Risk-Dominant Networks

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Abstract

In noncooperative game theory, it has been proved that the risk-dominant equilibrium is selected by stochastic evolutionary dynamics. Our paper aims to define an equivalent notion but in a framework of network formation. We first define the notion of risk-dominant network and study its properties. In particular, it is shown that when a risk-dominant network exists it is unique and it is selected by a stochastic evolutionary dynamics. In addition, we show that a risk-dominant network is a pairwise stable network and thus it refines this latter notion.

We then propose a set-valued extension: the risk-dominant set of networks, that aims to avoid the problem of lack of existence of the risk-dominant network. We show that this set exists. We also provide an evolutionary foundation to the risk-dominant set of networks.

Interestingly, we show that the risk-dominant set of networks can contain no pairwise stable network. We see this as a drawback of pairwise stability, and we provide examples where a set of networks is more "stable" than a pairwise stable network.

Keywords: Risk-dominance, Network formation, Pairwise stability, Stochastic stability.

JEL Classification: C70, D20.

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1 Introduction

The organization of individual agents into networks and groups or coalitions has an important role in the determination of the outcome of many social and economic interactions. For instance, networks of personal contacts are important in obtaining information about job opportunities. Goods can be traded and exchanged through networks, rather than markets, of buyers and sellers. The partitioning of societies into groups is also important in many contexts, such as the provision of public goods and formation of alliances, cartels and federations. Finally, amongst the most important ongoing processes in modern economies is the unregulated development of the Internet. In networks, the relationships between individuals are represented by links. The principal distinction between coalitions and networks concerns the nature of collaboration structures: coalition formation requires that every player belong to one group only, while network formation allows for cooperative relationships that are intransitive. The understanding of how and why networks and groups form and the precise way in which network structures affect outcomes of social and economic interaction is the main focus of the recent literature on the formation of networks. Jackson (2004) provides a survey of models of network formation.

There are many possible approaches to model network formation. One is simply to model it explicitly as a non-cooperative game (see e.g. Aumann and Myerson, 1988). A different approach is to analyze the networks that one might expect to emerge in the long run and to examine a sort of stability requirement that individuals not benefit from altering the structure of the network. This is the approach that was taken by Jackson and Wolinsky (1996) when defining pairwise stable networks. A pairwise stable network can be seen as an extension of the (strict) Nash equilibrium concept to networks. A network is pairwise stable if given this network, no individual can increase his payoff by breaking unilaterally a link and if a link is not created this is justified by the fact that at least one player does not have an interest to create this link. But this notion suffers from some drawbacks that can be related to the ones of the Nash equilibrium. One of the most important is that the process that leads this network to emerge is not explicit.

One approach in noncooperative game theory to explicitly formalize the process by which agents eventually play a (strict) Nash equilibrium has been proposed through evolutionary models. One could think to Young (1993), Ellison (1993, 2000), Kandori, Malaith, Rob (1993) among others. For instance, Young's process works as follows. At each date, different agents are matched to play a finite game. Given their beliefs on what their opponent will play, they choose a best reply. Agents are boundedly rational since, in order to form their beliefs on what the other players will play, they look at what was played in a recent past and think (myopically) that the other players will play the same on average.

This process is called a best reply dynamics. This defines a (Markovian) process with - roughly speaking - absorbing states corresponding to the strict Nash equilibria of the game that is played. But even when such an equilibrium exists, the process might not converge to it. It can be characterized by cycles.

In addition to this best-reply dynamics, Young supposes that agents can do mistakes or experiments and so, with a small probability can choose an action that is not a best reply. In the long run, and as the probability of mistakes tends to zero, the action profiles that have a significant frequency are called stochastically stable. Again, it can be shown that it might be that the process never converges to a strict Nash equilibrium (see for instance, Hurkens (1995), or Young (1998)). However, many papers characterize the equilibria that might be selected by this general process. For instance, Young (1993) shows that in a two player, two action game, only the risk-dominant equilibrium (in the sense of Harsanyi and Selten (1988)) is stochastically stable. This result was generalized by Maruta (1997) to two players finite games. Then, when this equilibrium exists, it seems to be very robust and uniquely selected. While the extension of the idea of the (strict) Nash equilibrium to networks has been provided by Jackson and Wolinsky (1996), we do not know any attempt that aim to define what is a risk-dominant network and thus the picture remains incomplete. This paper aims to fill the gap.

The evolutionary approach has been followed by Jackson and Watts (2002) in order to explicitly formalize the way networks are formed. Their process is closely related to the one of Young (1993) and can be described as follows. At each date, a pair of players is randomly drawn. The (potential) link between these two players is the only link that can be altered at that time. If the link is already in the network, then the decision is whether to sever it, and otherwise the decision is whether to add the link. The players involved act myopically, adding the link if it makes each at least as well off and one strictly better off, and severing the link if its deletion makes either player better off. After the action is taken, there is some small probability that a mutation (or tremble, or mistake) occurs and the link is deleted if it is present, and added if it is absent. In the long-run and as the probability that a mutation occurs goes to zero, a network that has a significant frequency is said to be stochastically stable. Therefore, we would like a risk-dominant network to be selected by the process.

Let us present our definition of a risk-dominant network. Such a network is a refinement of the notion of pairwise stability. A pairwise stable network is such that if we add a link to this network (or sever a link) between two players then one of these players will destroy this link (both players will re-create this link) and we will come back to the initial network. The concept of risk-dominant network strengthens that notion. A network is said to be risk-dominant if when we add a set of links to this network (or sever a set of links) such that the number of changes is less than the half of the total possible changes, then if we allow players to successively create or delete links, they will come back to the initial network.

We show that when a risk-dominant network exists, it is unique. Moreover it is the only stochastically stable network in Jackson and Watts (2002) stochastic evolutionary process. But while our notion of a risk-dominant network inherits of the "good" properties of the risk-dominant equilibrium, it also inherits of its "bad" property: it does not always exist. In the noncooperative field, Tercieux (2003) has proposed a solution to this lack of existence with a set-valued concept. We follow this approach in defining a risk-dominant set of networks that is proved to exist and to coincide with the risk-dominant network when it exists. We also show that if a network is stochastically stable then it belongs to the risk-dominant set of networks. Interestingly, the risk-dominant set of networks can contain no pairwise stable network. We see this as a drawback of pairwise stability, and we provide examples where a set of networks is more "stable" than a pairwise stable network.

Another drawback of pairwise stability is that it considers only deviations by at most a pair of individuals at a time. It might be that some group of individuals could all be made better off by some more complicated reorganization of their links, which is not accounted for under pairwise stability. In response, Jackson and van den Nouweland (2004) have introduced the notion of strongly stable networks. A strongly stable network is a network which is stable against changes in links by any coalition of individuals. Strongly stable networks are Pareto efficient and maximize the overall value of the network if the value of each component of a network is allocated equally among the members of that component. Here, we propose a refinement of pairwise stability assuming that the behavior of individuals is driven by risk-dominance considerations.

There may be tensions between stability of networks and their efficiency in the sense of overall payoff maximization. Jackson (2003) has pointed out two reasons why individual incentives might not lead to overall efficiency. The first one is the presence of externalities across coalitions. The second one is that individuals could try to position themselves well in the network to affect their relative power and resulting allocation. By means of examples, we will show that risk-dominance may be another reason for inefficiencies in network formation.

The paper is organized as follows. In Section 2 we define the notion of risk-dominant network and we study its properties. In Section 3 we propose a set-valued extension, the risk-dominant set of networks. In Section 4 we provide an evolutionary foundation to the risk-dominant set of networks. In Section 5 we conclude.

2 Risk-Dominant Networks

Let $N = \{1, ..., n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph.[1] Individuals are the nodes in the graph and links indicate bilateral relationships between individuals. Thus, a network g is simply a list of which pairs of individuals are linked to each other. If we are considering a pair of individuals i and j, then $\{i, j\} \in g$ indicates that i and j are linked under the network g. For simplicity, we write ij to represent the link $\{i, j\}$, and so $ij \in g$ indicates that i and j are linked under the network g. Let g^N be the set of all subsets of N of size 2. $G^N = \{g \subseteq g^N\}$ denotes the set of all possible networks or graphs on N, with g^N being the complete network.[2] The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network obtained by deleting link ij from an existing network g is denoted $g - ij$. For any network g, let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g.

Different network configurations lead to different values of overall production or overall utility to players. These various possible valuations are represented via a value function. A *value function* is a function $v : G^N \to \mathbb{R}$. The set of all possible value functions is denoted $\mathcal{V}$. A value function only keeps track of how the total societal value varies across different networks. We also wish to keep track of how that value is allocated or distributed among the players forming a network. An allocation rule is a function $Y : G^N \times \mathcal{V} \to \mathbb{R}^N$ such that $\sum_i Y_i(g, v) = v(g)$ for all v and g. It is important to note that an allocation rule depends on both g and v. This allows an allocation rule to take full account of a player i's role in the network. This includes not only what the network configuration is, but also and how the value generated depends on the overall network structure.

In evaluating societal welfare, we may take various perspectives. A network g is *Pareto efficient* relative to v and Y if there does not exist any $g' \in G^N$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with strict inequality for some i. This definition of efficiency of a network takes Y as fixed, and hence can be thought of as applying to situations where no intervention is possible. A network $g \subseteq g^N$ is *strongly efficient* relative to v if $v(g) \geq v(g')$ for all $g' \in G^N$. This is a strong notion of efficiency as it takes the perspective that value is fully transferable.

A simple way to analyze the networks that one might expect to emerge in the long

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[1] Bala and Goyal (2000) have studied network formation in directed networks. See also Dutta and Jackson (2000).

[2] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. We also use the symbols $\vee$ and $\wedge$ which mean "or" and "and", respectively.

run is to examine a sort of equilibrium requirement that agents not benefit from altering the structure of the network. A weak version of such condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no player benefits from severing one of their links and no other two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly.

Definition 1 *A network g is pairwise stable with respect to v and Y if*

(i) for all $ij \in g$, $Y_i(g,v) \geq Y_i(g-ij,v)$ and $Y_j(g,v) \geq Y_j(g-ij,v)$, and

(ii) for all $ij \notin g$, if $Y_i(g,v) < Y_i(g+ij,v)$ then $Y_j(g,v) > Y_j(g+ij,v)$.

Let us say that g' is adjacent to g if $g' = g+ij$ or $g' = g-ij$ for some ij. A network g' defeats g if either $g' = g-ij$ and $Y_i(g',v) \geq Y_i(g,v)$, or if $g' = g+ij$ with $Y_i(g',v) \geq Y_i(g,v)$ and $Y_j(g',v) \geq Y_j(g,v)$ with at least one inequality holding strictly. Pairwise stability is equivalent to saying that a network is pairwise stable if it is not defeated by another (necessarily adjacent) network.

Example 1. Consider a situation where four players can form links. The payoffs they obtained from the different network configurations are (see Figure 1): $Y_i(g) = \#(g)$ if $i \in N(g)$ with $\#(g)$ being the number of links in g, $Y_i(g) = 0$ if $i \notin N(g)$, and $Y_i(g) = 10$ if g is the empty network. Both the empty network and the complete network are pairwise stable networks. The empty network is also the efficient network.

Jackson and van den Nouweland (2004) have introduced the notion of strongly stable networks. A strongly stable network is a network which is stable against changes in links by any coalition of individuals. Strongly stable networks are Pareto efficient and maximize the overall value of the network if the value of each component of a network is allocated equally among the members of that component. In this example, the empty network is the unique strongly stable network. But, what happens if we consider a refinement of pairwise stability driven by risk dominance considerations?

We aim to refine the notion of pairwise stability. We define a notion of distance between two networks. For $g, g' \subseteq g^N$ we denote by

$$d(g,g') \equiv \frac{\#\{ij \in g^N \mid (ij \in g \wedge ij \notin g') \vee (ij \notin g \wedge ij \in g')\}}{\#g^N}$$

the *distance* between g and g'. Let us first restate the definition of Jackson and Watts (2002) of an improving path. An improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the resulting network offers relative to the current network. Each network in the sequence differs by one link from

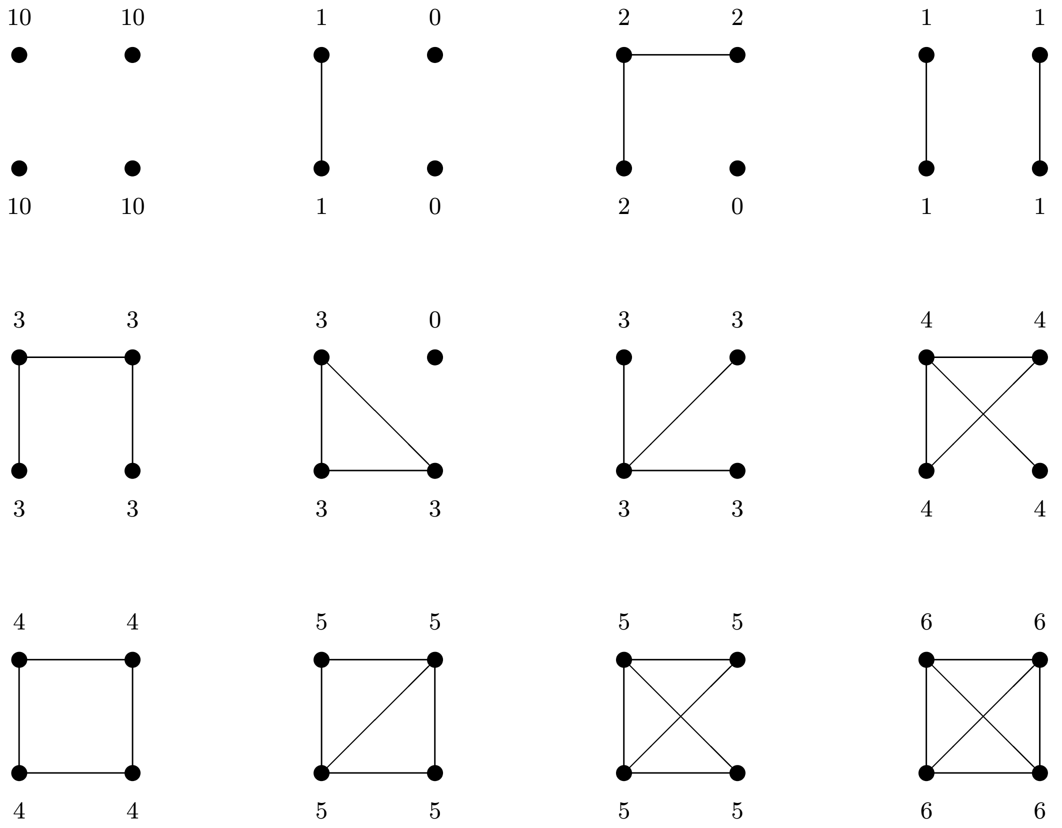

Figure 1: The empty and complete networks are pairwise stable (Example 1).

the previous one. If a link is added, then the two players involved must both agree to its addition, with at least one of the two strictly benefiting from the addition of the link. If a link is deleted, then it must be that at least one of the two players involved in the link strictly benefits from its deletion. Formally, an $K-$improving path from a network g to a network g' is a finite sequence of graphs $g_1, ..., g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, ..., K-1\}$ either:

(i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_k - ij) > Y_i(g_k)$, or

(ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_k + ij) > Y_i(g_k)$ and $Y_j(g_k + ij) \geq Y_j(g_k)$.

If there exists an improving path from g' to g, then as Jackson and Watts (2002) we use the symbol $g' \rightarrow g$. For a given network, g, let $im(g) = \{g' \subseteq g^N \mid g' \rightarrow g\}$. This is the set of networks for which there is an improving path leading from g' to g. If there exists $\overline{K}$ such that for all $K \geq \overline{K}$, any $K-$improving path from g' go to g, then we use the symbol $g' \longmapsto g$. For a given network g, let $IM(g) = \{g' \subseteq g^N \mid g' \longmapsto g\}$. In the sequel, we note $\phi(p)$ the largest number smaller or equal than p such that $\phi(p) \cdot \#g^N$ is an integer. This notation will be useful in defining a risk-dominant network.[3]

Definition 2 *Let $p \in [0,1]$. A network g is $p-$stable with respect to allocation rule Y and value function v if for all $g' \subseteq g^N$ such that $d(g', g) \leq 1 - \phi(p)$, we have $g' \in IM(g)$. If g is $\frac{1}{2}-$stable then it is said to be a risk-dominant network.*

Any network g that is $p-$stable is $p'-$stable for $p' \geq p$. The notion of $p-$stability is a refinement of pairwise stability in the following sense. A network g is pairwise stable if and only if it is $1-$stable. Thus, any network g that is $p-$stable is pairwise stable.

Proposition 1 *If there exists a $\frac{1}{2}-$stable network then it is the unique $p-$stable network for $p \leq \frac{1}{2}$.*

Proof. We proceed by contradiction. Assume that g^1 and g^2 are two distinct $p-$stable networks where $p \leq \frac{1}{2}$. Then, they are $\frac{1}{2}-$stable. If $d(g^1, g^2) \leq \frac{1}{2}$. Then we have a straightforward contradiction. (Since we must have $g^1 \in IM(g^2)$ and $g^2 \in IM(g^1)$ which is not possible since $g^1 \neq g^2$.)

If $d(g^1, g^2) > \frac{1}{2}$, we delete elements in $\{(ij \in g^1 \wedge ij \notin g^2)\}$ and add elements in $\{(ij \notin g^1 \wedge ij \in g^2)\}$ so that we obtain a network g' satisfying $d(g', g^1) = 1 - \phi(\frac{1}{2})$. By construction, this network g' satisfies $d(g', g^2) = \phi(\frac{1}{2})$. Then, since g^1 and g^2 are $\frac{1}{2}-$stable, we have that $g' \in IM(g^1)$ and $g' \in IM(g^2)$ which is not possible since $g^1 \neq g^2$.

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In Example 1 the complete network is the unique $\frac{1}{2}-$stable network, and so is a risk-dominant network. The reason is that from any network g' with $\#(g') \geq 3$ any (sufficiently long) improving paths go to the complete network g^N, but none goes to the empty network.

Example 2. The symmetric connection model (Jackson and Wolinsky, 1996).

This model is described as follows. Players form links with each other in order to exchange information. If player i is connected to player j, by a path of t links, then player i receives a payoff of δ^t from his indirect connection with player j. It is assumed that $0 < \delta < 1$, and so the payoff δ^t decreases as the path connecting players i and j increases; thus information that travels a long distance becomes diluted and is less valuable than information obtained from a closer neighbor. Each direct link ij results in a cost

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[3] Note that in all examples of the paper, we will choose the number of players N so that $\#g^N = \frac{N(N-1)}{2}$ is even. This will allow us to have $\phi(\frac{1}{2}) = \frac{1}{2}$.

c to both i and j. This cost can be interpreted as the time a player must spend with another player in order to maintain a direct link. Formally, the payoff player i receives from network g is $Y_i(g) = \sum_{j \neq i} \delta^{t(ij)} - \sum_{j:ij \in g} c$, where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j). Here the value of network g is $v(g) = \sum_i Y_i(g)$. The incentives in forming links come from the consideration of direct costs and benefit, as well as the benefits of indirect connections.

Jackson and Wolinsky (1996) have shown that, for $c < \delta - \delta^2$, the complete network is the unique pairwise stable network. The reason is that, in this cost range, any two players who are not directly connected benefit from forming a link. It follows that the complete is also $\frac{1}{2}-$stable, and so is a risk-dominant network. Notice that the complete network is the strongly efficient network if $c < \delta - \delta^2$.

Proposition 2 *In the symmetric connection model, for $c < \delta - \delta^2$, the complete network g^N is a risk-dominant network.*

Example 3. Consider a situation where four players can form links. The payoffs they obtained from the different network configurations are (see Figure 2): $Y_i(g) = [\#(g)]^2 - c \cdot \#\{j \in N$ such that $ij \in g\}$ if $i \in N(g)$, $Y_i(g) = 0$ if $i \notin N(g)$, and $Y_i(g) = 0$ if g is the empty network. The parameter $c > 0$ is the individual cost of forming a link. For $c < 11$ the complete network is pairwise stable, for $c > 1$ the empty network is pairwise stable. For $c < 5$ our refinement will select the complete network which is the unique $\frac{1}{2}-$stable network. For $c > 7$ the empty network is the unique $\frac{1}{2}-$stable network. But, if $5 < c < 7$ then a $\frac{1}{2}-$stable network fails to exist. The reason is that at $g' = \{12, 13, 34\}$ players 2 and 4 have incentives to form the link 24 but at the same time players 1 or 3 has an incentive to sever the link he has with 2 or 4. So, from g' some improving paths go to the empty network, while others go to the complete network. It follows that no risk dominant network exists.

Example 4. Suppose that five players can form links. In the complete network, $Y_i(g) = 8$ for all i. In any network g players $i \notin N(g)$ have a payoff $Y_i(g) = 0$. In networks g such that $\#(g) \in [3, 9]$, we have $Y_i(g) = 9 - \#(g)$ if $i \in N(g)$. In any g such that $\#(g) = 1$ or 2 and players 4 or 5 belong to $N(g)$ then $Y_i(g) = 0$ for all i. In any g such that $\#(g) = 2$ and players 4 and 5 do not belong to $N(g)$, we have that $Y_i(g) = 7$ for $i \in N(g)$. Finally, let $Y_1(\{12\}) = Y_3(\{13\}) = Y_2(\{23\}) = 6$, $Y_2(\{12\}) = Y_1(\{13\}) = Y_3(\{23\}) = 8$. In this example there is a unique pairwise stable network, the complete network. But, there does not exist a risk-dominant network. Indeed, from any g' such that $d(g', g^N) \geq \frac{1}{5}$, no improving path goes to g^N.

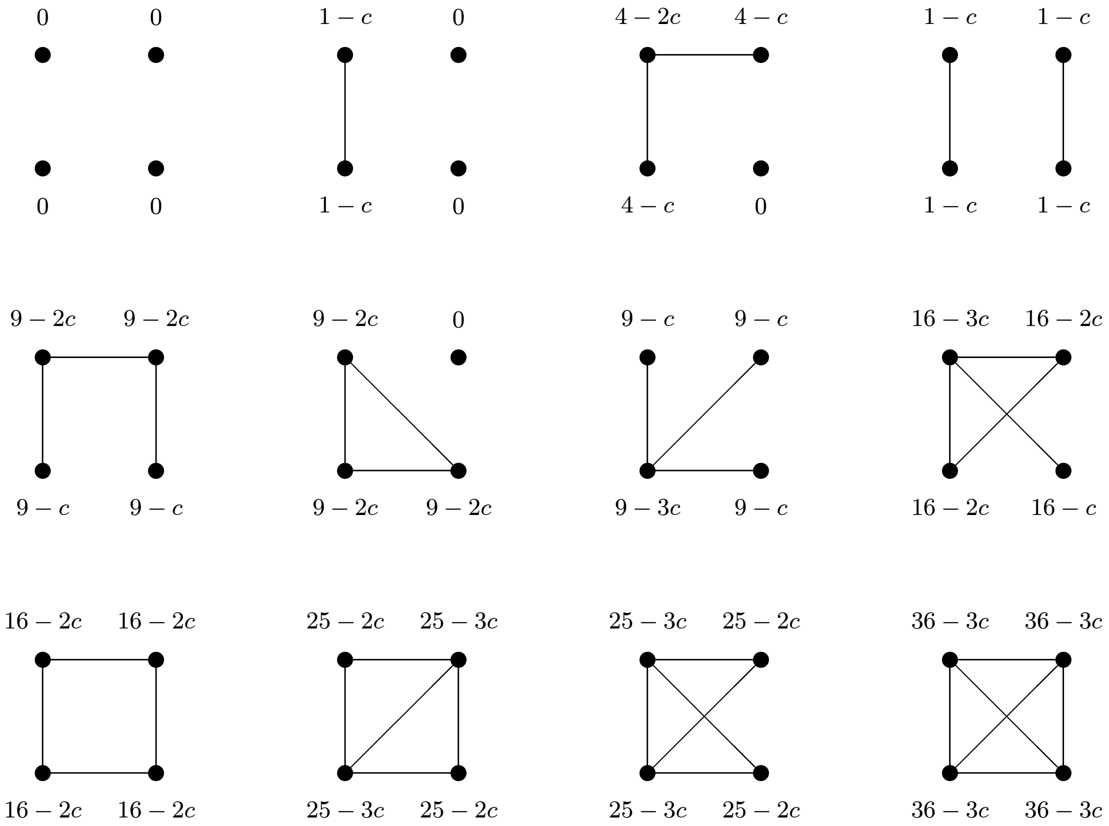

Figure 2: Non-existence of $\frac{1}{2}-$stable networks (Example 3).

Thus it is now clear that our definition of a risk-dominant network is subject to the same criticism than the risk-dominant equilibrium in noncooperative game: it does not always exist. Tercieux (2003) has proposed a solution to this problem in a noncooperative framework using a set-valued extension. We follow this approach in order to resolve the lack of existence. Interestingly, such an approach will put into relief that a set of networks that are not pairwise stable can be more "stable" than a pairwise stable network.

3 Risk-Dominance Sets of Networks

In the following we aim to solve the problem of non-existence risk-dominant networks by providing a set-valued extension. Let us first restate the definition of an improving path. An $K-$improving path from a network g to a set of networks $G \subseteq G^N$ is a finite sequence of graphs $g_1, ..., g_K$ with $g_1 = g$ and $g_K \in G$ such that for any $k \in \{1, ..., K-1\}$ either:

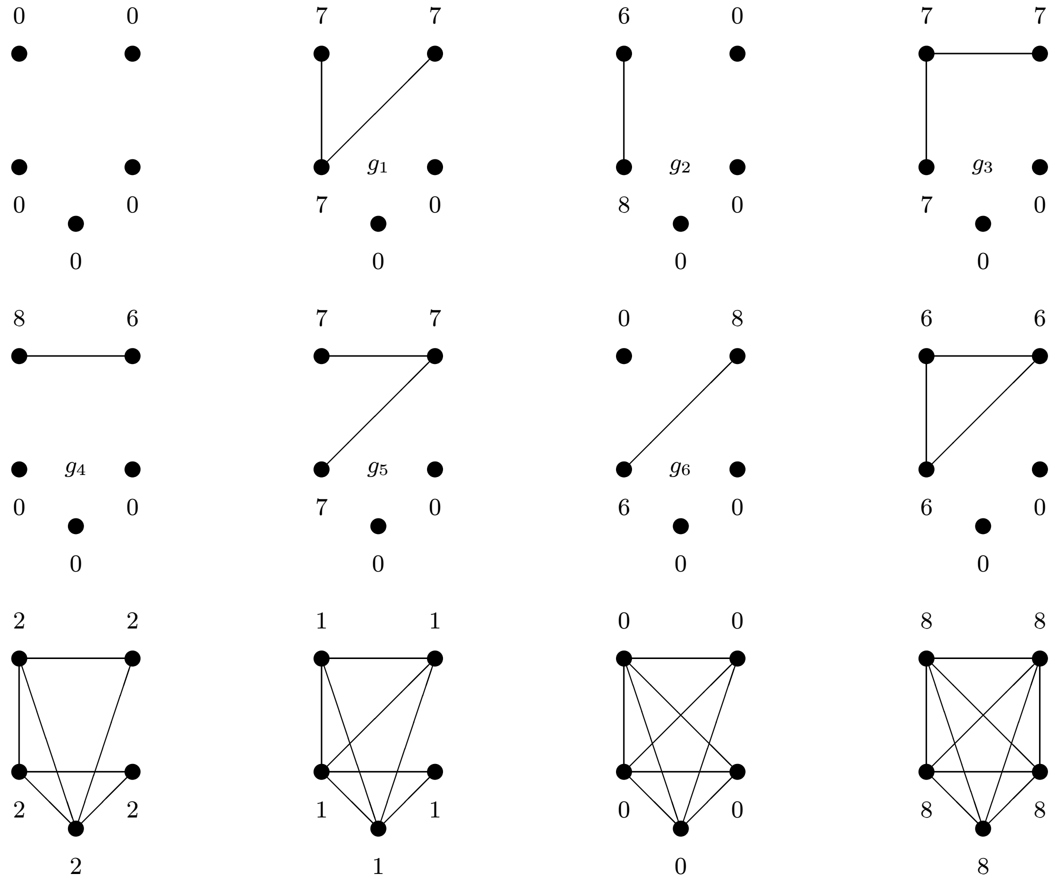

Figure 3: Another example of non-existence of $\frac{1}{2}-$stable networks (Example 4).

(i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_k - ij) > Y_i(g_k)$, or

(ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_k + ij) > Y_i(g_k)$ and $Y_j(g_k + ij) \geq Y_j(g_k)$.

If there exists $\overline{K}$ such that for all $K \geq \overline{K}$, any $K-$improving path from g' go to G, then we use the symbol $g' \longmapsto G$. For a given set of network G, let $IM(G) = \{g' \subseteq g^N \mid g' \longmapsto G\}$.

Definition 3 *Let $p \in [0,1]$. A set of networks $G \subseteq G^N$ is $p-$stable with respect to allocation rule Y and value function v if*
(1) *for all $g' \subseteq g^N$ such that $d(g', g) \leq 1 - \phi(p)$, for some $g \in G$, we have $g' \in IM(G)$,*
(2) *there does not exist $G' \subsetneq G$ such that G' satisfies **(1)**.*
If G is $\frac{1}{2}-$stable then it is said to be a risk-dominant set of networks.

Remark 1 *Note that the set G^N (trivially) satisfies **(1)** in Definition 3 for any $p \in [0,1]$.*

As underlined earlier, the main drawback of our definition of risk-dominant networks is that existence may fail also when a pairwise stable network exists. We now show that our set-valued notion of risk-dominant set always exists. As will become clear (see Example 4), when there does not exist any risk-dominant network, our notion allows to eliminate many possibilities. Moreover, it is possible that the risk-dominant set of networks does not contain any pairwise stable network (see Example 4). We claim that this last point is very important and underlines an important drawback of pairwise stability. The selection result we will introduce in the next section will give a foundation to this informal argument since we will prove that any network outside the risk-dominant set is not robust in a precise sense.

Note that, in the following, for $G \subseteq G^N$, and $g' \subseteq g^N$ we will note $d(g', G) \leq 1 - p$, if $d(g', g) \leq 1 - p$ for some $g \in G$.

Proposition 3 *Fix $p \in [0, 1]$. There always exists at least one $p-$stable set of networks.*

Proof. Let us proceed by contradiction. Let $p \in [0, 1]$ and assume that there does not exist any set of networks $G \subseteq G^N$ that is $p-$stable. This means that for any $G^0 \subseteq G^N$ that satisfies **(1)** in Definition 3 (there always exist such a G^0, see Remark 1), we can find a proper subset G^1 that satisfies **(1)**. But again for G^1, we can find a proper subset G^2 that satisfies **(1)**. Iterating the reasoning we can build an infinite (decreasing) sequence $\{G^k\}_{k \geq 0}$ of **distinct** elements of G^N satisfying **(1)**. But since $\#G^N < \infty$, this is not possible; so the proof is completed.

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Note first that if g is a risk-dominant network then $\{g\}$ is a risk-dominant set of networks. What our next result shows in particular is that $\{g\}$ is the only risk-dominant set of networks and thus the two notions coincide in that special case.

Proposition 4 *Let $p \leq \frac{1}{2}$. There always exists a unique $p-$stable set of networks.*

Proof. We proceed in two steps. (1) First we show that for $p \in [0, 1]$, two (distinct) $p-$stable set of networks must be disjoint. (2) We prove uniqueness for $p \leq \frac{1}{2}$.

(1) We proceed by contradiction. Assume that G and G' are two (distinct) $p-$stable sets of networks and $G \cap G' \neq \emptyset$. Then, for all $g' \subseteq g^N$ such that $d(g', g) \leq 1 - \phi(p)$, for some $g \in G$, we have $g' \in IM(G)$. But since this assertion is also true for G', we have that for all $g' \subseteq g^N$ such that $d(g', g) \leq 1 - \phi(p)$, for some $g \in G \cap G'$, $g' \in IM(G \cap G')$. Thus $G \cap G'$ satisfies **(1)** in Definition 3, contradicting the fact that G (and G') are $p-$stable set of networks, i.e. the minimality is violated (point **(2)** in Definition 3 of $p-$stable sets).

(2) We proceed again by contradiction. Assume that G^1 and G^2 are two distinct $p-$stable networks where $p \leq \frac{1}{2}$. Then, they satisfy **(1)** in Definition 3 for $p = \frac{1}{2}$.

If $d(G^1, G^2) \leq \frac{1}{2}$. Then we have a straightforward contradiction. (Since from $g \in G^1$ we must have $g \in IM(G^1)$ and $g \in IM(G^2)$ which is not possible since $G^1 \cap G^2 = \emptyset$.)

If $d(G^1, G^2) > \frac{1}{2}$, we take $g^1 (\in G^1)$ and $g^2 (\in G^2)$. We delete elements in $\{(ij \in g^1 \wedge ij \notin g^2)\}$ and add elements in $\{(ij \notin g^1 \wedge ij \in g^2)\}$ so that we obtain a network g' satisfying $d(g', G^1) = 1 - \phi(\frac{1}{2})$. By construction, this network g' satisfies $d(g', G^2) = \phi(\frac{1}{2}) \leq \frac{1}{2}$. Then, since G^1 and G^2 are $p-$stable for $p \leq \frac{1}{2}$ (i.e. they both satisfy **(1)** in Definition 3 for $p = \frac{1}{2}$), we have that $g' \in IM(G^1)$ and $g' \in IM(G^2)$ which is not possible since $G^1 \cap G^2 = \emptyset$.

■

In Example 3 we have that, for $5 < c < 7$, there does not exist a risk-dominant network, but the set formed by the complete and empty networks is the risk-dominant set of networks. In Example 4, the complete network is the unique pairwise stable network and there is no risk-dominant network. However, the $\frac{1}{2}-$stable set of networks is $G' = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ (see Figure 4), which does not include the complete network, because there is a cycle $g_1 \rightarrow g_2 \rightarrow g_3 \rightarrow g_4 \rightarrow g_5 \rightarrow g_6 \rightarrow g_1$ and all improving paths from any g' such that $d(g', g \in G') \leq \frac{1}{2}$ go to G'. Finally, notice that $\{g^N\}$ is $1-$ stable set and has no intersection with the risk-dominant set G'.

Our set-valued notion generalizes many existing concepts of the literature. We can easily link this to two definitions the first one is the well-known definition of pairwise stable networks of Jackson and Wolinsky (1996). The second one is the one of closed cycle provided by Jackson and Watts (2002).

The following straightforward proposition is stated without proof.

Proposition 5 *$\{g\}$ is a $p-$stable set if and only if g is a $p-$stable network. And so, $\{g\}$ is a $1-$stable set if and only if it is a pairwise stable network.*

The following definition is due to Jackson and Watts (2002, p.273). A set of networks G, form a *cycle* if for any $g \in G$ and $g' \in G$, there exists an improving path connecting g to g'. A cycle G is a *closed cycle* if no network in G lies on an improving path leading to a network that is not in G.

Proposition 6 *G is a $1-$stable set if and only if it is a closed cycle.*

Proof. The proof can be found in Appendix A. ■

4 Evolutionary Selection

It has been shown (see for instance, Young (1993), Kandori, Mailath and Rob (1993) and Maruta (1997)) that the non-cooperative game's concept of risk-dominant equilibrium is selected in many stochastic evolutionary processes. In the following, we show that our notion of risk-dominant networks (and risk-dominant set of networks) is relevant in the stochastic evolutionary process proposed by Jackson and Watts (2002).

4.1 The Process

Let us recall first the Jacskon and Watts (2002)'s process. At a discrete set of times, $\{1, 2, 3, ...\}$ decisions to add or sever a link are made. At each date, a pair of players ij is randomly identified with probability $p(ij) > 0$. The (potential) link between these two players is the only link that can be altered at that time. If the link is already in the network, then the decision is whether to sever it, and otherwise the decision is whether to add the link. The players involved act myopically, adding the link if it makes each at least as well off and one strictly better off, and severing the link if its deletion makes either player better off. After the action is taken, there is some small probability $\varepsilon > 0$ that a mutation (or tremble, or mistake) occurs and the link is deleted if it is present, and added if it is absent.

These mutations may arise for a variety of reasons. Mutations may be due to exogenous unmodeled factors that are beyond player's control. Alternatively, players may simply make errors in calculating whether adding or severing a link is beneficial. Finally, we could think to players having a limited information. Thus they occasionally experiment to see if adding or severing a link will make them better off (endogenous mutations has been formalized in several papers, see for instance van Damme and Weibull (2002), Blume (1999) and Maruta (2002)).

The above process defines a (finite) Markov chain with states being the network in place at the end of a given period. Note that with mutations as part of the process, each state of the system is reachable with positive probability from every other state. The Markov chain is said to be irreducible and aperiodic, and thus has a unique corresponding stationary distribution (see for instance Freidlin and Wentzel, 1984). As ε goes to zero, the stationary distribution converges to a unique limiting stationary distribution. A network that is in the support of the limiting (as ε goes to zero) stationary distribution of the above-described Markov process is said to be *stochastically stable*. Intuitively, a stochastically stable network is one that is observed infinitely many more times than others when the probability of mutations is infinitely small. Jackson and Watts (2002) provides a characterization of stochastically stable networks where those networks are the easiest to reach from other networks, with "easiest" interpreted as requiring the fewest mutations.

4.2 Relationship between Risk-Dominance and Stochastic Stability

In this section, we provide a characterization of the set of stochastically stable networks using refinements we have introduced earlier. The following theorem shows that under the process we have just described, the only networks that will arise with a significant frequency in the long run (i.e., the stochastically stable one) are in the risk-dominant set.

Theorem 1 *Let G be the risk-dominant set of networks. The set of stochastically stable networks is included in G.*

Proof. See Appendix B.

■

Thus any network outside that set must be considered as a non-robust network. To be more precise, the stochastic process presented above can be thought of as a check on the robustness of pairwise networks or cycles. Although a number of networks may be pairwise stable, they can differ in how resilient they are to the random mutations. For instance, it may be relatively hard to leave and easy to get back to some networks, our above theorem tells us that such networks are included in the risk-dominant set of networks. This result also tells us that any network that is not in the risk-dominant set is relatively easy to leave and hard to get back.

In order to understand these points, note that once the process has reached the risk-dominant set of networks G, it cannot leave it without further mutations. On the first hand, in order to get off that set, it is necessary that mutations occur. Indeed, by definition of G, it is necessary that strictly more than $\frac{\#g^N}{2}$ mutations occur (notice that in order to give the intuition of our result, we skip some technical points in assuming that N is such that $\frac{\#g^N}{2}$ is an integer). If it is not the case, the process will come back to G with no further mutation. On the other hand, as it will become clear, if the process has reached a network that is outside G, it is sufficient that less than $\frac{\#g^N}{2}$ mutations occur to allow the process to reach a network that belong to G. In order to see why it is so, note that from a network g' that does not belong to G, with (less than) $\frac{\#g^N}{2}$ mutations, one can reach a network $\bar{g}$ such that $d(g, \bar{g}) \leq \frac{1}{2}$ where g belongs to G. Thus, by definition, the process will move to G without any further mutations. To see how we can build $\bar{g}$, we just have to add links to g' that belong to g and not to g' or to delete links that do not belong to g but belong to g'. By repeating this procedure less than $\frac{\#g^N}{2}$ times, we can reach such a $\bar{g}$. Thus there exists networks in G which are the easiest to reach from other networks, where - again - "easiest" is interpreted as requiring the fewest mutations. These networks are stochastically stable. The formal argument is given in the appendix.

Of course, we would like to have a full characterization of the set of stochastically stable networks. In order to do so, we provide several sufficient conditions that go in that sense. These results are corollary of Theorem 1. The first one shows that if there exists a risk-dominant network then it must be the unique stochastically stable network. This result is a parallel to the one of Young (1993) [Theorem 3, p.72] in noncooperative games.

Corollary 1 *Assume that a network g is the risk-dominant network. Then g is the unique stochastically stable network.*

The following two corollaries directly come from the fact that if g is stochastically stable then g is part of a $1-$stable set of networks. Furthermore, if $g \in G$ is stochastically stable and G is a $1-$stable set then all $g' \in G$ are stochastically stable (this follows from Lemma 2 in Jackson and Watts (2002) together with our Proposition 6 that establishes the equivalence between a $1-$stable set and a closed cycle).

Corollary 2 *Let G be the risk-dominant set of networks. If G is $1-$stable then G is the set of stochastically stable networks.*

Indeed, we can go one step further with the following corollary.

Corollary 3 *Let G be the risk-dominant set of networks. If $G' \subseteq G$ is the unique $1-$stable set in G then G' is the set of stochastically stable networks.*

Example 5. Suppose that three players can form links. In the complete network, $Y_i(g) = 3$ for all i. In any network g players $i \notin N(g)$ have a payoff $Y_i(g) = 0$. In any g such that $\#(g) = 2$, $Y_i(g) = 2$ if $i \in N(g)$. Finally, let $Y_1(\{12\}) = Y_3(\{13\}) = Y_2(\{23\}) = 1$, $Y_2(\{12\}) = Y_1(\{13\}) = Y_3(\{23\}) = 4$. In this example there is a unique pairwise stable network, the complete network. There does not exist a risk-dominant network, $\{g^N\}$ is the $1-$stable set, and all networks except the empty one belong to the risk-dominant set of networks.

In Example 5, the complete networks is the unique pairwise stable network and there is no risk-dominant network because of the cycle $g_1 \to g_2 \to g_3 \to g_4 \to g_5 \to g_6 \to g_1$. The $\frac{1}{2}-$stable set of networks is $G' = \{g_1, g_2, g_3, g_4, g_5, g_6, g^N\}$ but this set is not $1-$stable. Indeed, $\{g^N\}$ is the unique $1-$ stable set.

5 Conclusion

In this paper, we have tried to provide a definition of a risk-dominant network. This notion extends the idea of the risk-dominant equilibrium of Harsanyi and Selten (1988) to

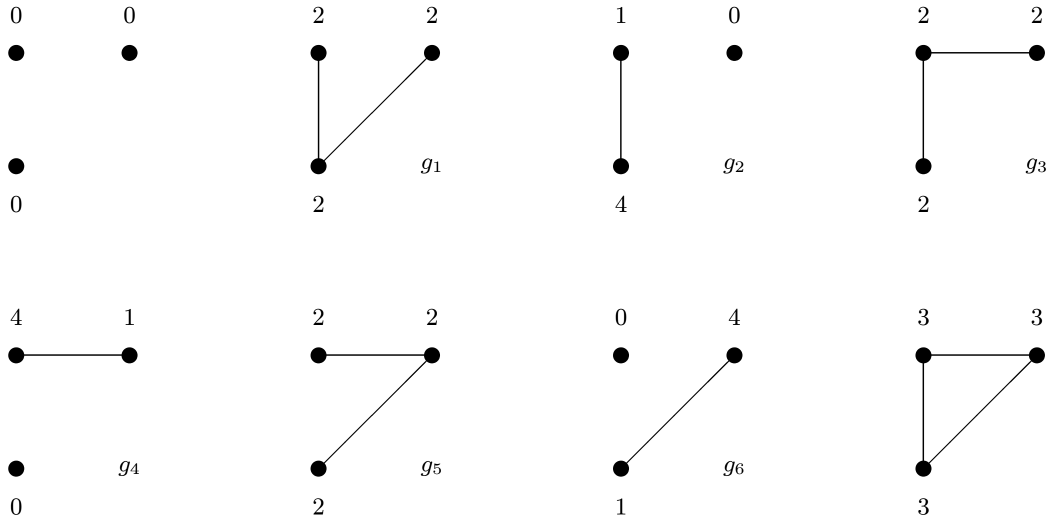

Figure 4: Risk-dominant set and stochastically stable networks (Example 5).

networks. We have proved that our concept and the one provided by Harsanyi and Selten were very close, for instance, as the risk-dominant equilibrium, the risk-dominant network is selected in stochastic evolutionary processes.

In noncooperative frameworks, it has been shown that a risk-dominant equilibrium could be also selected when agents are perfectly rational. Carlson and van Damme (1993) have proved such a result, in a two player, two action game, where there exists a risk-dominant equilibrium. To be more precise, Carlson and van Damme show that by introducing a "slight" incomplete information in that game, the unique rationalizable equilibrium (in the sense of Bernheim (1984) and Pearce (1984)) of the incomplete information game consists in playing at each state of the world the action profile of the risk-dominant equilibrium (of the associated complete information game). Then, the noncooperative concept of Harsanyi and Selten seems to be also very robust when agents are perfectly rational. Such an eductive foundation could certainly be given to our concept of risk-dominant network. This work is left for further research.

Appendix

A Proof of Proposition 6

In this part we prove Proposition 6 that establishes the equivalence between our notion of 1−stability and the notion of a closed cycle proposed by Jackson and Watts (2002). In order to do so, we first state and prove some useful lemmas. The following simple lemmata is stated without proof.[4]

Lemma 1 *If G is such that for all $g \in G$, $g \in IM(G)$ (note that this is (1) in Definition 3 of a $1-$stable set) then there exists $C \subseteq G$ that is a closed cycle.*

Our next lemma provides a first step in establishing a link between $1-$stability and closed cycles.

Lemma 2 *If C is a closed cycle then there exists $G \subseteq C$ that is $1-$stable.*

Proof. Since C is a closed cycle, we know that for all $g \in C$, $g \in IM(C)$. Then C satisfies **(1)** of Definition 3 of a $1-$stable set. Now assume that there does not exist any $G \subseteq C$ that is $1-$stable. Then any $G \subseteq C$ has a proper subset that satisfies **(1)** in the definition of $1-$stable sets. Now, as in the proof of Proposition 3, this implies that there exists an infinite decreasing sequence $\{C^k\}_{k \geq 0}$ where $C^0 = C$ and $C^{k+1} \subsetneq C^k$ for all $k \geq 0$. But since $\#G^N < \infty$, this is not possible; so the proof is completed. ■

Now we are ready to complete the proof of Proposition 6. We first prove the **"if" part**. Suppose that G is a closed cycle but G is not $1-$stable and show that this lead to a contradiction. This last point can be due to the violation of **(1)** or **(2)** in the definition of a $1-$stable set. Assume first that **(1)** is violated. This implies that there exists $g \in G$ and $g' \notin G$ such that $g \to g'$. Which contradicts the definition of a closed cycle. Assume now that **(2)** is violated. This means that there exists $G' \subsetneq G$ that satisfies **(1)** in the definition of a $1-$stable set i.e., for all $g' \in G'$, $g' \in IM(G')$. But by Lemma 1, we know that there exists a closed cycle $C \subseteq G' \subsetneq G$. Then, we have the following: first, because G is a (closed) cycle, we have that for all $g, g' \in G$, $g \to g'$. But we also have, because C is a closed cycle, that for all $g \in C(\subsetneq G)$ and $g' \in G - C$, $g \to g'$ is wrong. Thus we obtain a contradiction.

We now prove the **"only if" part**. We know by Lemma 1 that since G is $1-$stable, there exists $C \subseteq G$ that is a closed cycle. We must prove that $C = G$. So let us proceed by contradiction and assume that $C \subsetneq G$. We know by Lemma 2 that there exists $G' \subseteq C \subsetneq G$ that is $1-$stable. This leads to a straightforward contradiction since it contradicts **(2)** (the minimality) in the $1-$stability of G. This completes the proof of Proposition 6.

B Proof of Theorem 1

In order to prove Theorem 1, we first introduce some useful definitions and notations.

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[4] A complete proof would mimic the proof of Lemma 1 in Jackson and Watts (2002, p.273).

B.1 Definitions

For a given network g, remember that $im(g) = \{g' \subseteq g^N$ |there exists an improving path from g' to $g\}$. A path $p = \{g_1, ..., g_K\}$ is a sequence of adjacent networks. The resistance of a path $p = \{g_1, ..., g_K\}$ from g' to g, denoted $r(p)$, is computed by $r(p) = \sum_1^{K-1} I(g_i, g_{i+1})$, where

$$I(g_i, g_{i+1}) = \begin{cases} 0 & \text{if } g_i \in im(g_{i+1}) \\ 1 & \text{otherwise} \end{cases}.$$

Resistance keeps track of how many mutations must occur along a special path to follow that path from one network to another. A mutation is necessary to move from one network to an adjacent one whenever it is not in the relevant player's interests to sever or add the link that distinguishes the two adjacent networks.

Let $r(g', g) = \min\{r(p) \mid p$ is a path from g' to $g\}$ and set $r(g, g) = 0$. Note that $r(g', g) = 0$ iff $g' \in im(g)$ or $g' = g$. Thus if $g, g' \in G$ where G is $1-$stable, then $r(g', g) = 0$.

Given a network g, a $g-$tree is a directed graph which has as vertices all networks and has a unique directed path leading from each g' to g. Let $T(g)$ denote all the $g-$trees, and represent a $t \in T(g)$ as a collection of ordered pairs of networks, so that $g'g'' \in t$ if and only if there is a directed edge connecting g' to g'' in the $g-$tree t.

The resistance of a network g is computed as $r(g) = \min_{t \in T(g)} \sum_{g'g'' \in t} r(g', g'')$. The main result of Jackson and Watts (2002) that is closely related to the technics developed in Young (1993) is that the set of stochastically stable networks is the set $\{g \mid r(g) \leq r(g')$ for all $g'\}$. We will use this characterization in order to prove our main results.

B.2 The Proof

The proof is divided into two parts: (1) we bound the resistances of paths that begin at $g \in G$ and the resistances of paths ending at $g \in G$; (2) we show that some $g \in G$ minimizes stochastic potential.

(1) We give a lower bound on the resistance of the transitions that begin at $g \in G$ and end at any $g' \notin G$. Recall that by definition of $p-$stability for $p \leq \frac{1}{2}$, $r(g, g') > (1 - \phi(\frac{1}{2})) \cdot \#g^N \geq \frac{\#g^N}{2}$.

We give now an upper bound on the resistance of paths that begin at any $g' \notin G$ and end in G. Pick $g \in G$. We delete elements in $\{(ij \in g' \wedge ij \notin g)\}$ and add elements in $\{(ij \notin g' \wedge ij \in g)\}$ so that we obtain a network $\bar{g}$ satisfying $d(\bar{g}, g') = 1 - \phi(\frac{1}{2})$. By construction, this network $\bar{g}$ satisfies $d(\bar{g}, g) = \phi(\frac{1}{2})$ where $g \in G$. But G is a $p-$stable set of networks for $p \leq \frac{1}{2}$, thus with less than $(1 - \phi(\frac{1}{2})) \cdot \#g^N$ mutations, we will reach a network in G (note that once the process has reached G, we cannot leave it without mutations). Therefore, $r(g', \tilde{g}) \leq (1 - \phi(\frac{1}{2})) \cdot \#g^N$ for some $\tilde{g} \in G$. We will note such a $\tilde{g}$ by $\Psi(g')$. Thus for every $g' \notin G$, $r(g', \Psi(g')) \leq (1 - \phi(\frac{1}{2})) \cdot \#g^N$.

(2) Suppose by contradiction that $g' \notin G$ is stochastically stable. Denote by t' (one of) the $g'-$tree(s) ($t \in T(g')$) that minimizes resistance.

We know that there is a sequence $g_1, ..., g_n$ with $g_1 = \Psi(g')$ ($\in G$) and $g_n = g'$ such that:

- $g_l g_{l+1} \in t'$ for every $l = 1, ..., n-1$

- there is a $k \in \{1, ..., n-1\}$ such that $g_k \in G$ and $g_{k+1} \notin G$. Delete this edge and add one from g' to $\Psi(g')$. We obtain a tree $t'' \in T(g_k)$ where $g_k \in G$.

By construction, $r(g_k) = r(g') - r(g_k, g_{k+1}) + r(g', \Psi(g'))$. But as proved above, we have $r(g_k, g_{k+1}) > (1 - \phi(\frac{1}{2})) \cdot \#g^N \geq r(g', \Psi(g'))$. Hence, $r(g^k) < r(g')$. This contradicts the fact that g' minimizes stochastic potential. This completes the proof.

References

[1] Aumann, R. and R. Myerson, "Endogenous Formation of Links between Players and Coalitions: an Application of the Shapley Value," in The Shapley value, A. Roth Ed., Cambridge University Press, 175-191 (1988).

[2] Bala, V. and S. Goyal, "Self-Organization in Communication Networks," *Econometrica* 68, 1131-1230 (2000).

[3] Bergin, J. and B.L. Lipman, "Evolution with State-dependent Mutations," *Econometrica* 64, 943-956 (1996).

[4] Bernheim, D., "Rationalizable Strategic Behavior," *Econometrica* 52, 1007-1028 (1984).

[5] Blume, L.E., "How Noise Matters," *Games and Economic Behavior* 44, 251-271 (1999).

[6] Carlsson, H. and E. van Damme, "Global Games and Equilibrium Selection," *Econometrica* 61, 989-1018 (1993).

[7] van Damme, E. and J. Weibull, "Evolution in Games with Endogenous Mistake Probabilities," *Journal of Economic Theory* 106, 296-315 (2002).

[8] Dutta, B. and M.O. Jackson, "The Stability and Efficiency of Directed Communication Networks," *Review of Economic Design* 5, 251-272 (2000).

[9] Dutta, B. and S. Mutuswami, "Stable Networks," *Journal of Economic Theory* 76, 322-344 (1997).

[10] Ellison, G., "Learning, Local interaction, and Coordination," *Econometrica* 61, 1047-1071 (1993).

[11] Ellison, G., "Basins of Attraction, Long-Run Stochastic Stability, and the Speed of Step-by-Step Evolution," *Review of Economic Studies* 67, 17-45 (2000).

[12] Freidlin, M. and A. Wentzell, in "Random Perturbations of Dynamical Systems," pp.161-211, Springer-Verlag, New York (1984).

[13] Harsanyi, J. and R. Selten, A General Theory of Equilibrium in Games, Cambridge: MIT Press (1988).

[14] Hurkens, S., "Learning by Forgetful Players," *Games and Economic Behavior* 11, 304-329 (1995).

[15] Jackson, M.O., "The Stability and Efficiency of Economic and Social Networks," in Networks and Groups: Models of Strategic Formation, edited by B. Dutta and M.O. Jackson, Springer-Verlag: Heidelberg (2003).

[16] Jackson, M.O., "A Survey of Models of Network Formation: Stability and Efficiency," in Group Formation in Economics: Networks, Clubs and Coalitions, edited by G. Demange and M. Wooders, Cambridge University Press (2004).

[17] Jackson, M.O. and A. van den Nouweland, "Strongly Stable Networks," forthcoming in *Games and Economic Behavior* (2004).

[18] Jackson, M.O. and A. Watts, "The Evolution of Social and Economic Networks," *Journal of Economic Theory* 71, 44-74 (2002).

[19] Jackson, M.O. and A. Wolinsky, "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory* 71, 44-74 (1996).

[20] Kandori, M., G. Mailath, and R. Rob "Learning, Mutation, and Long Run Equilibria in Games," *Econometrica*, 61, pp.29-56 (1993).

[21] Maruta, T., "On the Relationship Between Risk-Dominance and Stochastic Stability," *Games and Economic Behavior* 19, 221-234 (1997).

[22] Maruta, T., "Binary Games with State Dependent Stochastic Choice," *Journal of Economic Theory* 103, 351-376 (2002).

[23] Pearce, D., "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica* 52, 1029-1050 (1984).

[24] Tercieux, O. "Existence, Uniqueness and Stability of a Set-Valued Concept," Cahiers de la MSE, n°2003-130 (2003).

[25] Watts, A., "A Dynamic Model of Network Formation," *Games and Economic Behavior* 34, 331-341 (2001).

[26] Young, P., "The Evolution of Conventions," *Econometrica* 61, 57-64 (1993).

[27] Young, P., Individual Strategy and Social Structure. An Evolutionary Theory of Institutions. Princeton: Princeton University Press (1998).