Network Formation With Endogenous Decay. [1]

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Abstract

This paper considers a communication network characterized by an endogenous architecture and an imperfect transmission of information as in Bala and Goyal (2000). We propose a similar network's model with the difference that it is characterized by an endogenous rate of information's decay. Endogenous decay is modelled as dependent on the result of a coordination game, played by every pair of directly linked agents and characterized by 2 equilibria: one efficient and the other risk dominant. Differently from other models, where the network rapresents only a channel to obtain information or to play a game, in our paper the network has an intrisic value that depend on the chosen action in the coordination game by each participant. Moreover the endogenous network structure affects the play in the coordination game as well as the latter affects the network structure. The model has a multiplicity of equilibria and we produce a full characterization of those are stochastically stable. For sufficiently low link costs we find that in stochastically stable states network structure is ever efficient; individuals can be coordinated on efficient as well as on risk dominant action depending on the decay difference among the two equilibria in the single coordination game. For high link costs stochastically stable states can display networks that are not efficient; individual are never coordinated on the efficient action.

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1. Introduction

In recent years there is a growing interest about social and economic networks. The reason is that these models fit better the real environment to explain many economic phenomena. For example, we remember the diffusion of new technologies, products and conventions, labor search, diffusion of information and social learning, internal and external firm's relations, customized equipment between a group of firms, and so on. The fundamental idea is that, against the assumption that agents in an economic system are anonymous, there is the empirical evidence that agents must have a link to do any kinds of transaction and that in an economic system interaction among agents (or the collection of information) has a cost. These costs together with the relative benefits influence the interaction structure that, in her turn, affects the final result. Indeed in a market two agents can exchange the goods if and only if they are in touch (or they are linked) and the pattern of links affects the competition for goods and profits. For these reasons many authors have examined the role of interaction structure and its evolution in an economic contest.

The literature on social networks concentrates on the one hand, upon the contrast between efficiency and stability of various possible structures and, on the other, upon the cost–benefit relation deriving from those networks. Jackson and Wolinsky [14] show how efficiency and stability do not always coincide. Their model considers a situation where each individual transmits a value into the network which is discounted according to the number of links involved. Moreover, they study those allocation rules which permit both efficiency and stability simultaneously. Later Bala and Goyal [1] consider a similar model, but with the differences that the links can also be one-sided. They study both those cases in which the benefit flows in one direction only, and those in which the benefit flows in two directions. Furthermore, they consider the dynamics of the link formation. Concerning this last is a work by Watts [21] that considers the dynamics of network formation in the case of the connection model of Jackson and Wolinski [14].

An open question in these papers is that the generated payoff is independent of any other action that is different from the creation or deletion of links in the networks. In detail decay is always considered exogenous or, from another point of view, independent of any action of the agents. We try to solve this problem considering a network characterized by an imperfect transmission of information as in the Bala and Goyal [1] but assuming that the rate of decay into the network is endogenous: the rate of decay in a given link depends on the results of a social game between the 2 (directly) linked agents. There are 2 possible actions: the first one produces a zero decay if both players choose it and a maximum decay if the players choose different actions; the second one produces an intermediate value of decay indifferently from the partner's choice. In this way we model a trade off between complexity (and efficiency) and compatibility. This trade off is illustrated by the following example: an individual has to pass a message and can write it in word format or ascii format. The first choice is more efficient only if the reader has the Word software. The second choice is less efficient but all the people can read it. This model has other two important features: the value of each individual depends on her position inside the network and the interaction between two agents affects all agents in the network. We are able to produce a full characterization of stochastically stable states: the networks characterized by the efficient action are stochastically stable for relatively low link cost, otherwise are stochastically stable those networks characterized by the risk dominant action.

Other related papers are those of Jackson and Watts [12], Goyal and Vega [9], Droste, Gilles and Johnson [4]. These consider the payoff generated by links as determined by the interaction strategy between individuals directly or indirectly linked. The fundamental idea in these papers is that individuals establish links to play a coordination game. As in our paper, to the choice as to whether to link or not is added the choice as to which interaction strategy to use with other individuals. Differently from our paper, these models have in common the idea that the payoff is generated only from the direct link or, as in the paper of Vega and Goyal [9], the payoff is also generated from the indirect link, but by means of a simplification, namely that the benefit deriving from the interaction of two individuals is independent of the kind of link (direct or indirect).

The paper is organized in the following way: in section 2 we describe the model. Section 3 contains the main results. Section 4 concludes the discussion and provide possible directions for futher research.

2. The Model

Let $N = \{1,2,...,n\}$ be a set of agents where $n \geq 3$. We assume that every agents is endowed with one unit of private information of value 1 as well as of a quantity of information deriving from other agents in the network. Each agent can choose a subset of other players with whom to establish links and to play a bilateral game. Let $g_i = (g_{i,1},..g_{i,i-1}, g_{i,i+1},..g_{i,n})$ be the set of links supported by player i where $g_{ij} \in \{0,1\}$ for each $j \in N \setminus \{i\}$. We say agent i supports a link with agent j if $g_{ij} = 1$. The set of all players link decisions, denoted by $g = (g_1, g_2, \ldots g_n)$, defines a direct graph $\{N, \Gamma\}$ called network. The network will be denoted by g. Specifically, the network g has the set of players N, as its set of vertices, and its set of arrows, $\Gamma \subset N \times N$, is defined as follows:

(2.1) $$\Gamma = \{(i, j) \in N \times N : g_{ij} = 1\}$$

Given a network g, we say that 2 players are directly linked if at least one of them has established a link with the other one, i.e. $\max\{g_{ji}, g_{ij}\} = 1$. To describe the direct links with no regard who support them, we define the closure $\bar{g}_{ij} = \max\{g_{ij}, g_{ji}\}$. Let $\bar{g}_i = (\bar{g}_{i,1}, .. \bar{g}_{i,i-1}, \bar{g}_{i,i+1}, .. \bar{g}_{i,n})$ be the set of direct links of agent i. Then $\bar{g} = (\bar{g}_1, \bar{g}_2, \ldots \bar{g}_n)$ describes the graph with no regard who support the links. Let $N^d(i;g) \equiv \{j \in N : g_{i,j} = 1\}$ be the set of players in network g with whom player i has established links, while $v^d(i;g) \equiv |N^d(i;g)|$ is its cardinality. In a similar way, let $N^d(i;\bar{g}) \equiv \{j \in N : \bar{g}_{i,j} = 1\}$ be the set of players in network g with whom player i is connected, while $v^d(i;\bar{g}) \equiv |N^d(i;\bar{g})|$ is its cardinality. We say there is a path in g between i and j if either $\bar{g}_{ij} = 1$ or there exists a set of agents $\{j_1, j_2 \ldots j_m\} \in N / \{i, j\}$ such that $\bar{g}_{ij_1} = \bar{g}_{j_1 j_2} = \ldots = \bar{g}_{j_m j} = 1$. By T_{ij} we denote the set of all paths between agents i and j. The distance in g between agents i and j, denoted as $d(i, j; g)$, is defined as the number of links of the shorter path in T_{ij}. The shorter path is that with the lower number of direct links. A sub-network $g' \subset g$ is called a component of g if for all $i, j \in g'$, $i \neq j$, there exists a path in g' connecting i and j, and there does not exist a path between an agent in g' and one in $g \setminus g'$. A network with only one component is called connected. Given any g, the notation $g + ij$ denotes the network obtained with the formation of a new link between i and j in the network g. Similarly, $g - ij$ refers to the network obtained deleting the link g_{ij} in g. By minimally connected we denote a connected network g such that $g - ij$ is a no connected network for all $i, j \in g$ such that $g_{ij} = 1$. Finally we introduce the following notation. A network is called:

- empty and denoted by g^e if $\bar{g}_{i,j} = 0$ for $\forall i, j \in N$;
- star and denoted by g^s if there exists some $i \in N$ such that, for all $k, j \in N \setminus \{i\}$, $j \neq k$, $\bar{g}_{i,j} = 1$ and $\bar{g}_{k,j} = 0$;
- complete and denoted by g^c if $\bar{g}_{i,j} = 1$ for $\forall i, j \in N$;
- essential if $g_{ij} \cdot g_{ji} = 0$ for $\forall i, j \in N$

Among the star networks we denote with g^{cs} the star with all the links supported by the central agent, with g^{ps} the star with all links supported by peripheral agents and with g^{ms} all the intermediate cases. The links are costly: every agent pays a cost $k > 0$ for each link she supports. In our model, as in Bala and Goyal [1] or Goyal and Vega Redondo [9], link formation is one-sided and non-cooperative: the formation of a link requires only the consensus of the supporting player.

In our model decay is endogenous. We assume that every pair of directly linked agents plays a 2 x 2 symmetric game in strategic form with a common action set given by $A = \{a, b\}$. For each pair of actions $a, a' \in A$, the share of information received by a player choosing *a* when the partner plays *a'* is denoted by $d(a, a')$ and is given by the following table:

(2.2)

	a	*b*
a	*1*	*0*
b	*e*	*e*

where $0{,}5 \leq e \leq 1$

Then, the quantity of information received by a player choosing *a* when the partner plays *a'* is given by $d(a, a') \cdot x$, where x is the information owned by the partner. There are 2 Nash equilibria in pure strategies: (a, a) and (b, b). The first one is efficient, the second one is risk dominant. Each agent plays the game with all directly linked agents and have to use the same action in all engaged bilateral games. In the following we indicate an agent(s) choosing action *a* by *a*-agent(s) where $a \in \{a, b\}$.

For a generic agent *i* the strategy space is identified with $S_i = G_i \; x \; A$, where G_i is the set of possible link decisions and *A* is the common action space of the underlying bilateral game. In the following we consider that $G_i = G \;\; \forall i \in N$.

Then, given the strategies of other players, $s_{-i} = (s_1,...s_{i-1}, s_{i+1},....s_n)$, the payoff of player i deriving from her participation to the game playing some strategy $s_i = (g_i, a_i)$ is given by:

(2.3) $$\Pi(s_i, s_{-i}) = \sum_{j \in N_i} \left[\prod_{l,k \in \bar{t}_{ij}} d(a_l, a_k) \right] - k \cdot v^d(i; g)$$

where $N_i = \{ j : T_{ij} \neq \emptyset \}$ and $\bar{t}_{ij}$ represents the path between players *i* and *j* such that $\bar{t}_{ij} = \underset{t_{ij}}{argmax} \prod_{l,k \in t_{ij}} d(a_l, a_k)$.

Time is modelled discretely, $t = 1,2,3,....$ At time t the state of the system will be given by strategy profile $s(t) = \{g(t), a(t)\}$ specifying the action chosen and links established by each player ($s_i(t) = \{g_i(t), a_i(t)\}$). At every period t one agent obtains, by a probability p, a chance to revise her strategy. When an agent receives this opportunity, she select a best response to strategy profile in the previous periods:

(2.4) $$s_i(t) \in \text{argmax}_{s_i \in S} \Pi[s_i, s_{-i}(t-1)];$$

If there are several best responses, then any one of them is chosen with equal probability. This strategy revision process defines a Markov chain on $S \equiv S_1 \times S_2 \times ... \times S_n$. As we will see, in our framework, this Markov chain could be characterized by several absorbing states. Then, the equilibria are depending on the initial conditions.

To select among all possible equilibria, we employ the standard techniques used by Kandory, Mailath and Rob [15] and Young [21]. We suppose, conditional on the chance to revise her strategy, players make mistakes. In this case, player chooses her strategy at random with some small probability $e > 0$. For any $e > 0$, the process defines an aperiodic and irreducible Markov chain that has a unique invariant probability distribution m_e. We analyze the structure of m_e as the probability of mistakes e converges to zero. We define $lim_{e \to 0} m_e = \hat{m}$, then a state s is called stochastically stable if $\hat{m}(s) > 0$.

3. Results

In this section we characterize the efficient states, study the characteristics of equilibria when dynamic is not perturbed by mistakes and, finally, examine the stochastic stability of different configurations.

3.1 Efficiency

We use the utilitarian concept of efficiency: the efficient state is that producing the higher total net payoff (gross payoff less cost of links).

***Proposition 1**: If $k \leq n$, in all efficient states all agents are coordinated on action $\boldsymbol{a}$ and networks are minimally connected. If $k > n$, only empty networks are efficient.*

As the intuition provided below is simple, a formal proof is omitted. Any network architecture with agents coordinated on action $\boldsymbol{b}$ is dominated by an equal network with agents coordinated on action $\boldsymbol{a}$. When all agents are coordinated on $\boldsymbol{a}$, from proposition 4.3 in Bala and Goyal [1] follows that for $k \leq n$ minimally connected networks are efficient otherwise, if $k > n$, the efficient networks are empty.

There are several efficient network architectures : all kinds of star, the line and, more in general, all architectures minimally connected and with minimum number of links: to connect *n* agents are necessary at least *n-1* links. This is possible because there are not differences in payoff between a direct link and a indirect one, when all individuals are coordinated on action $\boldsymbol{a}$ (remember $p(a,a) = 1$). If $p(a,a) < 1$, we could restrict the set of efficient networks because the share of information arriving from a player to another is decreasing with the number of links that have to pass through. In according to proposition 1 in Jackson and Wolinsky [13], we find efficient networks are complete or star or empty depending on the link cost k.

3.2 Static equilibria.

The following proposition describes the general characteristics of strict Nash equilibria in a network with endogenous decay.

***Proposition 2:** Let $\bar{s} = \{g, a\}$ be a strict Nash equilibrium. Then the network is essential, connected and all agents are coordinated on the same action. Moreover if $k < 1$ the chosen action could be $\boldsymbol{a}$ or $\boldsymbol{b}$, otherwise only $\boldsymbol{b}$ can be chosen.*

The proposition describes two important features of strict Nash equilibria: aggregation and conformity. A state with two or more separated components is not a strict Nash equilibrium as well as a state with agents coordinated on different actions. Note that we do not exclude the existence of Nash equilibria in which agents are choosing different actions in the same component (aggregation without conformity). We only say that these equilibria are not strict Nash. More in detail, for k<1, networks with agents coordinated on a could be strict Nash equilibria as well as networks with agents coordinated on b. On the contrary, for k>1 only networks with agents coordinated on b could be strict Nash. A sub-network $g' \subset g$ is called *a*-group, where $a \in \{a, b\}$, if $\forall i \in g'$ is an *a*-agents and, for all $i, j \in g'$, $i \neq j$, there exists a path in g′ connecting *i* and *j* and does not exist a direct link between an agent in g' and one *a*-agent in $g \setminus g'$. With this definition in hand we can prove the proposition.

Proof: The proof goes in two steps. In the first one we prove that all strict Nash equilibria with conformity are essential and connected. In the second step we show that a strict Nash equilibrium without conformity does not exists. Step 1. Assume a strict Nash equilibrium where all agents are choosing action a. From proposition 4.2 in Bala and Goyal [1] we know that when $k<1$ only g^{cs} is a strict Nash equilibrium. In this state every agents obtain a strictly positive payoff. Changing action an agent could obtain a payoff's proportional reduction at least of $1-e$. Then the considered state is a strict Nash equilibrium. For $k>1$, from proposition 4.2 in Bala and Goyal [1] we know that the unique candidate to be a strict Nash equilibrium is g^{e}. But in our model this network is never strict Nash because the player can switch to other action obtaining the same (zero) payoff. Now, assume a strict Nash equilibrium where all agents are choosing action b. The proof for essentiality and connecteness derives directly from proposition 5.3 in Bala and Goyal [1]. If an agent switches to action a, she obtain a zero payoff. Then the considered state is a strict Nash equilibrium. Step 2. Consider any strategy profile without conformity where n' agents are choosing action a, n'' agents are choosing b and n'>1 [3]. Using the same arguments as in proposition 4.2 in Bala and Goyal [1] we know that, if k<1, in a

[3] The special case with $n'=1$ cannot be a strict Nash equilibrium because the unique a-agent obtains zero payoff: she could change action obtaining at least a zero payoff (for example, if she changes action and does not link with anyone obtain zero payoff).

strict Nash equilibrium, all a-agents have to link among themselves in a g^{cs}. Suppose that b-agents are in one or more separated components. This state cannot be a strict Nash equilibrium because any b-agent, forming a link with an a-agent, could obtain a payoff of $n' \cdot e - k$ that is strictly positive given (2.2). All connected states where a-agents support links with b-agents cannot be strict Nash equilibrium either because a-agents obtain a negative net payoff from their links with b-agents [4]. Finally, all connected states where b-agents support links with a-agents cannot be strict Nash equilibrium because b-agents are indifferent among which a-agent to be tied. In the case for $k \geq 1$, in a strict Nash all agents must be b-agents because a strict Nash equilibrium for a-agents does not exist. QED.

3.3 Dynamic

In this section we describe the dynamic properties of different equilibria. First we study the no-perturbed dynamic in the complete game. After we provide a complete description of all equilibria that are stochastically stable.

Proposition 3: *There exists a $\bar{n}(k,e)$ such that for $n > \bar{n}(k,e)$ the dynamic process converges with probability 1 to a state characterized by an essential and connected network and with all agents coordinated on the same action or to a state with an empty network.*

If k<1 the system goes in a state where all agents are coordinated on a and the network is essential and minimally connected or in a state with all agents coordinated on b and connected network. If k>1 the system goes in an empty network or in a state with all agents coordinated on b and connected network.

Proof: The proof relies on showing that from any network there is a positive probability to transit to an absorbing set in a finite time. The results will follow from the standard theory of Markov chains. The proof goes on two steps. In the first one we show as, starting from any initial state, the process transits to a connected network or to g^e in finite time. In the second step we show as, starting from any connected network with heterogeneity, the system goes in a state with conformity.

<u>Step 1</u>: In this step we use the result stated in the following lemma.

[4] an a-agent supporting one link with a b-agent obtains a negative payoff of -k.

Lemma 1: *Let be $k > e$. Consider any initial state in which all agents chose action b. Then if:*

a) $e < k < 1$ the dynamic process converges to a connected state with all agents coordinated on action a or b;

b) if $k > 1$ the dynamic process coverges to a connected state with all agents coordinated on action b or to g^e;

The proof is in appendix.

Assume $k < e$. In any state the best response for b-agents is to be tied, directly or indirectly, with all others, while for a-agents is to be tied with all other a-agents. Then a no-connected network cannot be an absorbing state. Assume a network with h components ($h \geq 1$), n' agents choosing action a and n'' agents choosing action b. If $n' \geq 2$[5] and $e < k < 1$ the best response of a-agents is to be tied in unique a-group or to switch to action b. If the case is the first one the best response of b-agents is to be directly or indirectly tied to the a-group (switching action or not) and the system goes in a connected network. If the case is the second the system could go in a state with only b-agents. Using the result stated in the part a of lemma 1 we are able to demonstrate the convergence in a connected state. Assume $k > 1$. Giving repeatedly the chance to revise the strategy only to a-agents, they delete their links between them. The proof of this result is omitted because use similar arguments than in theorem 4.1 in Bala and Goyal [1]. Then, from the result stated in part b of lemma 1 we know that the system will go in connected state with all agents coordinated on action b or in g^e.

<u>Step 2</u>: In this step we use the result stated in the following lemma.

Lemma 2: *Let be $e - e^2 < k < e$. Then in a b-group of an absorbing state, the maximum distance between 2 player is bounded above by the minimum integer value of l such that $k < e - e^{l+1}$.*

The proof is in appendix.

Now we prove as, starting from a connected network g with n' á-agents and n'' b-agents, the system converges towards the conformity for different levels

[5] In the special case with $n' = 1$, the unique a-agent has in the set of her best responses to switch action, given that her payoff is zero and that switching action obtain at least zero payoff; therefore the proof follows lemma 1.

of link cost with positive probability. Then we show that the probability to go from a state with conformity to another without, is zero. From step 1 results that only if $k < 1$ the system could converge to a connected network characterized by no conformity. Therefore we discuss only the case for $k < 1$. Assume that $k < e - e^2$. The dynamic process converges in a state where every b-agent is directly linked with all other b-agents and with the a-group. We note that b-agents are indifferent to choose a specific a-agents or another to form a link and any a-agent is chosen with equal probability. Therefore exists a positive probability that best response dynamic converges to a state where all b-agents are linked to the same a-agent. The payoff of an a-agent receiving links from all b-agents and supporting x links, from choosing action a is:

(2.5) $$\Pi(a) = (n' - 1) - x \cdot k$$

while choosing action b obtains:

(2.6) $$\Pi(b) = (n' + n'' - 1) \cdot e - x \cdot k$$

From the stability condition, $\Pi(a) > \Pi(b)$, we obtain:

(2.7) $$n' > \frac{n'' \cdot e}{1 - e} + 1$$

Now we consider a b-agent. Among all complete network architectures the larger possible payoffs is:

(2.8) $$\Pi(b) = (n' + n'' - 1) \cdot e - k$$

Switching to action a the payoff is:

(2.9) $$\Pi(a) = n' - k$$

From the stability condition, $\Pi(b) > \Pi(a)$, we obtain:

(2.10) $$n' < \frac{(n'' - 1) \cdot e}{1 - e}$$

that is incompatible with the conditions (2.7). Finally we note how in a state with conformity to switch action is never a best response because it decrease the payoff (from a to b) or produce a zero payoff (from b to a).

Assume that $e - e^2 < k < e$. In this case the dynamic process converges in a state characterized from only one b-group given that to form a link with a no connected agent yields a positive payoff. The network architecture of b-group is not well defined. The stability condition for a generic b-agent i, such that she has no incentive to switch action, is:

(2.11) $$\Pi_i(b)+d\cdot n'>n'-1$$

where $\Pi_i(b)$ is the net payoff deriving from the b-group and $d\le e$ depending on i is directly or indirectly linked with the a-group. Follows that n' is bounded above by:

(2.12) $$n'<\frac{\Pi_i(b)+1}{1-d}$$

The stability condition for a generic a-agent j such that she has no incentive to switch action, is:

(2.13) $$n'-1>(n'-1)\cdot e+\Pi_j(b)$$

where $\Pi_j(b)$ is the better possible net payoff deriving from the b-group.

Then, n' is bounded below by:

(2.14) $$n'>1+\frac{\Pi_j(b)}{1-e}$$

Increasing the number of agents in the network have to increase both n' and n''. We note that it is impossible to increase n'' without increase n'. Indeed from lemma 2 we know that in an absorbing set payoff deriving from the b-group due to an agent more is bounded below by[6] $e-k$. Then increasing n'', to satisfy condition (2.14), n' have to increase too. Viceversa, increasing n', to satisfy condition (2.12), n'' have to increase too. Using this consideration we find that exists a value of n, denoted by $\bar{n}(k,e)$, such that (to satisfy the equilibrium conditions) must be $n'>\frac{k}{e-e^2}$ for all $n>\bar{n}(k,e)$. In this situation all b-agents have as best response to form a direct link with a-group. The stability condition for an a-agent receiving links from all b-agents and that for a b-agent with the greater possible payoff are the same than in previous case, (2.7) and (2.10), that are not compatible.

Finally we have to consider the case $e<k<1$. Using the same argumentation of previous case, we can say that exists a value of n, denoted by $\hat{n}(k,e)$, such that for all $n>\hat{n}(k,e)$ to satisfy the equilibrium conditions must be $n'>\frac{k}{e-e^2}$. In

[6] From lemma 2, adding an agent more at a distance larger than l from agent i, this can improve her payoff at least of e-k supporting a new link. If the new agent stays at a distance smaller than l from agent i, this receive a payoff larger than e-k.

this state all b -agents have as best response to form a direct link with a -group. Then, computing as in previous case the equilibrium conditions, we obtain the incompatible conditions (2.7) and (2.10). QED.

The system converges to an equilibrium state in according to initial conditions. But for a given interval of relevant parameters *(k, e, n)* we are not able to produce a full description of equilibrium states. To select among all possible equilibria, we use the concept of stochastic stability: conditional on the chance to revise their strategy, players make mistakes and choose their strategy at random with some small probability $e > 0$.

We denote a minimally connected network by g^m. The state characterized by a g^x network with all agents coordinated on action *a*, $a \in \{a, b\}$, is denoted by $g^x(a)$. We describe the result regarding the stochastically stable states in the following theorem.

Theorem 1: *There exists a* $\hat{n}$ *such that for all* $n > \hat{n}$ *:*

a) If $k < e - e^2$ *there exists a* $\hat{k}_1(e,n)$ *such that for* $k > \hat{k}_1$ *only* $g^m(a)$ *are stochastically stable and for* $k < \hat{k}_1$ *only* $g^c(b)$ *are stochastically stable.*

b) if $e - e^2 < k < e$*, there exists* $\hat{k}_2(e,n)$ *such that for* $k > \hat{k}_2$ *only* $g^m(a)$ *are stochastically stable and for* $k < \hat{k}_2$ *only* $g^{cs}(b)$*,* $g^{ms}(b)$ *and* $g^{ps}(b)$ *are stochastically stable.*

c) If $e < k < 1$ *there exists* $\hat{k}_3(e,n)$ *such that for* $k > \hat{k}_3$ *both* $g^{cs}(a)$ *and* $g^{ps}(b)$ *are stochastically stable and for* $k < \hat{k}_3$ *only* $g^{cs}(a)$ *are stochastically stable.*

d) If $k > 1$ *there exists* $\hat{k}_4(e,n)$ *such that for* $k > \hat{k}_4$ *only* g^e *is stochastically stable, and for* $k < \hat{k}_4$ *both* g^e *and* $g^{ps}(b)$ *are stochastically stable.*

To demonstrate this theorem we use the definition of recurrent set in the sense of Samuelson (Def. 7.4 pag 220). Then we use the result of proposition 7.7 of Samuelson (pag 221): when an absorbing state belonging to a recurrent set is stochastically stable so are all other states in the recurrent set. This result permits us to simplify the computation of stochastic potential needs to find the set of stochastically stable states: we can consider only the states belonging to a recurrent set; then if exists only one recurrent set, all states belonging to it are stochastically stable otherwise, if more than one recurrent set exists, we have to compute the stochastic potential considering only the transition from any state in a recurrent set to any other state in the other recurrent set that requires the minimum number of mutation. Let S_a and S_b be 2 recurrent sets. The stochastic potential of a state $s \in S_a$ is given by $|S_a| - 1 + m_{ba} + |S_b| - 1$ where m_{ba} denotes the minimum number of mutations needs to induce a transitions from S_b to S_a and $|S_x|$ denotes the number of elements in S_x. The stochastic potential of a state $s \in S_b$ is given by $|S_a| - 1 + m_{ab} + |S_b| - 1$. Therefore, to determine the stochastically stable states is enough to consider only the terms m_{ab} and m_{ba}.

Before to continue the proof of Theorem 1 we introduce some convenient notation. Given any agent in a network *g*, we denote by q_h the number of active links she supports to players choosing action *h*, where $h \in \{a, b\}$. Similarly, r_h stands for the number of passive links received from players choosing action *h* where $h \in \{a, b\}$.

Consider part *a* of Theorem 1, that is $k \leq e - e^2$. In this range of link cost, using the result stated in proposition 3, we identify 2 candidates to be recurrent sets: S_b characterized by $g^c(b)$ and S_a characterized by $g^m(a)$.

1. Now we compute the minimum number of mutations needs to move the system from S_a to S_b and denote it by m_{ab}. The payoff from choosing action b for a player *i* is given by:

   (2.15) $$\Pi_i(b) = (n-1) \cdot e - (L + q_b) \cdot k$$

   where: L denotes the number of a-groups do not linked to agent *i*. On the other hand, the payoff from choosing a is equal to:

   (2.16) $$\Pi_i(a) = (n - 1 - m_{ab}) - L \cdot k$$

   The agent *i* prefers action b only if the following is true:

   (2.17) $$\Pi_i(b) - \Pi_i(a) = m_{ab} - q_b \cdot k - (n-1) \cdot (1-e) \geq 0$$

   The more favorable condition to induce the transition is when all b-agents are supporting a link with agent *i*, that is $q_b = 0$. Solving (2.17) in m_{ab} as an equality we find the minimum number of b-agents (mutants) needs to induce the transition to S_b that is given by:

   (2.18) $$m_{ab} = (n-1) \cdot (1-e)$$

2. Now we compute the minimum number of mutations to lead the system from S_b to S_a and denote it by m_{ba}. The payoff from choosing action b for a player i is given by (2.15). The payoff from choosing a is equal to:

(2.19) $$\Pi_i(a) = m_{ba} - L \cdot k$$

The agent *i* prefers action a only if the following is true:

(2.20) $$\Pi_i(a) - \Pi_i(b) = m_{ba} - (n-1) \cdot e + q_b \cdot k \geq 0$$

The more favorable condition to induce the transition is for $q_b = n - 1 - m_{ba}$, that is when agent i supports links with all b-agents. Solving (2.20) in m_{ba} we obtain the minimum number of mutations needs to induce the transition to S_a that is given by:

(2.21) $$m_{ba} = (n-1) \cdot \frac{(e-k)}{1-k}.$$

The states in S_a are stochastically stable if and only if $m_{ab} > m_{ba}$. Using expressions (2.18) and (2.21), we write this condition as:

(2.22) $$k > 2 - \frac{1}{e}$$

Otherwise the stochastically stable states are those in S_b. The necessary condition for only one of these sets to be stochastically stable is a sufficiently large number of agent such that the minimum number of mutation to lead the system from a recurrent set to another is greater than one and $\left| m_{ab} - m_{ba} \right| > 1$.

Consider part *b* of Theorem 1, that is $e - e^2 \leq k \leq e$. In this case, using the result stated in proposition 3 and that in Theorem 1 in Feri (2003)[7], if *n* is sufficiently large, we identify two candidates to be recurrent sets: S_a defined in point *a*, and S_b characterized by $g^s(b)$.

1. Now we compute the minimum number of mutation to lead the system from S_a to S_b and denote it by m_{ab}. The payoff from choosing action b for a player *i* is given by:

(2.23) $$\Pi_i(b) = (r_a + r_b + q_a + q_b + x) \cdot e + (y + z) \cdot d - (q_a + q_b) \cdot k$$

[7] by this we are able to concentrate ourattention only on star networks.

where x denotes the number of a-agents indirectly linked with agent i through others a-agents, y is the number of a-agents indirectly linked with agent i through b-agents, z is the number of b-agents indirectly linked with agent i through b-agents and $e-k \le d \le e^2$. On the other hand, the payoff from choosing a is equal to:

(2.24) $$\Pi_i(a) = n-1-m_{ab} - \left(q_a + L_y\right)\cdot k$$

where L_y denotes the number of a-groups do not linked to agent i (or linked through b-agents). The agent i prefers action b only if the following is true:

(2.25)

$$\Pi_i(b) - \Pi_i(a) =$$

$$\left(r_a + q_a + x\right)\cdot e + y\cdot d + L_y \cdot k + r_b \cdot e + q_b \cdot (e-k) + z\cdot d - \left(n-1-m_{ab}\right) \ge 0$$

The more favorable condition to induce this transition is for $q_b = 0$, $z = 0$, $d = e^2$, $L_y = y$ and $r_a + q_a + x = 0$. Then (2.25) becomes:

(2.26) $$r_b \cdot e + y\cdot\left(e^2 + k\right) - \left(n-1-m_{ab}\right) \ge 0$$

Note that $r_b = m_{ab}$ and $y = n-1-m_{ab}$. Then, solving (2.26) in m_{ab} we find the minimum number of b-agents (mutants) needs to induce the transition to S_b that is given by:

(2.27) $$m_{ab} = (n-1)\cdot \frac{1-e^2-k}{1+e-e^2-k}$$

We note that for $k \ge 1-e^2$ it is not possible to find a sufficiently large value of n such that $m_{ab} > 1$.

3. Now we compute the minimum number of mutations to lead the system from S_b to S_a and denote it by m_{ba}. The payoff from choosing action b for a player i is given by (2.23). The payoff from choosing a is equal to:

(2.28) $$\Pi_i(a) = m_{ba} - \left(q_a + L_y\right)\cdot k$$

The agent i prefers action a only if the following is true:

(2.29)

$$\Pi_i(a) - \Pi_i(b) =$$

$$= m_{ba} - \left(r_a + q_a + x\right)\cdot e - L_y \cdot k - y\cdot d - r_b \cdot e - q_b \cdot (e-k) - z\cdot d \ge 0$$

We note that in (2.23) agent i is optimally indirectly tied with the y a-agents if $y \cdot e - L_y \cdot k \leq y \cdot d$. Therefore $y \cdot e \leq L_y \cdot k + y \cdot d$. Then the more favorable condition for the transition is given by $y = 0$, $r_b = 0$ and $z = 0$. In this case, noting that $r_a + q_a + x = m_{ba}$ and $q_b = n - 1 - m_{ba}$, we find that the minimum number of a-agents (mutants) needs to induce the transition to S_a is given by:

$$(2.30) \qquad m_{ba} = (n-1) \cdot \frac{e-k}{1-k}$$

The states in S_a are stochastically stable if and only if $m_{ab} > m_{ba}$. Using expressions (2.27) and (2.30), we write this condition as:

$$(2.31) \qquad k > \frac{e + 2 \cdot e^2 - e^3 - 1}{2 \cdot e - 1}$$

Otherwise the stochastically stable states are those in S_b. The necessary condition for only one of these sets to be stochastically stable is a sufficiently large number of agent such that the minimum number of mutations to lead the system in an equilibrium out of the recurrent set is greater than one and $\left| m_{ab} - m_{ba} \right| > 1$.

Consider part c of Theorem 1 that is for $e \leq k \leq 1$. In this case, using the result stated in proposition 3 and that in Theorem 1 in Feri (2003)[8], for a sufficiently large value of n, we have 2 possible recurrent sets: S_a defined in point a and S_b characterized by $g^{ps}(b)$.

1. To lead the system out of S_b is sufficient only one mutation of a peripheral agent i that chooses to form links with all others. If all other agents have the chance to revise before i, the system converges towards a $g^{cs}(b)$. This structure is unsustainable and the dynamic process leads the system to a g^e. From this state there is a positive probability that system goes in S_a.
2. Now we compute the range of link cost such that is sufficient only one mutation to lead the system from S_a to S_b. First we consider the

[8] by this we are able to concentrate our attention only on star networks with links supported by peripheral agents.

conditions to lead the system from S_a to S_b. The payoff from choosing action b for a player *i* is given by:

(2.32) $\Pi_i(b) = (r_a + r_b + q_a + q_b + x)\cdot e + (y + z)\cdot d - (q_a + q_b)\cdot k$

On the other hand, the payoff from choosing a is equal to:

(2.33) $\Pi_i(a) = (n - 1 - m_{ab}) - (q_a + L)\cdot k$

where L denotes the number of a-groups do not tied with *i* or indirectly linked through b-agents. The agent *i* prefers action b only if the following is true:

(2.34)

$$\Pi_i(b) - \Pi_i(a) =$$

$$= (r_a + q_a + x)\cdot e + y\cdot d + L\cdot k - (n - 1 - m_{ab}) + r_b\cdot e + q_b\cdot(e - k) + z\cdot d \geq 0$$

The more favorable condition for the transition is when $q_b = 0$, $z = 0$, $d = e^2$, $L = y$, $r_a + q_a + x = 0$. Inserting these conditions in (2.34) and arranging for $L = n - 1 - m_{ab}$ and $r_b = m_{ab}$, we obtain:

(2.35) $(n - 1 - m_{ab})\cdot(e^2 + k - 1) + m_{ab}\cdot e \geq 0$

Solving (2.35) in k assuming $m_{ab} = 1$ we find the range of link cost where to induce the transition to S_b is sufficient only one mutation, that is given by:

(2.36) $k \geq 1 - e^2 - \dfrac{e}{n-2}$

That for large values of n can be approximated by $k \geq 1 - e^2$.

Follows that if the condition (2.36) is not satisfied only states in S_a are stochastically stable, while when this condition is satisfied are stochastically stable either states in S_a and S_b.

Consider part *d* of Theorem 1 that is for $k \geq 1$. In this case, using the result stated in proposition 3 and that in Theorem 1 in Feri (2003)[9], for a sufficiently large value of *n*, we have 2 possible recurrent sets: S_e characterized by g^e and S_b characterized by $g^{ps}(b)$. To lead the system from a state in S_b to an empty network is sufficient only one mutation (see part c). On the contrary to conduce

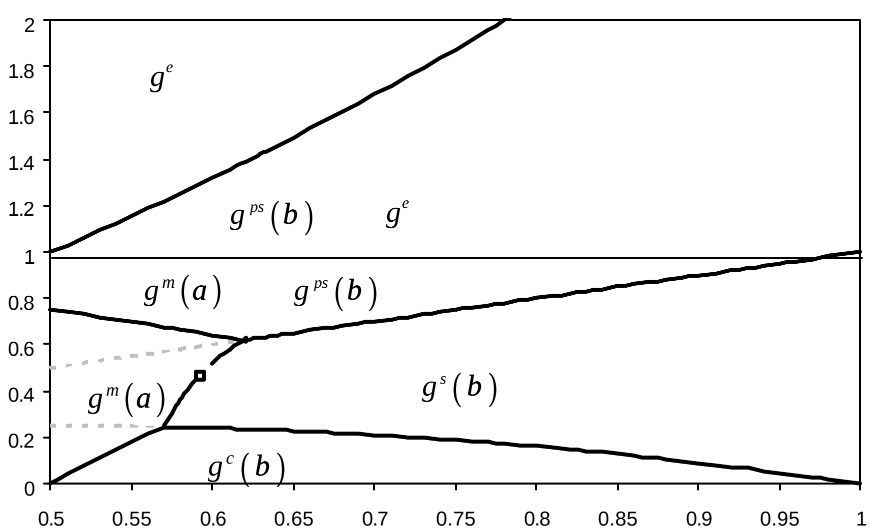
[9] by this we are able to concentrate ourattention only on star networks.

the system from an empty network to a state in S_b needs only one mutation if and only if the cost k is relatively lower. Consider an empty network. A mutating agent i chooses action b and forms links with n_1 agents. If $n_1 \geq \frac{k-e}{e^2}$ all no connected agents will choose to form a link with agent i as soon as possible. If there are at least other n_1 agents without links revising their strategy before than the central agent will do, the system will have a star network with at least n_1 links supported by peripheral agents. From this state the no perturbed dynamic leads the system to a star network with links supported by peripheral agents. The condition on the link cost k that permits this transition with only one mutation is:

(2.37) $$k \leq e + \frac{n-1}{2} e^2 .$$

Therefore if (2.37) is satisfied, both $g^{ps}(b)$ and g^e are stochastically stable states, otherwise only g^e are stochastically stable states.

We consider the stochastically stable states as the states where the system spends most of the time. With this consideration in hand we explain the main characteristics of the model using the following figure where we display the stochastically stable states in according to the levels of e and k (x-axis displays e and y-axis displays k).



We note as efficient states are stochastically stable mainly for small values of *e*. Intuition is that if the premium to play efficient action is small the rational individuals spend most of the time on the risk dominant action or that is more probable that the system converges in a state characterized by individuals coordinated on risk dominant action. The second feature of the model is that, given a sufficiently low value of *e*, efficient states are stochastically stable only for intermediate values of k. Indeed for small values of k, the advantage to be coordinated on the efficient states, deriving from a smaller number of links, is lower. On the other side for large values of k coordination problems seem to play an important role to rule out the efficient states.

4. Conclusion

In this paper we have analyzed in a stylized form a social network characterized by an endogenous network architecture. We have considered the variety of networks with decay and we have modeled this as endogenous. The main result regard the analysis of equilibria. In our frameworks decay is assumed to depend on the actions chosen by agents participating to the network. We assume that agents can choose among two actions: one is efficient while the other is risk dominant. The empirical counterpart is the trade off between efficient technologies and compatible technologies.

In this model we have a greater number of equilibria and we are not able to produce a full description of them; on the contrary we are able to produce a full characterization of the subset of stochastically stable states. Differently from the results in Jackson and Watts [12] and Goyal and Vega Redondo [9], in our model the network structure depends on which action individuals coordinate. Moreover efficient states are stochastically stable for intermediate levels of link cost and not for high levels. The first difference in the results is due to the fact that the social game determines the decay level and not only the payoff. Therefore this result follows those in Jackson and Wolinsky [13] and Bala and Goyal [1]. The fact to have different network structures in according to the action on which individuals coordinate, affects the dynamic selection given that every network structure produces different esternalities on the participants. This explains the difference in the results regarding the stochastically stable states too.

Further development can be made in many directions. First, we might consider a model with two-side link formation: this is more similar to real world and it may change the result on stochastic stability. Second, we might use a setting where small deviations from the best response are more probable that the large ones. Third, we can model the endogenous decay with different social game that can be more respondent to different empirical situations. Finally, we could study applications regarding the diffusion of technologies and the hierarchical and social structure in the enterprises and firms.

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Appendix.

Proof of lemma 1.

Given any *g*, let be $M(i;g)=\{i\in N: g_{ij}=1 \text{ at least for one } j\in N, j\neq i\}$ and $L(i;g)=\{i\in N: g_{ij}=0\ \forall\ j\in N\}$. Give the chance to revise only to agents $i\in M$. After each revision $|M|$ decreases or does not change while $|L|$ increases or does not change. Therefore the system goes in a state where M is empty (g^e) or in a state where $|M|$ and $|L|$ do not change. In this last case $\forall i\in M$ obtains a positive net payoff from her (link) strategy and if $|M|\geq 2$ all agents $i\in M$ are in the same component. Suppose more than 1 component; each agent in one given component can add to its current links the links supported by any player in another component and, by doing so, obtains an additional payoff. We note as g^e never is an equilibrium for $k<1$. In this state agents with the chance to revise choose the action randomly and when happen that one chooses action a, all revising agents will have as best response to choose action a and to be tied in an unique a-group (connected network). On the contrary, when $k>1$ g^e is an equilibrium. QED.

Proof of lemma 2.

Suppose that in a generic state there are 2 agents indirectly linked through $l+1$ links. Then, one of two agents can improve her payoff forming a direct link with the other agent given that $e^{l+1} < e - k$ .QED.