum_{P \ni S, P \in \mathcal{P}} \varphi_i(N, v_{S,P}) = \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) \beta_i(S, n) = \beta_i(S, n)$ for all $S \subseteq N$. Hence, $\sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) = 1$, for all $S \subseteq N$, that is, the weights $\alpha(S, P)$ are indeed weights in a proper sense.

Finally, we claim that the value φ can be constructed through the average approach, using the weights $\alpha(S, P)$. Indeed,

$$\begin{aligned} \varphi_i(N, v) &= \sum_{(S,P) \in ECL} \varphi_i(N, v_{S,P}) v(S, P) = \sum_{(S,P) \in ECL} \alpha(S, P) \beta_i(S, n) v(S, P) \\ &= \sum_{S \subseteq N} \beta_i(S, n) \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P) v(S, P). \end{aligned}$$

■

Theorem 1 provides additional intuition and support for the strong symmetry axiom: under the two basic axioms of linearity and dummy player, it is equivalent to the possibility of using the average approach. Moreover, the average approach is a simple way of constructing values satisfying the three axioms. Hence, it is natural to ask for the conditions on the weights that make sure that the value obtained through such a procedure does indeed satisfy all the axioms. The next corollary clarifies this point.

Corollary 1 *A value φ satisfies linearity, strong symmetry, and dummy player if and only if it can be constructed through the average approach with weights satisfying the following condition:*

$$\alpha(S, P) = \sum_{P' \in \mathcal{P}(i, P)} \alpha(S \setminus \{i\}, P') \text{ for all } (S, P) \in ECL \text{ with } |S| > 1. \quad (3)$$

Proof. The necessary condition is immediately implied by Theorem 1 and Proposition 2. For the sufficient part, note that if the value is constructed through the average approach, then linearity and efficiency are immediate. Also, strong symmetry is direct consequence of the fact that $\beta_i(S, n)$ only depends on the players being in or outside the set S . Finally, it is also easy to check that, given the other axioms, the dummy axiom

holds if and only if condition (2) holds, which corresponds to (3) when written in terms of the weights $\alpha(S, P)$. ■

The three requirements of linearity, strong symmetry, and dummy player do not yield a unique value. We provide in the following tables the parametrized family of values that satisfy the three axioms for games with three and four agents. We write in the table the weight of each embedded coalition structure (S, P) as well as the value that each agent in the coalition S obtains in the basis game $(N, v_{S,P})$. Note that, by strong symmetry, each player in $N \setminus S$ pays an equal share of the amount received by the players in S .

(S, P) for $n = 3$	$\alpha(S, P)$	$\varphi_i(S, P)$ for $i \in S$
$(\{i\}, (\{i\}, \{j\}, \{k\}, \emptyset))$	$1 - a$	$\frac{1}{3}(1 - a)$
$(\{i\}, (\{i\}, \{j, k\}, \emptyset))$	a	$\frac{1}{3}a$
$(\{ij\}, (\{i, j\}, \{k\}, \emptyset))$	1	$\frac{1}{6}$
$(\{N\}, (N, \emptyset))$	1	$\frac{1}{3}$

Any real number a will generate a different value satisfying for $n = 3$ all three axioms. However, let us informally discuss some facts. For $a > 1$ player i in the basic game $(\{i\}, (\{i\}, \{j\}, \{k\}, \emptyset))$ would receive a negative payoff (he has to pay the other players). The same would happen for $a < 0$ in the basis game $(\{i\}, (\{i\}, \{j, k\}, \emptyset))$. One may argue that this is not a convincing feature in these games. Indeed, the only coalition that may generate some profits is $\{i\}$, hence it does not seem sensible that player i ends up paying the other players. It may be more sensible to consider values such that $a \in [0, 1]$. In fact, since the issue we are interested in is related to the existence of externalities one may argue that $a = 0$ and $a = 1$ will also not generate interesting values since they are ignoring somehow the power of some coalition structures.⁴

⁴Note that for $n = 3$, substituting symmetry by strong symmetry does not imply any additional restriction on the value.

(S, P) for $n = 4$	$\alpha(S, P)$	$\varphi_i(S, P)$ for $i \in S$
$(\{i\}, (\{i\}, \{j\}, \{k\}, \{l\}, \emptyset))$	$1 - b - 2c$	$\frac{1}{4}(1 - b - 2c)$
$(\{i\}, (\{i\}, \{j\}, \{k, l\}, \emptyset))$	c	$\frac{1}{4}c$
$(\{i\}, (\{i\}, \{j, k, l\}, \emptyset))$	$b - c$	$\frac{1}{4}(b - c)$
$(\{i, j\}, (\{i, j\}, \{k\}, \{l\}, \emptyset))$	$1 - b$	$\frac{1}{12}(1 - b)$
$(\{i, j\}, (\{i, j\}, \{k, l\}, \emptyset))$	b	$\frac{1}{12}b$
$(\{i, j, k\}, (\{i, j, k\}, \{l\}, \emptyset))$	1	$\frac{1}{12}$
$(N, (N, \emptyset))$	1	$\frac{1}{4}$

The parameters b and c can be any real numbers. Again, as discussed before one may include constraints on these parameters that will allocate a positive outcome to the player(s) generating positive profits in the basis games. For $n = 4$ this would imply $b \in (0, 1)$, $c \in (0, \frac{1}{3})$ and $b \in (c, 1 - 2c)$. Still many values remain possible.

In the next section we introduce the final axiom and obtain a unique value.

5 The similar influence axiom and the value

The fourth axiom that we propose addresses the issue that similar environments should lead to similar outcomes for the agents. To understand why the three previous axioms are not sufficient to guaranty this property, take $N = \{1, 2, 3\}$ and consider any value φ of the class defined by the three axioms. Consider the following two games: $(N, v_{S,P})$ and $(N, v_{S,P'})$ where $S = \{1\}$, $P = (\{1\}, \{2, 3\}, \emptyset)$ and, $P' = (\{1\}, \{2\}, \{3\}, \emptyset)$. The two games are very similar. In both only agent 1 alone can produce some benefits. The only difference is that in the first game agents 2 and 3 should be together for the benefits to the player 1 to be realized while in the second game agents 2 and 3 should be separated. The outcomes for the three players in these games are: $\varphi(N, v_{S,P}) = (\frac{(1-a)}{3}, -\frac{(1-a)}{6}, -\frac{(1-a)}{6})$ and $\varphi(N, v_{S,P'}) = (\frac{a}{3}, -\frac{a}{6}, -\frac{a}{6})$. Note that the payoff of agents 2 and 3 (hence, the payoff of agent 1 as well) can differ very much depending on whether they influence the worth of agent 1 by staying together or separated. To introduce the similar influence axiom, which precludes this sort of behavior, we first define the notion of “similar influence”. We say that a pair of players $\{i, j\} \subseteq N, i \neq j$, has *similar influence* in games (N, v) and (N, v')

if $v(T, Q) = v'(T, Q)$ for all $(T, Q) \in ECL \setminus \{(S, P), (S, P')\}$, $v(S, P) = v'(S, P')$, and $v(S, P') = v'(S, P)$, where the only difference between P and P' is that $\{i\}, \{j\} \in P \setminus S$ while $\{i, j\} \in P' \setminus S$.

4. Similar influence: A value φ satisfies the similar influence axiom if for any two games (N, v) and (N, v') and for any pair of players $\{i, j\}$ that has similar influence in those games, we have $\varphi_i(N, v) = \varphi_i(N, v')$ and $\varphi_j(N, v) = \varphi_j(N, v')$.

Note that when applied to the basis games, the similar influence axiom reduces to the requirement that for any $(S, P), (S, P') \in ECL$, where the only difference between P and P' is that a pair of players $j, k \in N \setminus S, i \neq j$, are singletons in P and are a pair in P' (or the other way around), we have $\varphi_i(S, P) = \varphi_i(S, P')$ and $\varphi_j(S, P) = \varphi_j(S, P')$.

To see the restrictions of this axiom for games with small number of players, notice that it implies that for games with three agents, the parameter a that we introduced at the end of last section will take the level $a = 1/2$. Similarly, for games with four agents, the parameters defining the value are $b = 1/2$ and $c = 1/6$.

In the next theorem we show there is a unique value satisfying the four axioms, and provide an explicit and simple formulae to calculate it.

Theorem 2 *There is a unique value φ^* satisfying linearity, strong symmetry, dummy player, and similar influence. For basis games $(N, v_{S,P})$ for $(S, P) \in ECL$ the value is given by:*

$$\varphi_i^*(S, P) = \beta_i(S, n) \frac{\prod_{T \in P \setminus S} (|T| - 1)!}{(n - |S|)!} \text{ for all } i \in N.$$

The value for any game is given by extending the previous expression linearly.

Proof. See the Appendix ■

The easy part of the proof is to show that this value satisfies all the axioms. To show this is the only way to satisfy the requirements is more difficult. This part of the proof proceeds by induction on coalition sizes starting from the grand coalition of size n and ending with singletons.

We now give a first interpretation about the value φ^* . Remember that $\beta_i(S, n)$ is the outcome allocated to player i by the Shapley value in the game (with no externalities)

in which the worth of the coalition S (independently of the partition structure) is 1 and all the other coalitions obtain nothing. $\varphi_i^*(S, P)$ is the outcome of the players in the basic game $(N, v_{S,P})$. It seems reasonable that the outcome $\varphi_i^*(S, P)$ of the players in S should be smaller than $\beta_i(S, n)$, since the coalition S only achieves value when precisely the partition P forms. The ratio that multiplies $\beta_i(S, n)$ in the expression for $\varphi_i^*(S, P)$ measures how to “discount” the outcome for the players, depending on the partition P .

For simplicity, we will refer from now on to the value φ^* identified in Theorem 2 as *the value*. Since it satisfies the three axioms we used in Theorem 1, we can construct the value in an easy and intuitive way through the average approach. Next corollary, whose proof is immediate, identifies the weights that must be used to precisely obtain the value.

Corollary 2 *The value φ^* can be constructed through the average approach by using the following weights:*

$$\alpha^*(S, P) = \frac{\prod_{T \in P \setminus S} (|T| - 1)!}{(n - |S|)!} \text{ for all } (S, P) \in ECL.$$

According to Corollary 2, in the computation of the average value of a coalition $S \subseteq N$, more weight is given to those partitions with large coalitions than to partitions with a large number of small coalitions.

To gain more intuition about the value, and to see how it relates to the original Shapley value for games with no externalities, we now provide another way of writing and computing the value as an average of marginal contributions. In order to introduce the expression, consider any agent $i \in S$, and any $(S, P) \in ECL$. Let us denote by $(S \setminus \{i\}, P_{i,R})$ the element of ECL resulting from moving agent $i \in S$ from S to $R \in P \setminus S$ (remember that $\emptyset \in P$). For example, if $S = \{1, 2\}$, $i = 1$, $R = \{3, 4\}$ and $P = (\{1, 2\}, \{3, 4\}, \{5\}, \emptyset)$, then $(S \setminus \{i\}, P_{i,R}) = (\{2\}, (\{2\}, \{1, 3, 4\}, \{5\}, \emptyset))$. Also, we take the convention that $|\emptyset| = 1$ and we write:

$$MC_i(S, P) \equiv v(S, P) - \sum_{(S \setminus \{i\}, P_{i,R}) \in ECL} \frac{|R|}{(n - |S| + 1)} v(S \setminus \{i\}, P_{i,R}).$$

That is, $MC_i(S, P)$ is a marginal contribution of player $i \in S$ to the coalition S , given the coalition structure P , where the worth of the coalition $S \setminus \{i\}$ is some average of the

worth of this coalition in all the possible coalition structures that can emerge by moving i in P . Then we can write φ^* as follows:

$$\varphi_i^*(N, v) = \sum_{\substack{(S, P) \in ECL \\ S \ni i}} \frac{\prod_{T \in P} (|T| - 1)!}{n!} MC_i(S, P) = \sum_{\substack{S \subset N \\ S \ni i}} \beta_i(S, n) \sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha^*(S, P) MC_i(S, P).$$

The expression (??) is similar to the formula (1) for the Shapley value, once we interpret $\sum_{\substack{P \ni S \\ P \in \mathcal{P}}} \alpha^*(S, P) MC_i(S, P)$ as the (average) marginal contribution of player $i \in S$ to the coalition S .

6 Comparison with previous values and further properties

Two previous solutions for the problem of sharing surplus with externalities were proposed by Myerson (1977) and Bolger (1989). Myerson (1977) adapts the Shapley's value axioms to environments with externalities and derives an extension, that we will denote φ^M , of the Shapley value for this class of environments. The three axioms that uniquely characterize the Myerson's extension are linearity, symmetry, and a carrier axiom requiring that the surplus is shared just among the members of the carrier. The Myerson value of a player is given by:

$$\varphi_i^M(N, v) = \sum_{(S, P) \in ECL} (-1)^{|P|-1} (|P| - 1)! \left[\frac{1}{n} - \sum_{\substack{T \in P \setminus S \\ i \in N \setminus T}} \frac{1}{(|P| - 1)(n - |T|)} \right] v(S, P).$$

The carrier axiom implies both efficiency and a dummy concept much stronger than the one assumed in our analysis. A set S of agents is a *carrier* if: $v(\tilde{S}, P) = v(\tilde{S} \cap S, P \wedge \{S, N \setminus S\})$ for all (\tilde{S}, P) where $P \wedge Q = \{S \cap T \mid S \in P, T \in Q, S \cap T \neq \emptyset\}$. The *carrier axiom* states that if S is a carrier in the game (N, v) , the sum of values assigned to the members of S equals $v(N, (N, \emptyset))$. We can say that a player $i \in N$ is a dummy player, in the Myerson sense, in the game (N, v) , if there exists a carrier set S with $i \notin S$. Given that all the dummy players are symmetric, they all get zero according to the Myerson value.

A problematic aspect of carrier axiom is that in many cases a dummy player (in the Myerson's sense) might, through changes in his behavior, affect the outcome reached. Take for example the game with three agents $(\{1, 2, 3\}, v)$, where the characteristic function is given by the expressions: $v(\{1\}, (\{1\}, \{2, 3\}, \emptyset)) = v(\{1, 2, 3\}, \{1, 2, 3\}, \emptyset) = 1$ and $v(S, P) = 0$ otherwise. In this game, agent 1 is a carrier and hence agents 2 and 3 are dummy players. Therefore, $\varphi_1^M(N, v) = 1$ and $\varphi_2^M(N, v) = \varphi_3^M(N, v) = 0$. However, agent 2 can affect the outcome since if he does not join agent 3, agent 1 will get zero rather than one. Thus we feel agent 2 is not “really” a dummy player.

We also note that, due to the carrier axiom, the Myerson value yields very different outcomes to games that are quite similar. Consider the game $(\{1, 2, 3\}, w)$, where the worth of the coalitions is given by $w(\{1\}, (\{1\}, \{2\}, \{3\}, \emptyset)) = w(\{1, 2, 3\}, \{1, 2, 3\}, \emptyset) = 1$ and $w(S, P) = 0$ otherwise. This game is similar to the game (N, v) proposed previously in the sense that the “worth” of the agents is similar. However, players 2 and 3 are not dummy now and the Myerson value for these agents is surprising since it does not allocate any worth to player 1 : $\varphi_1^M(N, w) = 0$ and $\varphi_2^M(N, w) = \varphi_3^M(N, w) = 1/2$.

Bolger (1989) obtains a unique value φ^B characterized by our properties of linearity, symmetry, and dummy player, and an additional requirement based on the behavior of the value in “voting” games. To be more specific, Bolger (1989) considers simple games satisfying that the worth of any coalition is either one or zero. He says that the coalition S wins with respect to (S, P) if $v(S, P) = 1$. Now, consider an embedded coalition $(S \setminus i, P_{i,R})$ obtained from (S, P) by moving agent $i \in S$ from S to $R \in P \setminus S$. In Bolger's terminology this is called a move for agent i . Such a move is called a pivot move if S wins with respect to (S, P) and $S \setminus i$ loses with respect to $(S \setminus i, P_{i,R})$. The additional property Bolger (1989) introduces states that for simple games, a player i has the same value in two games (N, v) and (N, v') if he has the same number of pivot moves in both games. There is no closed form expression for φ^B .

The values φ^M and φ^B while satisfying our basic properties, cannot be constructed through the average approach, as shown in the next proposition.

Proposition 3 *The values φ^M and φ^B fail to satisfy the strong symmetry axiom.*

Proof. Consider the basis game $v_{S,P}$ where $S = \{1\}$ and $P = (\{1\}, \{2, 3\}, \{4\}, \emptyset)$.

The Myerson and Bolger values for this game are given by $\varphi_1^M = -1/3$, $\varphi_2^M = \varphi_3^M = 1/6$, and $\varphi_4^M = 0$; whereas $\varphi_1^B = 7/144$, $\varphi_2^B = \varphi_3^B = -1/72$, and $\varphi_4^B = -1/48$. Hence both values violate the strong symmetry axiom and thus cannot be constructed through the average approach. ■

Straightforward calculations show that Bolger and Myerson do not satisfy the similar influence property as well.

Finally, we would like to comment on the behavior of the various values with respect to an alternative axiom concerning the dummy player. Our dummy (as well as Bolger's) axiom imposes that a dummy agent should obtain zero, but it does not require that this agent does not influence the outcome obtained by the other agents. In games with no externalities, the basic axioms do indeed imply this additional property. However, this is not necessarily so in games with externalities. We call this property the "strong dummy" property:

- 3'. Strong Dummy Player: A value φ satisfies the strong dummy player axiom if for any dummy player α in the game (N, v) , $\varphi_i(N, v) = \varphi_i(N \setminus \{\alpha\}, v)$ for all i in $N \setminus \{\alpha\}$.

We note that, given that a value is efficient, the strong dummy player axiom implies the dummy player axiom. The next proposition shows that even though the strong dummy property was not imposed as a requirement, our value does satisfy it.

Proposition 4 *The value φ^* satisfies the strong dummy player property.*

Proof. See the Appendix. ■

Bolger's value (as pointed out in Bolger, 1989) violates the strong dummy player property. On the other hand, Myerson (1977) satisfies the requirement; his dummy player property (implied by his carrier axiom) is still much more demanding than the strong dummy player property. Hence, the strong dummy player axiom is not sufficient to characterize, together with symmetry and linearity, a unique value. In fact, the class of values satisfying these three axioms is still large. Even if we substitute symmetry by strong symmetry, or if we add the similar influence axiom, a value is still not singled out.

7 The value allocation for a world climate change program

Eyckmans and Tulkens (2003) analyze global climate negotiations from a game-cooperative point of view. They consider six regions: USA (player 1), Japan (player 2), European Union (player 3), China (player 4), Former Soviet Union (player 5) and the Rest of the World (player 6). They determine the agreements reached and the payoffs accruing to the various regions under alternative modes of cooperation and analyze the stability properties of alternate transfer systems.

We approach the problem from a normative point of view and calculate the payoff configuration generated by our value for this “climate change game”. The game is given by the payoffs received by each coalition for any of the 203 possible partitions of the six player set.⁵ As is obvious from the data this game is characterized by wide-spread externalities. We report a few payoff configurations (the rest are available upon request).

When the partition consists of six singletons the payoffs to the six players respectively are: 78,352, 42,909, 102,729, 9,140, 23,793, and 81,134 yielding a total world welfare of 338,057. If all six regions form one coalition they receive 339,831, which is the largest payoff the world can receive. When the US, Japan and the EU form one coalition whereas China, the FSU and ROW stay as singletons the payoffs are given by 224,005 to the three member coalition and 9,142, 23,798 and 81,154 to players 4, 5 and 6 respectively. If players 4, 5 and 6 form a coalition as well then the payoff to the coalition $\{1, 2, 3\}$ is 224,913 and it is 114,394 to the coalition $\{4, 5, 6\}$, which implies that in this case there are positive externalities.

Our value takes into account these and all other externalities and yields the following payoff configuration to the six players respectively:

78,692, 43,070 103,203, 9,293, 23,933 and 81,639.⁶

Note that the value prediction turns out to be individually rational. Our value recommendation can be attained by having the players carry out efficient climate change policy, and a transfer formula assuring that each region receives its value. This sugges-

⁵We are grateful to Johan Eyckmans for providing the data set.

⁶We thank Nina Kirma for computational assistance.

tion naturally ignores the possibility of manipulation and strategic behavior on part of the regions. It also takes for granted the complete information assumption. We think of it as a desirable outcome, that an international court or body of nations can justify as an arbitration offer to settle international disputes.

8 Conclusion

We set out to provide a normative solution concept for environments with externalities. The construction proceeded in stages. We first took the natural extensions of the Shapley axioms to our environment and studied their implications. They generated a large family of possible values. We then strengthened the symmetry axiom and showed it is equivalent to an average approach for resolving the value problem.

The average approach amounts to calculating a value for a game with externalities by associating with it a game with no externalities, where each coalition is assigned a payoff which is an average of its payoff over all possible partitions containing it. The Shapley value of the average game is then taken to be the value of the original game. There are several restrictions on the weighting method, but still many values remain as possible solutions. We provided parametric families of such solutions for the three and four agents case.

The final axiom we added regarded the behavior of the value in very similar games. This was called the similar influence axiom, since the only difference between the games was the pairing of two singletons in one partition into a pair in the other game. We wanted the value assigned to each of the two concerned agents to be the same in both games. We showed there is a unique value that satisfies all these axioms. This value, given by a simple formula, can be easily calculated and generates a payoff vector for any environment with externalities.

There have been two previous attempts to provide a normative solution to the sharing problem in environments with externalities, by Myerson (1977) and Bolger (1986). The main advantages of our value are the plausibility of its payoff recommendations and its ease of calculation. It uses a reasonable dummy property, and furthermore satisfies the strong dummy property whereby the removal of a dummy player has no effect on the

values of the other players.

Our value can be used to resolve distributional problems in very general settings. It can determine a bench mark result arbitrators might consider as a good compromise. We apply it to a climate change game based on actual data and generate a payoff configuration that can serve as a basis for a system of transfers across regions.

There are several open questions regarding the axioms characterizing the value. It is not clear which, if any, of the basic axioms can be relaxed by the introduction of the strong dummy axiom. It is also of interest to study whether or not there exist axioms different than the similar influence axiom, which lead to a unique value. In actual applications it might also be that certain suggestions generated by just a subset of the axioms form an appropriate solution.

The approach in this paper has been normative and it would be interesting to construct game forms that implement the value constructed. Pérez-Castrillo and Wettstein (2001) provided a deterministic mechanism that implements the Shapley value in pure strategy Subgame Perfect Equilibrium.

The analysis throughout the paper proceeded under the assumption of transferable utility. The transition to environments without side payments remains an interesting topic of further research.

9 Appendix

Proof. of Theorem 2. We start by showing that the value φ^* satisfies the four axioms. It obviously satisfies linearity, strong symmetry, and similar influence. To show dummy note that if $i \in N$ is a dummy player in a game (N, v) , then

$$\begin{aligned}
\varphi_i^*(N, v) &= \sum_{(S,P) \in ECL} \varphi_i^*(S, P)v(S, P) \\
&= \sum_{\substack{(S,P) \in ECL \\ S \ni i}} \left[\varphi_i^*(S, P)v(S, P) + \sum_{P' \in \mathcal{P}(i,P)} \varphi_i^*(S \setminus \{i\}, P')v(S \setminus \{i\}, P') \right] \\
&= \sum_{\substack{(S,P) \in ECL \\ S \ni i}} \left[\varphi_i^*(S, P) + \sum_{P' \in \mathcal{P}(i,P)} \varphi_i^*(S \setminus \{i\}, P') \right] v(S, P).
\end{aligned}$$

Substituting in the expressions for φ_i^* , we get $\varphi_i^*(S, P) + \sum_{P' \in \mathcal{P}(i, P)} \varphi_i^*(S \setminus \{i\}, P') = 0$ and hence $\varphi_i^*(N, v) = 0$ whenever player i is a dummy player in (N, v) .

We now prove that if a value φ satisfies the four axioms, then $\varphi = \varphi^*$. Since φ can be constructed through the average approach, let us denote $\alpha(S, P)$ the weights associated to φ . Proving that $\varphi = \varphi^*$ is equivalent to proving that the weights $\alpha(S, P)$ are the same as the weights $\alpha^*(S, P)$ associated to φ^* , that is,

$$\alpha(S, P) = \alpha^*(S, P) = \frac{\varphi_i^*(S, P)}{\beta_i(S, n)} = \frac{\prod_{T \in \mathcal{P} \setminus S} (|T| - 1)!}{(n - |S|)!} \text{ for all } (S, P) \in ECL. \quad (4)$$

By strong symmetry, $\alpha(S, P)$ only depends on the sizes of the coalitions in P . Hence, denoting $s = |S|$, we can write $\alpha(S, P) = \rho(s; t)$, where $t = (t_1, \dots, t_h)$ with $\sum_{k=1}^h t_k = n - s$, is the vector of sizes of the coalitions in P different from S . We prove (4) if we show that:

$$\rho(s; t) = \frac{\prod_{k=1}^h (t_k - 1)!}{(n - s)!} \text{ for all } s \leq n, \text{ for all } t = (t_1, \dots, t_h), \text{ with } \sum_{k=1}^h t_k = n - s. \quad (5)$$

We prove that the expression (5) holds through an induction argument on the size of the coalition S , going from $s = n$ to $s = 1$.

($s = n$) If $s = n$, equation (5) holds since the only embedded coalition structure for $S = N$ is $(N, (N, \emptyset))$ and $\alpha(S, P) = \rho(n; 0) = 1$.

(s) We make the induction argument that (5) holds for all $(s'; t')$ with $s' > s$. We then prove that it also holds for every $(s; t)$. Denote

$$\mathcal{T}' = \left\{ t' = (t'_1, \dots, t'_h) \mid t'_1 \geq \dots \geq t'_h \text{ and } \sum_{k=1}^h t'_k = n - s - 1 \right\}.$$

By the induction argument, we know that $\rho(s+1; t')$ satisfies (5) for all $t' \in \mathcal{T}'$. We endow the set \mathcal{T}' with the following complete lexicographic order: $t' = (t'_1, \dots, t'_h) \succeq \hat{t}' = (\hat{t}'_1, \dots, \hat{t}'_h)$ if $t'_h > \hat{t}'_h$, or $t'_h = \hat{t}'_h$ and $t'_{h-1} > \hat{t}'_{h-1}$, and so on, or $t' = \hat{t}'$. Also, we say $t' \succ \hat{t}'$ if $t' \succeq \hat{t}'$ and $t' \neq \hat{t}'$. We rename the vectors in \mathcal{T}' as t^1, t^2 , etc., so that $t^1 \prec t^2 \prec \dots$

For all $t' \in \mathcal{T}'$, we denote by $\mathcal{T}(t')$ the set of vectors t of the form $t = (t'_{-k}, t'_k + 1)$ or $t = (t', 1)$. Notice that the sum of the components of the vectors in $\mathcal{T}(t')$ is $n - s$. Moreover, all vector t with $\sum_{k=1}^h t_k = n - s$ belongs to $\mathcal{T}(t')$ for some $t' \in \mathcal{T}'$. Hence, we

prove property (5) for s if and only if we prove the following property:

$$\rho(s; t) = \frac{\prod_{k=1}^h (t_k - 1)!}{(n - s)!} \text{ for all } t = (t_1, \dots, t_h) \in \mathcal{T}(t^m), \text{ for all } t^m \in \mathcal{T}'. \quad (6)$$

We prove (6) through a second induction argument. This time we do the induction on the number of elements of \mathcal{T}' . We will use Corollary 1 that we rewrite as

$$\rho(s + 1; t') - \sum_{k=1}^h \rho(s; t'_{-k}, t'_k + 1) - \rho(s; t', 1) = 0 \text{ for all } t' \in \mathcal{T}'. \quad (7)$$

($s, m = 1$) It is clear that $t^1 = (1, \dots, 1)$, that is, t^1 is a vector with $n - s - 1$ unitary coordinates. The set $\mathcal{T}(t^1)$ is composed of $(n - s - 1)$ vectors of the form $(2, 1, \dots, 1)$ and a vector $(1, \dots, 1)$. By the strong influence axiom, $\rho(s; 2, 1, \dots, 1) = \rho(s; 1, \dots, 1)$. Hence, (7) implies that $\rho(s + 1; t^1) = (n - s)\rho(s; t^1)$, i.e.,

$$\rho(s; 1, \dots, 1) = \rho(s; 2, 1, \dots, 1) = \frac{1}{(n - s)} \rho(s + 1; t^1) = \frac{1}{(n - s)} \frac{\prod_{k=1}^h (t_k^1 - 1)!}{(n - s - 1)!} = \frac{1}{(n - s)!},$$

which corresponds to (6) for the vectors $(1, \dots, 1)$ and $(2, 1, \dots, 1)$, which are the two types of vector in $\mathcal{T}(t^1)$. Hence, the induction property holds for $m = 1$.

(s, m) We now suppose that (6) holds for up to $t^{(m-1)}$, and prove that it also holds for t^m . By the strong influence axiom, $\rho(s; t) = \rho(s; \hat{t})$ if \hat{t} is the vector equal to t except that each component $t_k = 2$ has been substituted in \hat{t} by two components with value 1. Let us write all the vectors in $\mathcal{T}(t^m)$ as vectors where each component with value 2 is substituted with two components with value 1, and let us denote the set of such vectors as $\mathcal{T}_1(t^m)$.

By using the induction hypothesis, we prove that $\rho(s; t)$ is known for all $t \in \mathcal{T}_1(t^m)$, except possibly the highest vector in $\mathcal{T}_1(t^m)$ (when we apply the lexicographic order \succeq defined above). To do the proof, let us write $t^m = (t^{m2}, t^{m1})$, where $t^{m1} = (1, \dots, 1)$ and $t_k^{m2} \geq 2$ for all $k = 1, \dots, r$, where $t^{m2} = (t_1^{m2}, \dots, t_r^{m2})$ (t^{m1} may not exist for $m > 1$). It is clear that the highest vector in $\mathcal{T}_1(t^m)$ is $(t_{-r}^{m2}, t_r^{m2} + 1, t^{m1})$. Consider now any other vector in $\mathcal{T}_1(t^m)$. We distinguish two possibilities:

1. If $t = (t_{-z}^{m2}, t_z^{m2} + 1, t^{m1})$ for some $z < r$, then let $t^p \equiv (t_{-\{z, r\}}^{m2}, t_z^{m2} + 1, t_r^{m1} - 1)$. Note that $t^p \prec t^m$, hence $p < m$, and also note that $t = (t_{-r}^p, t_r^p + 1)$. Hence, by induction hypothesis we already know $\rho(s; t)$.

2. If $t = (t^{m2}, t_{-z}^{m1}, t_z^{m1} + 1)$, or $t = (t^m, 1)$, then $\rho(s; t) = \rho(s; t^m, 1)$. Let $t^p \equiv (t_{-r}^m, t_r^m - 1, 1)$. Again, since $(t^m, 1) = (t_{-r}^p, t_r^p + 1)$, $\rho(s; t)$ is already known.

Hence, we have proven that $\rho(s; t)$ is known for all $t \in \mathcal{T}_1(t^m)$, except possibly the highest vector in $\mathcal{T}_1(t^m)$. Therefore, in the formulae (7) (for $t' = t^m$) we know all the values except possibly the one corresponding to the highest vector in $\mathcal{T}_1(t^m)$. Moreover, all the other values are different from zero, hence the solution for the remaining value is unique. Since $\rho(s; t) = \prod_{k=1}^h (t_k - 1)! / (n - |S|)!$ is an expression that does satisfy the equation, it is the unique solution.

This concludes the proof of Theorem 2. ■

Proof. of Proposition 4. We first introduce the following notation: Consider two sets N and M , with $N \subset M$, and $P \in \mathcal{P}(N)$ and $Q \in \mathcal{P}(M)$. We say $Q \sqsupset P$, or $P \sqsubset Q$ if the partition Q is equal to P when we take out of Q the players in $M \setminus N$. To show the strong dummy property, let $\alpha \in N$ be a dummy player in the game (N, v) . For every player $i \in N$ we have

$$\begin{aligned} \varphi_i(N, v) &= \sum_{(S', P') \in ECL(N)} \varphi_i(S', P') v(S', P') \\ &= \sum_{(S, P) \in ECL(N \setminus \{\alpha\})} \sum_{\substack{S' \supseteq S \\ P' \sqsupset P \\ (S', P') \in ECL(N)}} \varphi_i(S', P') v(S', P') \\ &= \sum_{(S, P) \in ECL(N \setminus \{\alpha\})} v(S, P) \sum_{\substack{S' \supseteq S \\ P' \sqsupset P \\ (S', P') \in ECL(N)}} \varphi_i(S', P'). \end{aligned}$$

Substituting in the expressions for φ_i , we get $\sum_{\substack{S' \supseteq S, P' \sqsupset P \\ (S', P') \in ECL(N)}} \varphi_i(S', P') = \varphi_i(S, P)$ and hence $\varphi_i(N, v) = \sum_{(S, P) \in ECL(N \setminus \{\alpha\})} v(S, P) \varphi_i(S, P) = \varphi_i(N \setminus \{\alpha\}, v)$ for every $i \in N$. ■

References

- [1] Aumann, R. and J. H. Dreze (1974), “Cooperative games with coalition structures,” *International Journal of Game Theory* 3, 217-237.
- [2] Bloch, F. (1986), “Sequential formation of coalitions in games with externalities and fixed payoff division,” *Games and Economic Behavior* 14, 90-123.

- [3] Bloch, F. (2002), "Coalitions and networks in industrial organization," *Manchester School* 70, 36-55.
- [4] Bolger, E.M. (1989), "A set of axioms for a value for partition function games," *International Journal of Game Theory* 18, 37-44.
- [5] Eyckmans, J. and H. Tulkens (2003), "Simulating coalitionally stable burden sharing agreements for the climate change problem," forthcoming in *Resource and Energy Economics*.
- [6] Hart, S. and M. Kurz (1983), "Endogenous formation of coalitions," *Econometrica* 1047-1064.
- [7] Maskin, E. (2003), "Bargaining, coalitions, and externalities," presidential address of the Econometric Society.
- [8] Myerson, R.B. (1977), "Values of games in partition function form," *International Journal of Game Theory* 6, 23-31.
- [9] Pérez-Castrillo, D. and D. Wettstein (2001): "Bidding for the Surplus: A Non-Cooperative Approach to the Shapley Value," *Journal of Economic Theory*, 100, 274-294.
- [10] Ray, D. and R. Vohra (1999), "A theory of endogenous coalition structures," *Games and Economic Behavior* 26, 286-336.
- [11] Ray, D. and R. Vohra (2001), "Coalitional power and public goods," *Journal of Political Economy* 109, 1355-1384.
- [12] Shapley, L.S. (1953), "A value for n -person games," in *Contributions to the theory of games II* (H.W. Kuhn and A.W. Tucker eds.), 307-317.