Matching and Playing Games

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Abstract: We examine a new class of games where players not only choose strategies but also choose with whom they play. A group of players who are dissatisfied with the play of their current partners can join together and play a new equilibrium. This imposes interesting new refinements of equilibrium play, where play depends on the relative populations of players in different roles among other things.

We also examine finite repetitions of games where players may jointly choose to rematch in any period. Some equilibria of fixed-player repeated games cannot be sustained as equilibria in a repeated matching game. We show that all renegotiation-proof equilibria of fixed-player games have corresponding matching equilibria if all roles have the same size population of players. If roles have different size populations of players then it is possible that no renegotiation proof equilibrium has a corresponding matching equilibrium. The set of matching equilibria also includes some plays that are not part of any equilibrium of the corresponding fixed-player repeated games.

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1. Introduction

In many social and economic interactions, players have choices not only of what actions to play, but also with whom they interact. For instance, if an employee does not like the behavior of his or her employer, he or she can quit and work for another firm. Similarly, dissatisfied employers can fire their employees and hire new ones. This ability to rematch has strong implications for behavior within the relationships. While this is a relatively obvious statement, we have no systematic method of modeling the play within a game when such play depends on players' ability to rematch. In this paper we introduce such a methodology and show that it has strong and intuitive implications for behavior. We examine a new class of games where players not only choose strategies but also choose with whom they play.

We examine two situations: one where the choice of matching is made just once, and another where the interaction occurs over a finite number of periods and players may rematch in any period. In the one-shot version of the game, a ``matching equilibrium'' consists of a matching of players into various groups who will each play the game together, as well as a description of what each player will play. This must satisfy two requirements: first, the play of each group must be a Nash equilibrium; and second, no set of players would all improve by leaving their current groups, forming a new group, and playing some other Nash equilibrium. In the finitely repeated version of the game, a ``matching equilibrium'' includes both a specification of what each player will play given each possible history of matching and play (by all players), as well as a specification of who is matched with whom given each possible history. The equilibrium definition is an inductive one. It requires that no group of players could jointly deviate and play a different matching equilibrium in the continuation and all improve their payoffs. We provide two different definitions of repeated matching equilibrium depending on how we treat the possibilities for rematching of other players.

Our results explore the existence of rematching equilibria, as well as their structure. We show that rematching equilibria generally exist in bipartite settings - where the matchings of players are into pairs. However, when the matchings of players are into groups of three or more players rematching equilibria may not exist, unless players care only about the play of the game and not the identities of the players with whom they are matched. We also show that the requirement that no group of players desire to leave their current group and match together has strong implications for the play in the game. It implies that only Nash equilibria which are not strictly Pareto dominated by other Nash equilibria can be played, and in fact can result in a strict selection from that set, even when the population of players is completely evenly matched. We also show that play can depend on the relative populations of players available for different roles in the game, with the selection among equilibria favoring players who are less populous.

When the game is finitely repeated, and rematching is possible in any period, we show that existence depends on whether deviating players are allowed only to rematch once and for all, or whether they can suggest more complicated rematching plans that include other players outside of their group. The possibility of rematching results in a set of equilibria that is neither a superset nor a subset of the set of fixed-matching subgame perfect equilibria. In particular, through rematchings players can be rewarded and punished with payoff combinations that cannot be achieved in fixed-matching games. This changes the structure of play that can be sustained.

Our model relates to two strands of literature. One obvious strand is the matching literature that followed the seminal paper of Gale and Shapley (1962) and is detailed in Roth and Sotomayor (1989). That standard matching world is the special case of our model where the game played between players is degenerate so that each player has only one action available to them so that players' payoffs depend only on with whom they are matched. We show that well-known results on existence and the lattice structure of matchings from the bipartite matching world have analogs in our setting when the game is bipartite.

Another strand of literature that relates to our work concerns renegotiation-proof equilibria in finitely repeated games (e.g., Bernheim and Whinston (1987), Farrell and Maskin (1989), and Benoit and Krishna (1993)). This corresponds to the other extreme of the model where there is only one group of players. In that case, our definitions correspond exactly to renegotiation-proofness. When there are multiple possible matchings, then the relationship between our equilibria and renegotiation-proof equilibria depend on the setting. If it is possible to match all players at the same time and all players are homogeneous within each role, then our equilibria are a superset of the set of renegotiation-proof equilibria. However, in cases with an imbalance in the possible groupings of players, our equilibria can differ completely from the set of renegotiation-proof equilibria, and in fact include plays that are not supported by any previous equilibrium concept.

Other related papers include those that study network formation in coordination and anti-coordination games (e.g. Jackson and Watts (2002), Droste, Gilles, and Johnson (2003), and Goyal and Vega-Redondo (2004)). In those models a given player plays the same coordination game with every agent to whom he or she is connected to in the network. Players are allowed to occasionally update their links in the network and the strategy that they play, generally with some random mutations added. These network papers examine social coordination games where players want to coordinate actions with everyone they interact with on a network whereas here the type of game played between agents is more general and the game can be played between more than just two players, and as in the matching context (of marriages and hospital-interns) each agent plays at most one game at a time.

There are other papers showing that endogenous interactions can lead to efficient play. Mailath, Samuelson, and Shaked (2001) examine endogenous interactions in a local interactions model (where each agent plays the same game against multiple opponents) and show that if agents have the ability to seclude themselves from undesirable opponents then the asymptotically stable outcomes are both efficient and homogenous, assuming that the game played has one Nash equilibrium that strictly Pareto dominates all other correlated equilibria. Rob and Yang (2003) examine endogenous formation of long term relationships where partners play a Prisoner's Dilemma game with each other. They show that heterogeneity of types (where some types always cheat, some are always honest, and some are opportunistic) combined with endogenous interactions leads to the good equilibria.

2. The Basic Model

Given is a normal form game with **player roles** denoted by $i \in N=\{1,...,n\}$.

There is a **population**, P_i, of players who are of role i. For instance, P_1 would be all of the women and P_2 would be all of the men in the society if the game is the battle of the sexes. Let $P = P_1 \times ... \times P_n$ be set of all vectors of players consisting of one player of each role. We use i,j, and k to denote indices of different player roles. We use a,b, and c to denote generic players. We denote generic elements of P by p, and we abuse notation and write $c \in p$ to indicate that player c is in the vector of players p.

Let n_i be the cardinality of P_i, and order player roles so that $n_i \geq n_k$, whenever i>k.

Each player role i has a strategy set S_i, and a player $c \in P_i$ in that role gets a payoff $u_c(s,p)$ if s is the vector of strategies that is played and c is matched in group p, where s is in $S = S_1 \times ... \times S_n$. The payoff function u_a is a von Neumann-Morgenstern utility function. Mixed strategies for a player in role i are denoted m_i in $\Delta(S_i)$. We let $u_c(m_p,p)$ denote the expected utility for a player c who is in the vector of players p when the description of all players strategies is given by the vector of mixed strategies m, and m_p denotes the n-vector of mixed strategies played by the players in p.

Note that payoffs can depend on both the strategies and the set of players that a given player is matched with.

A **matching** is a mapping f from $\cup P_i$ into $(\cup P_i) \cup P$, such that
(i) either f (a)= a or $f(a)=p \in P$ such that $a \in p$, and
(ii) if f(a)=p and $b \in p$, then f(b)=p.

The interpretation is that f(a) is the set of players that a is matched with and will be playing with.

We normalize things so that the payoff to an unmatched player is 0. Given a mixed strategy m profile for all players and a matching function f, let $U_c(m,f)$ be the expected utility that player c gets if the matching f is in place and m is played.

A **matching equilibrium** is a mixed strategy profile m, and a matching function f such that

(a) if $f(c)=p\in P$ for some player c, then m_p is a Nash equilibrium and $u_c(m_p,p)\geq 0$, and
(b) there does not exist $p\in P$, and a profile of strategies m'_p for the players in p such that $u_c(m'_p,p)>U_c(m,f)$ for all $c\in p$ and such that m'_p is a Nash equilibrium.

We start with a simple example to illustrate the definition.

Example 1: Battle of the Sexes with Uneven Populations

There is one woman $P_1=\{1\}$ and two men $P_2=\{2,3\}$. The woman is in the row player role, while the men are in the column role and are both identical. The payoffs to the players is described by the following matrix (and the woman's payoff is independent of which man she plays with).

	A	B
A	1,3	0,0
B	0,0	3,1

There are three Nash equilibria to the game: the pure strategy equilibria (A,A) and (B,B), and a mixed strategy equilibrium where the man plays A with probability 3/4 and the woman plays A with probability 1/4.

There are two matching equilibria: one with a matching of f(1)=f(2)=(1,2), and another with a matching f'(1)=f'(3)=(1,3). In both equilibria the matched couple plays (B,B).

The other two Nash equilibrium strategies are not part of any matching equilibrium, as for instance, under f where 1 and 2 are matched, if the intended play is not (B,B) then players 1 and 3 can deviate to match and play (B,B) and both be better off.

Note that this example also illustrates that, in order to guarantee existence of equilibrium, it is necessary that a deviation can only block a proposed matching equilibrium if the deviating players are all strictly better off. With a weaker notion of blocking, where only some of the deviating players need to strictly benefit, equilibrium would fail to exist in the above game.

3. Matching Equilibria when Players Within Roles are Identical

We first examine the class of games where players *within* a given role are all identical. This means two things:

- if $b \in P_i$ and $c \in P_i$ then $u_b = u_c$, and
- $u_b(s,p)$ is independent of p.

Thus, players in the same role have the same preferences, and also players do not care about the players they are matched with - only how those players behave. We call these matching *games with homogeneous populations*. Note that this does not require that players from different populations be similar, only that players who might play the same role be similar.

3.1 Existence of Equilibrium

We now show that the set of matching equilibria is nonempty and compact. Compactness is important in establishing the existence of repeated matching equilibria, which we explore below.

Theorem 1: The set of matching equilibria of a matching game with homogeneous populations is nonempty and compact.

Proof: Let us first show that the set of matching equilibria is nonempty. Order player roles so that $n_i \geq n_k$, whenever i>k. Let NE_1 be the set of mixed strategy Nash equilibria that reach maximal payoff for the player role 1, subject to all other players getting at least 0. Let NE_2 be the subset of those that maximize player 2 types utilities, subject to being in NE_1. Inductively, let NE_k be the subset of those that maximize player role k's utility, subject to being in NE_{k-1}. If NE_n is empty, match all players to themselves. Otherwise, select any matching f such that players in P_1 are all matched and pick m such that all players play their role's component from some mixed strategy profile in NE_n (so that all players in the same role play the same mixed strategy). This forms a matching equilibrium.

Next, let us argue that the set of equilibria is compact. Given the finite set of possible matchings and the compact nature of the strategy spaces, we need only show that the set of strategy profiles that are part of an equilibrium for some given matching is compact. Let $m^r \rightarrow m$, where (m^r,f) is a matching equilibrium for every r. It is immediate that (a) is satisfied by m. We need only verify that (b) is satisfied by m. Suppose to the contrary that there exists $p \in P$, and a profile of strategies m'_p for the players in p such that $u_c(m'_p) > U_c(m,f)$ for all $c \in p$ and such that m'_p is a Nash equilibrium. Then for large enough r, it follows that $u_c(m'_p) > U_c(m^r,f)$ for all $c \in p$, which is a contradiction.◊

While the set of matching equilibria is compact, we note that the matching equilibrium correspondence (as payoffs are varied) is *not* upper hemi-continuous. This is in contrast with many equilibrium correspondences of fixed-player games, such as Nash or perfect equilibrium. The failure of upper hemi-continuity is illustrated in the following example.

Example 2: Failure of upper hemi-continuity

There are two players and two player roles. Player 1 (the row player) has only one strategy, while player 2 (the column player) has two strategies: {left, right}. The payoffs are as follows.

left	right
1 , 1	-1, 1+1/k

For any k, there is a unique matching equilibrium which is to have both players remain single, as the only Nash equilibrium is ``right'' which gives player 1 a negative payoff. In the limit, ``left'' is a Nash equilibrium, and there is a matching equilibrium where both players are matched and get payoffs of 1. This means that both players remaining single is no longer a matching equilibrium in the limit, as then (b) is violated.

3.2 Characterization of Equilibrium

Let PO(k) represent the set of k-vectors of utility $(u_1', u_2', \ldots, u_k')$ such that

- player roles {1,…,k} receive a nonnegative payoff, and
- if m is a Nash equilibrium such that all player roles get a positive payoff, then there exists $i \leq k$ such that $u_i' \geq u_i(m)$.

Given a matching and mixed strategy profile (f,m) of some matching game with homogeneous populations, for each i let

$$v_i(f,m)=\min_{\{c\in Pi,\ f(c)\neq\{c\}\}} u_c(m_{f(c)}).$$

Thus, v_i is the minimum utility obtained under (f,m) by any player in role i who is matched under f.

We now offer a complete characterization of the set of matching equilibria.

Theorem 2: Consider a matching game with homogeneous populations. Suppose that there exists at least one Nash equilibrium such that all player roles have a positive payoff[1] and let k be the smallest i such that $n_{i+1} > n_i$, letting k=n if there is no such i. Then (f,m) is a matching equilibrium *if and only if*

[1] The ``if'' part of the statement still holds if there exists a Nash equilibrium where all player roles get a nonnegative payoff (as opposed to positive). Clearly, if in every Nash equilibrium some player role gets a negative payoff then all matching equilibria have all players unmatched.

(i) all players in P_1 to P_k are matched,

(ii) each matched group p plays a Nash equilibrium m_p such that all players get a nonnegative payoff, and

(iii) $(v_1(f,m), v_2(f,m),\ldots, v_k(f,m)) \in PO(k)$.

Proof: Let us first show the ``if'' part. By (ii) m_p is a Nash equilibrium for any matched group p under f and payoffs are nonnegative and so (a) of matching equilibrium is satisfied. (b) is satisfied since by (iii) any deviating group p with deviating strategy m'_p must have some player role $i \leq k$ for whom $u_i(f,m) \geq v_i(f,m) \geq u_i(m'_p)$ (by the definition of PO(k) and since all players in roles 1 to k are matched).

Next, let us show the converse. If not all players in roles 1 to k are matched, then there must be some complete group p of players who are not matched. Consider any equilibrium m'_p that gives positive payoffs to all players (and such a Nash equilibrium exists by the assumption of the theorem). We then contradict (b) in the definition of matching equilibrium, as all players in p can strictly benefit by forming a group playing m'_p. Thus, all players in P_1 to P_k are matched and so (i) holds. Next, note that (ii) follows directly from (a) in th definition of matching equilibrium. Finally, let us show (iii). Suppose to the contrary that $(v_1(f,m), v_2(f,m),\ldots, v_k(f,m)) \notin PO(k)$. Consider a group p consisting of a player from each player role 1 to k who is obtaining $v_i(f,m)$. Since that $(v_1(f,m), v_2(f,m),\ldots, v_k(f,m)) \notin PO(k)$, along with players in roles above k that were unmatched under f (if k<n). It follows from the definition of PO(k) that there exists $m'_p \in NE$ that gives all players in p a positive payoff and makes all the players in the roles 1 to k in p strictly better off than under (f,m), and players in roles above k better off than being unmatched. This contradicts (b), and so the supposition was incorrect, implying (iii). ◊

We saw an illustration of Theorem 2 in Example 1, where the uneven matchings of the populations meant that the woman's favorite Nash equilibrium was the only possible play in a matching equilibrium. That example generalizes as follows.

Corollary: Consider a matching game with homogeneous populations where population 1 is the smallest ($n_1 < n_2$), and where one of player role 1's most preferred Nash equilibria gives all player roles a positive payoff. In any matching equilibrium, all players of role 1 are matched and all groups of players play one of player role 1's most preferred Nash equilibria.

The following example shows that in a matching equilibrium the play across groups is restricted even in situations where the populations of different player roles are exactly balanced.

Example 3: Matching Equilibrium with Evenly Matched Populations

This is an extension of the Battle of the Sexes example where there is a third strategy available, but now where the populations of players are evenly matched. Let there be two men and two women who play the following game.

	A	B	C
A	1,3	0,0	0,0
B	0,0	3,1	0,0
C	0,0	0,0	2,2

At a matching equilibrium all agents are matched and it is possible for both groups to play A,A or both to play B,B or both to play C,C. It is also possible for one group to play A,A and for the other to play C,C (or one to play B,B and the other to play C,C). However, if is not possible for one group to play A,A and the other to play B,B. Here the woman from the first group and the man from the second group can form a group and play B,B thus making both players strictly better off.

3.3 A Single Population of Players

We now briefly consider a variation of the model of games with homogeneous populations, where in fact there is just one population. That is, any set of n players may be grouped together and all players are ex ante identical. The definition of matching equilibrium extends in the obvious way.[2] We call these *single homogeneous population matching games.*

The following example shows that once we are in a world with a single population of players, even in two-player games existence of a matching is no longer guaranteed.

Example 4: Nonexistence in a single homogeneous population matching game:

There are three players and a game involves two players.

	A	B
A	0,0	1,2
B	2,1	0,0

There does not exist any matching equilibrium. Any equilibrium would necessarily have two players matched and play either (A,B) or (B,A) as the mixed strategy equilibrium is strictly Pareto dominated by either pure strategy equilibrium. However, in any such matching the player getting the lower payoff can deviate together with the unmatched player and both be made better off (by playing the equilibrium that is less favorable to the formerly unmatched player and more favorable to the previously matched player).

[2] Simply let P in the definition be the set of all vectors of n players.

This example shows that it is important for existence in single homogeneous population matching games that there exist a matching that include all players. In situations where there does exist a matching including all players, then there does exist an equilibrium, as described in the following proposition. Moreover, the proposition shows that such a matching equilibrium will have ``most'' groups playing the symmetric Nash equilibrium, if one exists and is not Pareto dominated by another Nash equilibrium.

Proposition 1: Consider a single homogeneous population matching game such that there exists a matching that includes all players.

- There exists a matching equilibrium:
  - If every Nash equilibrium yields a negative payoff for at least one player, then all matching equilibria have all players unmatched.
  - If there exists a Nash equilibrium with nonnegative payoffs for all players, then any situation where all players are matched and play some equilibrium (the same across groups) that is not strictly Pareto dominated by any other Nash equilibrium is a matching equilibrium.
- If the game has a Nash equilibrium that gives all players the same payoff u>0 and is not weakly Pareto dominated by another Nash equilibrium, then in every matching equilibrium all players are matched and all but at most n-1 groups play an equilibrium that gives all players a payoff of u.

The proof is straightforward and left to the reader.

4. **Heterogeneous Players within Populations**

Let us now return to the general case, where players within the same role can be heterogeneous. This means that players within a given role might have different utility functions, and also that players might care about the identity of the other players with whom they are matched. For instance, in a battle of the sexes game different men might have different preferences over plays of the game, and any given man might have preferences over the play of the game that depend on which woman he is matched with. Each group of players can have different sets of Nash equilibria and Nash equilibrium payoffs.[3] So, we return to the case where player's utility functions are indexed by player (not simply player role) and depend on the matching. where player c's payoff from strategy s and a matching of p is denoted $u_c(s,p)$.

Let NE(p) denote the set of Nash equilibria for the group of players p.

[3] This is clearly a generalization of the usual matching world. If each group had a single Nash equilibrium in their game, then we would be in the standard matching world. If, however, they have multiple Nash equilibria, then the problem becomes more intricate, as now the preferences are not uniquely defined.

Let $\geq_i$ be the partial order (reflexive, transitive and antisymmetric) over possible strategy-matching pairs defined by saying that $(m,f) \geq_i (m',f')$ if all players in role i weakly prefer (m,f) to (m',f').

Let us say that a matching equilibrium f is Player role i-optimal if $(m,f) \geq_i (m',f')$ for all matching equilibria f'.

Theorem 3: If there are two player roles, then there exists both a Player role 1-optimal matching equilibrium, and similarly there exists a Player role 2-optimal matching equilibrium.

Proof: Let us refer to player role 1 as men and player role 2 as women. To find the man-optimal matching equilibrium we extend the Gale-Shapley deferred-acceptance algorithm, where a man proposes to a woman and also proposes a Nash equilibrium to be played by the couple. Let each man rank all the Nash equilibria from playing the game with every possible woman, where the man discards any Nash/woman pair which gives him a negative payoff. Artificially break ties, so that we have a strict ranking over acceptable mates and equilibria for each man, and similarly for each woman. The algorithm is as follows. First, each man simultaneously proposes to his best Nash/woman pair (i.e., he proposes to this woman and proposes that they play this particular Nash equilibrium). Each woman then reviews her options and accepts the proposal of the man/Nash pair she likes best. If there is no proposal which gives her a nonnegative payoff, then all proposals are rejected. In the second round each currently unmatched man proposes to his second best Nash/woman pair. Again the women each accept their best acceptable proposal, where now a proposal from the first round is rejected if a woman receives a better proposal in the second round. This process continues iteratively, where each time a man is unmatched he proposes the best acceptable woman/Nash pair that he has not yet proposed, or else makes no proposal. The process ends when all unmatched men have exhausted their acceptable proposals. This process must end at a matching equilibrium: By construction, (a) of matching equilibrium is satisfied. The argument that (b) must also be satisfied is as follows. If there is a man who would prefer to be matched with someone else than his current mate and/or would prefer to play a different Nash equilibrium, then it must be that he already proposed this Nash to this woman and that at some prior step she turned him down, which means she had a better (or equivalent) offer. As the woman's ending match must be at least as good as the one she had at that time (by the structure of the algorithm), this woman would not be made better off by leaving her current Nash/man for this Nash/man pair. Thus, (b) is satisfied. A woman-optimal matching equilibrium can be similarly constructed. ◊

It is clear that Theorem 3 cannot be extended to settings with more than two player roles. This follows from similar reasoning as multi-partite matching literature, as illustrated in the following example.

Example: Consider 6 players in 3 player roles. Players 1 and 4 are in role 1, players 2 and 5 are in player role 2, and players 3 and 6 are in role 3. Let there be a single Nash equilibrium for each matched group of players. Let the payoffs from those Nash equilibria be as follows: (3,3,3) for groups (1,2,3) and (4,5,6); (4,4,4) for group (4,2,3); (1,1,1) for groups (1,5,6) and (4,5,3); (2,5,2) for (1,2,6); and (0,0,0) for all other groups. The only potential matchings are then (1,2,3)(4,5,6); or (4,2,3)(1,5,6); or (4,5,3)(1,2,6). Note that (4,2,3) blocks the first matching, (1,2,6) blocks the second matching, and (4,5,6) block the third matching. Thus, there is no matching equilibrium.

In the case of two player roles, there is a nice structure to the set of matching equilibria. This is well-known for the standard marriage-market setting, and turns out to extend to the matching-game setting. The following Proposition is an extension of a similar result by Conway (as reported by Knuth (1976)) for the marriage market problem. Of course, the marriage market problem is a special case of our setting where there is a unique Nash equilibrium for each matched pair of players.

Say that a strategy matching profile (m,f) is **plausible** if (a) in the definition of matching equilibrium is satisfied (m_p is an individually rational Nash equilibrium for any matched set of players p), and for any c and $f(c)=p\in P$, m_p is not Pareto dominated by any m' in NE(p).

Say that **players are never indifferent** if the payoffs to a player c from two plausible strategy-matching pairs (m,f) and (m',f') differ when either $p=f(c)\neq f'(c)$ or $m_p \neq m'_p$.

Proposition 2: If there are two roles and players are never indifferent, then

(a) for any two matching equilibria (m,f) and (m',f'): (m,f) $\geq_1$ (m',f') if and only if (m',f') $\geq_2$ (m,f), and

(b) the set of matching equilibria forms a distributive lattice (based on either $\geq_1$ or $\geq_2$).

Proof: Again, call the players in role 1 men and the players in role 2 women. First we prove part (a) of the proposition and show that if (m,f) $\geq_1$ (m',f') then (m',f') $\geq_2$ (m,f). Assume to the contrary that (m,f) $\geq_1$ (m',f') and that at least one woman, say W2, strictly prefers (m,f) to (m',f'). If W2's spouse at (m,f), say M1, also strictly prefers (m,f) to (m',f') then (m',f') is not a matching equilibrium since W2 and M1 prefer to sever their (m',f') ties and link with each other and play their (m,f) Nash. Thus it must be that M1 is indifferent between (m,f) and (m',f'). Since we assumed players are never indifferent this is only possible if M1 has the same spouse/Nash at both equilibria. But if this is true, then W2 would have the same spouse/Nash at both equilibria and thus would not strictly prefer (m,f). Thus the "if" statement of part (a) must be true. The "only if" statement follows from the above; simply replace the role 1 (2) players with the role 2 (1) players.

Next we prove part (b) of the proposition.

Let (m,f) and (m',f') be two matching equilibria. Define $\sup_1\{(m,f),(m',f')\}$ to be the strategy matching profile where each man is matched with the spouse/Nash pair he most prefers (or points to) from either his (m,f) or (m',f') spouse/Nash pair. Define $\inf_1\{(m,f), (m',f')\}$ to be the strategy matching profile where each man is matched with the spouse/Nash he least prefers from either his (m,f) or (m',f') spouse/Nash pair. We show that $\sup_1\{(m,f),(m',f')\}$ and $\inf_1\{(m,f),(m',f')\}$ are both plausible matching profiles and that they are both in fact matching equilibria. First we show that $\sup_1\{(m,f),(m',f')\}$ is plausible. It is enough to show that there do not exist two men, say M1 and M3, who both point to the same spouse, say W2, when they point to their preferred spouse/Nash pairs. Assume to the contrary that two such men exist and that M1 is matched to W2 at matching equilibrium (m,f) while M3 is matched to W2 at matching equilibrium (m',f'). Since players are never indifferent, W2 must prefer either her spouse/Nash at (m,f) or at (m',f'). Say she prefers M1 or her spouse/Nash at (m,f). But then (m',f') cannot be a matching equilibrium since W2 prefers her spouse/Nash at (m,f) and M1 also prefers his spouse/Nash at (m,f); thus M1 and W2 prefer to sever their (m',f') links and link to each other and play their (m,f) Nash. Thus $\sup_1\{(m,f),(m',f')\}$ must be plausible.

Next we show that $\sup_1\{(m,f),(m',f')\}$ is also a matching equilibrium. Assume to the contrary that $\sup_1\{(m,f),(m',f')\}$ is not a matching equilibrium, thus there exists a woman, say W2, who would like to sever her $\sup_1\{(m,f),(m',f')\}$ tie and link with a different spouse/Nash, say M1. At $\sup_1\{(m,f),(m',f')\}$ W2 must be linked with the same spouse/Nash she is linked with at either (m,f) or (m',f'), say it is (m,f). Since (m,f) is a matching equilibrium it must be that if W2 asks M1 to sever his (m,f) tie and link with her and play a certain Nash, M1 says no. Since M1 weakly prefers his spouse/Nash at $\sup_1\{(m,f),(m',f')\}$ to (m,f) he will also refuse to sever his $\sup_1\{(m,f),(m',f')\}$ link to link with W2. Thus even though W2 would like to sever her $\sup_1\{(m,f),(m',f')\}$ tie and link with another spouse/Nash she is unable to do so. Thus $\sup_1\{(m,f),(m',f')\}$ must be a matching equilibrium.

Next we show that $\inf_1\{(m,f),(m',f')\}$ is plausible and is a matching equilibrium. By part (a) we know that $\inf_1\{(m,f),(m',f')\}$ is the same as $\sup_2\{(m,f),(m',f')\}$. Thus from the above analysis $\sup_2\{(m,f),(m',f')\}$ must also be a matching equilibrium.

Lastly we show that if there exists (m'',f'') such that $(m'',f'')\geq_1(m,f)$ and $(m'',f'')\geq_1(m',f')$ then $(m'',f'')\geq_1 \sup_1\{(m,f),(m',f')\}$. This follows from the definition of $\sup_1\{(m,f),(m',f')\}$. Similarly if there exists (m'',f'') such that $(m'',f'')\leq_1(m,f)$ and $(m'',f'')\leq_1(m',f')$ then $(m'',f'')\leq_1 \inf_1\{(m,f),(m',f')\}$. Thus the set of matching equilibria must form a lattice (based on either $\geq_1$ or $\geq_2$).

To show that the lattice (based on either $\geq_1$ or $\geq_2$) is distributive involves some straightforward manipulations which we leave to the reader to verify. ◊

Proposition 3: Consider a matching game with two player roles such that players are never indifferent and all pairs of players (from different populations) have at least one Nash equilibrium which generates positive payoffs for both players. If $n_1<n_2$, then every matching equilibrium has the same set of players who are unmatched.

Proof: Again, call the players in role 1 men and the players in role 2 women. First, note that in any matching equilibrium all men must be matched (since $n_1<n_2$, and otherwise an unmatched man and woman can improve their situation by matching) and all matches must play a positive payoff equilibrium. Second, suppose to the contrary of the proposition that there exists a matching equilibrium (call it ME1) where some woman, say W2, is unmatched while some other woman, say W4, is matched and that there exists another matching equilibrium (call it ME2) where W2 is matched and W4 is unmatched.

In order for ME1 to be a matching equilibrium it must be that the man W2 is matched with at ME2, say M1, strictly prefers (this preference will be strict since we have assumed no indifference) his ME1 spouse/Nash to playing the ME2 Nash with W2 (otherwise at ME1, W2 and M1 will prefer to link and play their ME2 Nash). Similarly, in order for ME2 to be a matching equilibrium it must be that the woman M1 is matched with at ME1, say W5, strictly prefers her ME2 spouse/Nash to her ME1 spouse/Nash (otherwise at ME2, W5 and M1 will prefer to link and play their ME1 Nash). In order for ME1 to be a matching equilibrium it must be that the man W5 is matched with at ME2 strictly prefers his ME1 spouse/Nash to his ME2 spouse/Nash. If we keep repeating this process we will end up with all women who are matched at ME1 must strictly prefer their ME2 spouse/Nash. However this is not possible. To see this recall that all men must be matched at every matching equilibrium. Thus if there are n_1 men then there must be n_1 women who are matched at ME1 and who strictly prefer their ME2 spouse/Nash. However, since W2 is unmatched at ME1 but matched at ME2 and since we assumed no indifference, it must be that W2 also strictly prefers her ME2 spouse/Nash, thus there are (n_1+1) women who strictly prefer the ME2 equilibrium. Since only n_1 woman are matched at ME2 this is not possible. Thus it must be that the set of women who are unmatched is the same at both equilibria. ◊

Say one player b **weakly dominates** another c (in the same player role) if for every potential matching p with c in p and every Nash equilibrium m_p for p that gives all players a nonnegative payoff, there exists a Nash equilibrium $m_{p'}$ for p' where b replaces c that strictly Pareto dominates m_p for the players other than b in p' and gives b a nonnegative payoff.

We say that players are **well-ordered** if for every pair of players b and c in the same player role, either b weakly dominates c, or c weakly dominates b.

When players are well-ordered, we have an unambiguous ordering over the players from all players' perspectives. It would seem natural to expect that in this case, any matching equilibrium would turn out to be assortive (with highest ranking players matched with other highest ranked, etc.), or at least that there should exist one such equilibrium. This is true in the standard marriage-market setting, but turns out not to be true in the matching game setting! The following example shows a case where the only matching equilibrium involves mismatching high ranked players with low ranked players.

Example: Non Assortive Matching.

Let there be two player roles and two players in each role. Players 1 and 3 are those in role 1 and players 2 and 4 are those in player role 2. When players 1 and 2 are matched, they have two possible Nash equilibria, leading to payoffs of (4,2) and (2,4).[4] When

[4] Note that generically, there will be an odd number of Nash equilibria. The example is easily modified to include a third possible Nash equilibrium payoff for each set of players. For instance, have these be payoffs to Battle of the Sexes games, with a mixed strategy equilibrium that leads to lower payoffs for both player roles than either of the pure strategy equilibria.

players 1 and 4 are matched (and the same for 3 and 2 matched), they have two possible Nash equilibrium payoffs of (3,1) and (1,3). When players 3 and 4 are matched they can only generate a payoff of (0,0).

Note that here, players 1 and 2 unambiguously dominate their counterparts in the same roles, 3 and 4, by generating higher payoffs regardless of their matching. Nevertheless, the only matching equilibrium has player 1 matched with player 4 and player 2 matched with player 3! To see this, first note that if we try to match players 1 and 2, then their payoff must be either (2,4) or (4,2). Given the symmetry, let us assume, without loss of generality, that it is (2,4). Then players 1 and 4 can block and get (3,1), which is better for both players. Indeed, the only matching equilibrium has players 1 and 4 matched with payoff (3,1), and players 2 and 3 matched with payoffs (1,3) (with the higher payoff for player 2 who is in role 2).

Example: All Matching Equilibria are Man-Optimal.

Let there be two men and three women, with preferences that allow indifference. Assume each pair has two Nash equilibria that are not strictly Pareto dominated by other Nash in their game. Let the Nash payoffs be as follows: If M_1W_1 are matched then the game played results in two Nash equilibria with payoffs of (4,2) or (2,4). If M_2W_2 are matched then the game played results in two Nash equilibria with payoffs of (4,2) or (2,4). If M_1W_2 or M_2W_1 are matched then the Nash payoffs are (1,3) or (3,1). If M_1W_3 and M_2W_3 are matched then the Nash payoffs are (1,4) and (3,2). Thus M_1 prefers W_1 to W_2 or W3 in the sense that the man's best equilibrium gives M1 a higher payoff if he is matched to W_1 and the woman's best equilibrium gives M1 a higher payoff if he is matched to W_1. Similarly M_2 prefers W_2. Here there are four matching equilibria: M_1W_1 and M_2W_2 matched and both play the (4,1) Nash and W_3 unmatched; M_1W_2 and M_2W_1 matched and both play the (3,2) Nash and W_3 unmatched. (There are two other matching equilibria like the last one with W_1 or W_2 being unmatched, respectively.) So having more women than men guarantees that each man plays a man's favorite Nash, but it does not guarantee that each man receives his first choice of mate, even though this is feasible for both men.

Each group of players, p, generates a set of Nash equilibria represented by NE(p). Let PO(p) represent the set of nonnegative utility vectors for the group p such that if $(u'_a,\ldots,u'_c) \in$ PO(p) then for every $m_p \in$ NE(p) such that all players receive a positive payoff, there exists at least one agent $b \in p$ who has $u'_b \geq u_b(m_p,p)$. Thus PO(p) represents the set of utility vectors which are not strictly Pareto dominated by any positive Nash equilibrium utility vectors.

Theorem 4 Suppose that there exists at least one group of agents with a Nash equilibrium such that all players in the group receive a positive payoff. In every matching equilibrium (m,f): (i) at least some players are matched, (ii) for every matched group, p, $m_p \in NE(p)$ which generates a utility vector in PO(p) and (iii) any group of agents, p', who are not currently matched to each other receive some utility vector $(u'_a,\ldots,u'_c) \in PO(p)$.

Proof: First we show that if the assumptions of the Proposition hold true then we have a matching equilibrium. Since by (i) and (ii) every matched group plays a Nash equilibrium with nonnegative payoffs, (a) of matching equilibrium is satisfied. (b) is satisfied since any deviating group of agents, say $\{1_j,\ldots,n_i\}$, with deviating strategy m' must currently receive utility $(u_j,\ldots,u_i)$ such that $u_k \geq u_k(m')$ for at least one $k \in \{1_j,\ldots,n_i\}$ by definition of $PO_{i\ldots j}$. Thus at least one agent k in this group does not want to deviate.

Next we show that at any matching equilibrium assumptions (i), (ii) and (iii) must hold true. First we show that at least some players must be matched. Since by assumption there exists at least one group of agents with a Nash equilibrium for which all players in the group receive a positive payoff, it must be that these agents prefer to be matched and playing such a Nash, thus by (b) of matching equilibrium at least some of these agents must be matched. Next we show that every matched group, $\{1_i,\ldots,n_j\}$, plays an $m \in NE_{i\ldots j}$ which generates a utility vector $(u_i,\ldots,u_j) \in PO_{i\ldots j}$. By definition of matching equilibrium any matched group of agents must play a Nash equilibrium which generates nonnegative payoffs, thus every matched group must play an $m \in NE_{i\ldots j}$ which generates nonnegative payoffs. Also by definition of matching equilibrium it must be that no group of agents $\{1_i,\ldots,n_j\}$ who are currently matched and are playing m prefer to play $m' \neq m$ where $m' \in NE_{i\ldots j}$. (Note that if all agents prefer m', then m' must give all agents a positive payoff since m gives all agents a nonnegative payoff.) Thus it must be that there exists at least one agent $k \in (1_i,\ldots,n_j)$ for whom $u_k(m') \leq u_k(m)$, which implies that the utility vector generated by m must be an element of $PO_{i\ldots j}$ or that assumption (ii) holds true.

Lastly we show that (iii) must be true. Suppose there exists agents $\{1_j,\ldots,n_i\}$who are not currently matched to each other and for whom current utility equals $(u_j,\ldots,u_i) \notin PO_{j\ldots i}$ and where current utility is nonnegative for all agents. (If current utility is negative for any agent, this agent will prefer to sever his current tie, which contradicts the assumption that agents are in a matching equilibrium.) Since $(u_j,\ldots,u_i) \notin PO_{j\ldots i}$ there exists an $m \in NE_{j\ldots i}$ such that $u_k(m) > u_k \geq 0$ for all $k \in \{1_j,\ldots,n_i\}$, thus agents $\{1_j,\ldots,n_i\}$ prefer to sever their current ties, link to each other and play m. Thus at a matching equilibrium (iii) must hold true. ◊

5. Finitely Repeated Matching Games

We now extend the concept of matching equilibrium to finitely repeated games. For this analysis, we return to the setting of homogeneous populations so that players within a population are identical and players care only about the play of the game.

Consider a sequence of matching games played over the finite set of periods $\{1,2,3,...,t\}$.

Players receive the discounted sum of payoffs of per period plays, with a discount rate of $0<\delta<1$.

Rematchings are possible in any period.[5]

Let $h=[s_1,f_1;s_2,f_2;...s_t,f_t]$ denote a generic history of the game through some time t, which includes a list of the strategies played and the matches that were in place. Let H(t) denote the union of all histories of the game through time t, with the convention that H(0) is a singleton (empty) history which we denote by 0.

A t-period (behavioral) strategy for a player j in role i is a map $\sigma_j:H(t-1) \to \Delta(S_i)$. The behavioral strategy profiles for the t period game are denoted S(t).

So a strategy for a player indicates which mixed strategy they play following any finite history of length no more than t-1.

A t-period matching function is a mapping $F:H(t-1) \to MF$, which indicates the current period matching following any history h in H(t-1). Let F(h,i) denote i's match after history h. The set of all t-period matching functions is denoted MF(t), and so MF(1)=MF.

Let $U_c(\sigma,F)$ denote player c's discounted expected utility if the profile of strategies σ are played and the matchings are governed by F.

A **repeated matching equilibrium** of a t-period game, denoted RME(t), is defined inductively as follows.

Let RME(1) be the matching equilibria of the 1 period game.

Inductively, Let RPE(t) be the set of (σ,F) in S(t) x MF(t) such that

(i) $(\sigma(h),F(h))$ in RME(t-1) for all h in H(1), and

(ii) no player wants to deviate from $\sigma(0)$ given the current matching F(0) and anticipating the continuation governed by (σ,F).

Let RME(t) be the set of (σ,F) in RPE(t) such that

(a) there does not exist any $c \in N$ with $0>U_c(\sigma,F)$, and

(b) there does not exist any $p \in P$ and (σ',F') in RPE(t) such that $U_c(\sigma',F')>U_c(\sigma,F)$ for all c in p, and such that F'(h,c)=p for all c in p and h.

[5] If rematchings are only possible ex ante, then the analysis is similar to that in the previous sections (where one can modify definitions to require that play be according to subgame perfect equilibrium rather than just Nash equilibrium).

This definition has the same structure as the original definition of matching equilibrium, where the idea of Nash equilibrium is replaced by the notion of RPE(t) – an equilibrium that is a perfect equilibrium anticipating that the continuation will be a matching equilibrium of t-1 periods.

While there is little ambiguity in defining matching equilibrium in a single period, there are new issues that arise in the multiple period setting. In a single period, if some individual player prefers to go off alone, or some group of players all benefit from forming a new group and playing some Nash equilibrium, they can do this without any worry about how the other players will organize now or in the future. With multiple periods this is no longer the case. The above definition allows such an individual or group to propose a new matching (σ',F') which is consistent with equilibrium and dominates things for the individual or group.

Notice that in our definition of repeated matching equilibrium, the deviating group is just on its own, and cannot rely on rematching with any other players in any future continuation of the game. If such rematchings are allowed then existence of a repeated matching equilibrium is not guaranteed as the following definition and example show.

A **strong repeated matching equilibrium** of a t-period game, denoted SRME(t), is defined inductively as follows.

Let SRME(1) be the matching equilibria of the 1 period game.

Inductively, suppose that we have defined SRME through t-1.

Let SRPE(t) be the set of (σ,F) in S(t) x MF(t) such that
(i) (σ(h),F(h)) in SRME(t-1) for all h in H(1), and
(ii) no player wants to deviate from σ(0) given the current matching F(0) and anticipating the continuation governed by (σ,F).

Let SRME(t) be the set of (σ,F) in SRPE(t) such that
(a) there does not exist any $p \in P$, (σ',F') in SRPE(t), and $c \in N$ with F(c,0)=p, F'(0,c)=c and $U_c(\sigma', F') > U_c(\sigma, F)$, and
(b) there does not exist any $p \in P$ and (σ',F') in SRPE(t) such that $U_c(\sigma', F') > U_c(\sigma, F)$ for all c in p.

Notice that in the case of t=1, both repeated matching equilibrium definitions coincide with that of matching equilibrium.

In the case where there is just one group and no issue of individual rationality, both definitions coincide with the definition of finite period renegotiation-proof equilibrium (e.g., Benoit and Krishna (1993)).

Example: Nonexistence of Strong Repeated Matching Equilibrium

There are three player roles and six players {1,2,3,4,5,6}. Players 1 and 4 are in role 1, players 2 and 5 are in role 2, and players 3 and 6 are in role 3.

Player role 3 has only one strategy. Player role 1 is the row player and player role 2 is the column player. Payoffs are as follows

	left	center	right
up	1, 4, 1	-10, -10, -10	7, -10, -10
middle	-10, -10, -10	4,1,1	-10, -10, -10
down	-10, 7, -10	-10, -10, -10	5,5,5

Here there are several matching equilibria in a one period game. These are where all players are matched and each group plays either of the pure strategy equilibria: up, left or middle center. Note that any mixed strategy equilibrium gives a payoff of less than 1, and that down and right are weakly dominated strategies.

The following is in SRPE(2): in the first period:
players 1,2,3 are matched and play down, right and get payoffs of (5,5,5).
players 4,5,6 are matched and play up, left and get payoffs of (1,4,1).

In the second period --
If nobody from the first group deviated in the first period then:
players 1,5,6 are matched and play middle center and get payoffs of (4,1,1)
players 4,2,3 are matched and play up left and get payoffs of (1,4,1)

If someone from the first group deviated in the first period then
players 1,5,6 are matched and play up left and get payoffs of (1,4,1)
players 4,2,3 are matched and play middle center and get payoffs of (4,1,1)

In terms of verifying that this is in SRPE(2) – it is clear that nobody will wish to deviate in the first period. If one of players 1,2,3 deviates in the first period, they will all get a payoff of 1 in the second period, while if they do not deviate, they get a payoff of (4,4,1), respectively in the second period. So a deviating player will lose 3, but gain at most 2 in the first period.

Now, we can see that SRME(2) is empty – since the fact that the three players can get a payoff of (5,5,5)+(4,4,1)=(9,9,6); cannot be given to two groups at once.
It is the second period regroupings that give players in roles 1 and 2 the incentives to play the prescribed strategies, and these regroupings are necessarily asymmetric.

We now prove the existence of a repeated matching equilibrium.

Theorem 5: The set of repeated matching equilibrium is nonempty and compact for every t. Moreover, there exists such an equilibrium where the repeated matching function is constant.

Proof: We prove the theorem by induction. Recall that player roles are ordered so that $n_i \geq n_k$ whenever i>k.

First note that the set of possible histories is finite, and so strategies can be represented as a finite list of vectors, where each vector represents a mixed strategy to be played following a given history and thus belongs to a simplex. The set of repeated matching functions is finite.

Theorem 1 established existence and compactness (and constant matching) for t=1. Supposing that the claim has been established for all t<T, we show that it is true for T.

We first establish that RPE(T) is nonempty and compact and contains something with a constant matching function.

We first show that RPE(T) is nonempty, and in particular has something with a constant matching function. Let f be a matching that has some equilibrium in RME(T-1) for which the matching is constant and equal to f. Let F(0)=f. H(1) is a finite set of possible histories that can occur in the first period. Associate with each h in H(1) the same continuation matching equilibrium in RME(T-1) that has the constant matching f. This then defines (σ,F) except for $\sigma(0)$. Given that continuation payoffs are constant, and since RME(1) is nonempty and must have f as part of some equilibrium, there is at least one equilibrium play for $\sigma(0)$ that gives all players nonzero payoffs.

Next, let us argue that RPE(T) is compact. Let $(\sigma^k,F^k) \rightarrow (\sigma,F)$, where (σ^k,F^k) is in RPE(T) for each k. By the compactness of RME(T-1), it follows that (i) is satisfied. To see (ii), note that by the finiteness of the number of repeated matching functions, we can restrict attention to the case where F^k=F for each k. (ii) then follows, since any improving deviation from σ at the limit would also be an improving deviation from σ^k for large enough k.

We now argue that RME(T) is nonempty and compact. We first argue that it is nonempty.

Here we repeat the arguments of theorem 1, but using the constant matchings from RPE(T) that give all nonzero payoffs (we know that this set is nonempty as argued above). [fill in details] This must then satisfy (a) and (b).

Compactness of RME(T) now follows along similar lines as the proof of the corresponding claim in Theorem 1 (given the compactness of RPE(T)), as a violation of (a) or (b) at the limit of a sequence of equilibria would imply a violation far enough along the sequence. ◊

As we have remarked, the concept of repeated matching equilibrium incorporates some notions of renegotiation. That is, it is possible that the set of players p who consider changing equilibria in part (b) is actually already matched. It is very important to note, however, that although the definition of repeated matching equilibrium imposes some forms of renegotiation, it is neither a subset nor a superset of the set of renegotiation equilibria (except when t=1). This follows since changing the set of equilibria in a repeated game changes both the possibilities at a given date, and also the possible threat points offered for other dates. The changes in these combinations leads the sets to differ in a non-nested way when there are more than 2 periods.

Proposition 4: Let $n_1=n_2=\ldots=n_n$. Assume there exists a renegotiation proof equilibrium such that each player's expected utility is non-negative in every period both on and off the equilibrium path. Call this set of equilibrium RNE(t). A weak repeated matching equilibrium exists where each player is matched by some constant matching function f and where every matched group plays the same σ in RNE(t).

Proof: We use the definition of renegotiation proof equilibrium found in Benoit and Krishna (1993). First we show that matching every player and having every matched group play the same σ in RNE(1) must be a matching equilibrium of the 1 period game. By definition of renegotiation proof equilibrium, σ must be a Nash equilibrium of the 1 period game and by assumption $u_j(\sigma)\geq 0$ thus condition (a) of matching equilibrium is met. Next consider condition (b) of matching equilibrium. By definition of renegotiation proof equilibrium, players who are currently grouped together do not want to play a different Nash. Since we assumed every group plays the same renegotiation proof equilibrium, it must be that every player of type i receives the same payoff thus there is no group of agents who want to sever their current ties and form a new group and play a different Nash, and so condition (b) must hold true. Thus having every player matched and all groups play σ is a matching equilibrium of the 1 period game.

Next we show that having every player matched by some constant matching function f and having all groups play some σ in RNE(t) must be a weak repeated matching equilibrium of the t period game. First we show that (σ,f) is in RPE(t). By definition of renegotiation proof equilibrium, all continuation payoffs of (σ,f) are in RNE(t-1). And we have already showed that for t=1, (σ,f) is in RME(1). Thus for t=2 we know that all continuation payoffs of (σ,f) are in RME(1). Similarly, if (σ,f) is in RME (t-1) for the (t-1) period game, then all continuation payoffs of (σ,f) are in RME(t-1) for the t period game. Thus condition (i) is met. Condition (ii) is met by the definition of renegotiation proof equilibrium.

Next we show that (σ,f) is in RME(t). By assumption $u_j(\sigma,f)\geq 0$ and so assumption (a) of weak repeated matching equilibrium is met. Next we show that assumption (b) is met. Since every matched group plays the same σ, we know that every player in role i must have the same expected payoff (if one player i cheats, then all have incentive to cheat) and so we can just show that no group who is currently matched wants to change strategies as there will be no extra gain from a new group forming and changing strategies. By definition of renegotiation proof equilibrium, we know that (σ,f) is not Pareto dominated by another renegotiation proof equilibrium. Thus (σ,f) is not Pareto dominated by an equilibrium where players cannot switch partners or groups. But since we assumed all players play the same equilibrium, there is no incentive to switch partners or groups. Thus (σ,f) cannot be Pareto dominated by any element of RPE(t) either and condition (b) holds true. ◊

Remarks: If $n_1=n_2=\ldots=n_n$, then just like in the 1-period game, having some groups play one renegotiation proof equilibrium and other groups play another renegotiation proof equilibrium may not be a matching equilibrium.

Additionally, if $n_1\leq n_2\leq\ldots\leq n_n$ with at least one strict inequality, then the set of weak repeated matching equilibria may no longer contain any renegotiation proof equilibrium. This is because the minority player types will get to choose their favorite Nash equilibrium in the last period. Thus other threats will no longer be credible and the set of equilibria will change. For an example of this see below (Example where repeated matching equilibrium is neither renegotiation proof nor Pareto Optimal).

Finally, if players are identical and if it is possible for everyone to be simultaneously matched then players will play the symmetric Nash in the last period. So again other threats are no longer credible and not all renegotiation proof equilibria will be weak repeated matching equilibrium. For an example see below (Example comparing renegotiation proof equilibria to (weak) repeated matching equilibria).

[[add another example showing inefficiency.]]

Example comparing renegotiation proof equilibria to (weak) repeated matching equilibria.

This game is from Benoit and Krishna [1993]. Let there be an even number of at least 4 identical agents who play the following game which is repeated twice. Let the discount rate δ=1.

	A	B	C
A	0,0	1,3	0,0
B	3,1	0,0	6,0
C	0,0	0,6	0,0

This game has two Nash equilibria: A,B and B,A.
There are three renegotiation proof equilibria (as defined by Benoit and Krishna) here. They are:

(1) play B,A in period 1 and A,B in period 2. With a discount rate of 1 the expected payoffs from following this strategy are (4,4).
(2) play B,C in period 1 and play A,B in period 2. If agent 2 deviates in period 1 then play B,A in period 2. The expected payoffs are (7,3).
(3) Play C,B in period 1 and play B,A in period 2. If agent 2 deviates in period 1 then play A,B in period 2. The expected payoffs are (3,7).

However, the only matching equilibria is for all pairs (or all pairs except one) to play equilibrium 1. Assume to the contrary that two pairs of agents play equilibrium 2. Then there are two agents who expect a payoff of 3. They would be better off if they severed their current ties and linked with each other and played equilibrium 1. However it is possible for one pair of agents to play equilibrium 2 as long as all other pairs of agents play equilibrium 1. Then the agent who is receiving the payoff of 3 from equilibrium 2 will have no one to form a new link with.

Note also that repeating a Nash equilibrium for two periods is not a matching equilibrium. Assume to the contrary that some pair of agents is considering playing (B,A) in both periods. Their expected payoff would be (6,2). If all other pairs of agents are playing equilibrium 1 then condition (iii) of the definition of repeated matching equilibrium is violated since this first pair would be better off agreeing to play equilibrium 2 and receiving an expected payoff of 7,3. If there is another pair of agents already playing equilibrium 2 (or 3) then the player in the first pair who is receiving the 2 payoff and the player in the second pair who is receiving the 3 payoff would be better off linking and playing equilibrium 1 and receiving an expected payoff of (4,4) and again condition (iii) is violated. Thus repeating B,A twice is not a matching equilibrium.

Example where Repeated Matching Equilibrium is neither renegotiation proof nor Pareto Optimal

Now change the above example so that the discount rate δ=.9 and so that there are 3 agents who are not identical. Specifically, let there be one agent of type 1 and two agents of type 2. The set of renegotiation proof equilibria does not change. However the set of repeated matching equilibria does change. The unique repeated matching equilibrium is now for player 1 to be matched and for the matched pair to play B,A in both periods. To see this notice that in period 2, player 1 will always insist on playing B,A. In period 2, if player 1 is paired with someone who wants to play anything else, then 1 will sever his tie and link with the unlinked player of type 2 and play B,A. Player 1 would like to play B,C in period 1. However, his partner will always have incentive to cheat in period 1 (by cheating his partner will gain a payoff of 1, while if he agrees to play B,C in the first period and B,A in the second period he will receive only .9). Since B,C is not credible in period 1, player 1 will want to play B,A in period 1 which is credible. At this repeated matching equilibrium, the expected payoffs are (5.7, 1.9) which is Pareto dominated by

(6.9, 2.7) which are the expected payoffs from renegotiation proof equilibrium 2. Thus the repeated matching equilibrium is neither renegotiation proof nor Pareto optimal.

Example where Repeated Matching Equilibrium is not a Subgame Perfect Equilibrium (without matching)

Now change the above example so that the discount rate δ=1 and so that there is one agent of type 1 and two agents of type 2. We will slightly modify the payoffs in the 3x3 game so that the game is now:

	A	B	C
A	0,0	1,3	0,0
B	3,1	0,0	6,.1
C	0,0	.1,6	0,0

The unique repeated matching equilibrium is now for player 1 to be matched and for the matched pair to play B,C in period 1 and to play B,A in period 2. If the matched type 2 player cheats in period 1, then player 1 will sever his tie with this player and link with the other player of type 2 in period 2. To see that this is a repeated matching equilibrium, first notice that in period 2, player 1 will always insist on playing B,A. In period 1, the matched type 2 player will not cheat (if he cheats he receives an expected payoff of 1, while if he does not cheat his expected payoff is 1.1). The expected payoffs from this repeated matching equilibrium are (9, 1.1). This payoff is the best that player 1 can expect, so he will not want to change groups in period 1 and thus this strategy is a repeated matching equilibrium. However, this equilibrium is not a subgame perfect equilibrium if there is no rematching possible. Player 1 is able to achieve this repeated matching equilibrium with the threat of severing his tie and linking to the other type 2 player in period 2, however this threat does not exist without rematching. Without rematching, player 2 will always cheat in period 1, knowing that in period 2 he will receive a payoff strictly greater than .1 (since all Nash equilibria including the mixed Nash give player 2 a payoff of greater than .1).

Example: 2 Period Prisoners' Dilemma

Let there be three identical players who play the following Prisoner's Dilemma game which is repeated twice. Let the discount rate δ=1.

	C	D
C	3,3	-2,4
D	4,-2	2,2

The unique repeated matching equilibrium is for two players to be matched and for the matched pair to play C,C in period 1 and D,D in period 2. (Since this game has a unique Nash equilibrium, the matched pair must play D,D in period 2.) If one of the matched players cheats in period 1 then the other matched player will sever this tie and will link with the unmatched player in period 2. So if a player cheats in period 1 he receives an expected payoff of 4 while if he does not cheat he receives an expected payoff of 5. Notice that the unmatched player cannot offer either matched player a credible better deal in period 1. Thus cooperation is sustained in the first period, which differs from any other equilibrium concept.

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