An Algorithm for Stable and Equitable Coalition Structures with Public Goods

Fan-chin Kung*

Institute of Economics

Academia Sinica, Taiwan

fckung@econ.sinica.edu.tw

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Abstract

We study the formation of coalitions that provide public goods to members. Individuals are linked on a tree graph and those with similar preferences are connected on the tree. We present a solution that selects allocations belonging to the coalition structure core and that are also envy-free.

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1. Introduction

Individuals in a society face many collective decisions: for example, how much tax to levy, how much to spend on local public schools, and whether to build a

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community center. Collective decisions are carried out in groups (public coalitions). Depending on different classes of issues at stake, people belong to different groups, and there are usually multiple groups. Countries, local governments, unions, and political parties are such examples. When national security and foreign affairs are concerned, decisions are made for a country: the United States can have only one stance on these matters. When policies on abortion or gun control are concerned, states are the corresponding groups: each state chooses its own policies. When it comes to matters of local public schools, school districts act as public coalitions. In these situations, people form public coalitions to decide the provision of public goods: A coalitions is endowed with a feasible set of public alternatives. Each individual joins one and only one coalition, and each coalition chooses an alternative from its feasible set. A partition of individuals is a coalition structure.

The concept of a public coalition originates from the Tiebout equilibrium literature, which studies the competition of local tax jurisdictions. Jurisdictions compete for freely mobile individuals with tax-expenditure packages. A Tiebout equilibrium is an allocation where no one wants to switch to another jurisdiction (see, to name only a few, Tiebout 1956, Westhoff 1977, and Bewley 1981). The *coalition structure core*, which is a partition of individuals such that no coalition can block, is also applied to local public good economies. Wooders (1978, 1980, 1989) and Conley and Wooders (1996, 1997, 2001) study the existence of and equivalence between Tiebout equilibrium and the coalition structure core in production economies with local public goods and crowding effects in both consumption and production. They study Tiebout equilibrium as a competitive price equilibrium with endogenously formed jurisdictions. Guesnerie and Oddou (1981), and Greenberg and Weber (1986) study the coalition structure core in cooperative games with local public goods. The above work supplements the Tiebout literature by allowing jurisdictions to form endogenously. Greenberg and Weber (1993) and Demange (1994) study a stronger notion of stability: the intersection of coalition structure core and Tiebout equilibria. This intersection is shown to be nonempty when preferences are single-peaked on a line in the former and when preferences are *intermediate* on a tree graph in the latter. Coalition feasible sets are assumed to be monotonic in both. The formation of public coalitions covers a broad range of economic applications. The following are some examples: (i) Local jurisdictions offer public goods to attract residents who at the same time choose jurisdictions that provide desirable public goods. (ii) Political parties decide policy platforms, and individuals can join a party or form a new one. (iii) Buyers choose among different insurance policies in the market. People who purchase the same contract are bound by the same terms. (iv) Groups of people set up institutions, such as firms, governing their economic activities.

We study a public good game where individuals are linked on a tree graph and their preference profile has *connected support*. This means that individuals who strictly prefer one alternative to the other in any pair of alternatives form a connected set on the tree. People with the same preferences are connected. We also assume the feasible sets of coalitions to be monotonic; more alternatives are feasible to a coalition as new members join. Can we derive allocations that are both stable and equitable? We use the coalition structure core as our stability concept and *envy-free* as the equity standard. An allocation is envy-free if no one wants to switch places with another.

Our result relates closely to the following articles. Kaneko and Wooders (1982) develop a version of Scarf's (1967) balancedness condition for the nonemptiness of the core in partitioning games. Le Breton, Owen and Weber (1992) applies it to communication games on graphs where only connected coalitions are effective. Demange (1994) uses the same setting with public goods as ours and shows nonemptiness with Scarf's theorem. We add to the literature by presenting an algorithm, called the *hierarchical benevolence solution*, to this game. It selects allocations belonging to the coalition structure core and that are also envy-free. This solution also serves as a constructive proof for the nonemptiness of the desired intersection without using Scarf's balancedness condition in any form. We use a preference restriction that is weaker than *intermediate preferences* used in Demange (1994). The idea of the solution is to utilize the natural tree order associated with a root and set up a decision hierarchy. Individuals are assigned decision levels and they make decisions according to the hierarchy. One acts as a benevolent dictator for agents on the subtree originated from her. The dictator chooses an allocation for agents on the subtree assuring them of the welfare level they may attain when they are benevolent dictators.

Section 2 introduces the model. Section 3 presents the hierarchical benevolence solution. Section 4 shows that it selects from the intersection of the coalition structure core and envy-free allocations. Section 5 concludes.

2. Public Coalition Formation

Let N denote the set of all individuals in a society. Let $X \subset \mathbb{R}^m$ denote the set of potential alternatives. Each individual $i \in N$ has preferences R_i over X, which is a weak order. Let P_i denote strict preference and I_i indifference. The preference profile of individuals in set N is denoted by $R = \{R_i\}_{i \in N}$. A *coalition* is a subset $S \in 2^N$ (the power set of N). The set of feasible alternatives of coalition S is $\phi(S)$ where $\phi : 2^N \to 2^X$ is called a feasibility correspondence. (Note that a coalition may have an empty feasible set. However, individuals cannot form a coalition with an empty feasible set.) To eliminate triviality, we further assume that there exists an $S \in 2^N$ such that $\phi(S) \neq \varnothing$. A *society* is a list $\mathcal{S} = (N, X, R, \phi)$. We make the following assumptions: X is a closed set, R_i is continuous in X, ϕ is compact-valued, and N is a finite set.[1] Each individual joins one and only one coalition. A coalition chooses an alternative from its feasible set. A *coalition structure* C in society $\mathcal{S}$ is a partition of N, where $C \subset 2^N$ and (i) $S \cap S' = \varnothing$ for all $S, S' \in C$, $S \neq S'$; (ii) $\cup_{S \in C} S = N$; (iii) $\phi(S) \neq \varnothing$ for all $S \in C$. An *allocation* a is a mapping $a : N \to X$, which assigns alternative $a(i)$ to individual i. Allocation a is *feasible* if there is a coalition structure and a list of alternatives $\left(C, \{x_S\}_{S \in C}\right)$ with $x_S \in \phi(S)$ for all $S \in C$ such that $a(i) = x_S$ for all $i \in S$ and all $S \in C$. Moreover, there is a tree graph G on N. A *path* p in G is a sequence of distinct edges $\{i_0 i_1, i_1 i_2, ..., i_{k-1} i_k\}$; we also denote it by $p(i_0, i_k)$. A path is of *length* k if it contains k distinct edges. We restrict ourselves to societies with the following two properties.

- The feasibility correspondence ϕ is *monotonic*: $\phi(S) \subseteq \phi(S')$ for all $S, S' \in 2^N$ with $S \subset S'$.

- The preference profile R has *connected support* on tree G: for any pair $x, y \in X$, the set $\{i \in N \mid x P_i y\}$ is connected on G.

Connected support requires those who strictly prefer one alternative to the other in any pair form a connected set. It is not a restrictive assumption. For example, in

[1] These assumptions can be replaced with "X and N are both finite sets". All results still hold.

most models with public goods, if people are ranked by income (a linear tree), those who strictly prefer more public goods form a connected set. Connected support is more general than *intermediate preferences* (Grandmont 1978 and Demange 1994) which requires, in addition, those with the same weak preference over any pair also form a connected set. Based on the idea that individuals who are indifferent should not play a role in making collective decisions, connected support does not restrict their positions. When applied to a linear order, intermediate preferences is equivalent to *single-crossing* and *order restriction* (Rothstein 1990), which are commonly used in the literature (see Gans and Smart 1996 and Kung 2002 for the equivalence results). The following example illustrates the difference between connected support and intermediate preferences.

Example 1. There are five people linked according to their labels 1 to 5 with two preference profiles.

$$R = \begin{array}{c} xI_1yP_1z \\ xP_2yP_2z \\ xI_3yP_3z \\ zP_4yP_4x \\ zP_5xI_5y \end{array}, \quad R' = \begin{array}{c} xP_1yP_1z \\ xI_2yP_2z \\ xI_3yP_3z \\ zP_4xI_4y \\ zP_5yP_5x \end{array}.$$

R has consecutive support but does not satisfy intermediate preferences; R' satisfies intermediate preferences. The difference lies in those who are indifferent between x, y. Intermediate preferences requires indifferent individuals to be in-between those with strict preferences, while this is relaxed in consecutive support. ∎

Take individual $r \in N$ as the root of G; r is the highest (the first) level agent. The rooted tree G^r generates a *decision hierarchy* as follows: The distance between individual $i \in N$ to r is $\delta(r, i) = k$ if r and i are linked by a path of length k. We say that i is of level $k + 1$. Since N is finite, there exists a maximum distance from r. Let $\bar{k} = \max_{i \in N} \delta(r, i)$. Let $I(l) = \{i \in N \mid \delta(r, i) = l - 1\}$ for $l = 1, ..., \bar{k} + 1$. $I(l)$ is the set of level l agents. Note that $I(1) = \{r\}$ and there are $\bar{k} + 1$ levels. Let N^i be the subtree originated from i that contains i's lower level agents. Agent i is a higher-up of j $(i \neq j)$ if i is on the path linking j and r; or equivalently, if $N^j \subset N^i$.

The next lemma shows the subtrees originated from i and j are disjoint if i, j are of the same level.

Lemma 1. *$N^i \cap N^j = \varnothing$ for all $i, j \in I(l)$, $i \neq j$ for all $l = 1, ..., \bar{k}+1$.*

Proof. Note that any two vertices are linked by a unique path on a tree. Suppose $h \in N^i \cap N^j \neq \varnothing$. Then, $p(i,h) \cup p(j,h)$ contains a path linking i and j that only consists of agents of levels lower than l, except for i, j. Also, $p(i,r) \cup p(j,r)$ contains a path linking i and j that consists of agents of levels no lower than l. Apparently, we have found two distinct paths linking i, j; a contradiction. ∎

3. The Hierarchical Benevolence Solution

The hierarchical benevolence solution is defined by a recursive algorithm. Take a decision hierarchy G^r. Each individual is assigned an admissible set of coalition-alternative pairs. This set is obtained by allowing an individual to be a dictator for all agents on her subtree. The dictator can form a connected coalition in her subtree that affords one of her most preferred alternative, given that all members of the chosen coalition are made no worse off than when they are dictators themselves. The admissible set of i, acting as a benevolent dictator for N^i, is defined as follows.

Add, temporarily, a common worst alternative $\underline{z}$ to every individual's preferences; $xP_i\underline{z}$ for all $x \in X$ for all $i \in N$. We will show that $\underline{z}$ does not play a role in the final allocation later. Let $\bar{X} = X \cup \{\underline{z}\}$ and $\bar{\phi}(S) = \phi(S) \cup \{\underline{z}\}$ for all $S \in 2^N$. A coalition-alternative pair (S, x) is *admissible* to i if (i) S is a connected coalition in N^i that contains i, (ii) x is feasible to S, (iii) for all other members of S, x is at least as good as any alternative in their admissible sets, and (iv) x is one of i's most preferred alternatives among all (S, x) that satisfy i, ii, iii; moreover, we require that (v) there is no $S' \supset S$ such that i to iv are satisfied and $x \in \phi(S')$ (S is maximal for x). Let A^i denote the set of admissible pairs for individual i.

First, A^i is defined for the following two cases.

Case 1: $N^i = \{i\}$. Let $A^i = \left\{(\{i\}, x) \mid x \in \bar{\phi}(i) \text{ and } xR_ix', \forall x' \in \bar{\phi}(i)\right\}$.

Case 2: $N^i \neq \{i\}$ and A^j is defined for all $j \in N^i \backslash i$. Let $Z^j = \left\{x \in \bar{X} \mid \exists S \text{ s.t. } (S, x) \in A^j\right\}$ be the set of j's admissible alternatives. Let $\tilde{C}^i$ be the collection of all connected subsets of N^i containing i.

$$\tilde{C}^i = \left\{ S \in 2^{N^i} \mid i \in S,\, S \text{ is connected} \right\}.$$

Let

$$B^i = \left\{ (S,x) \in 2^{N^i} \times \bar{X} \mid S \in \tilde{C}^i,\, x \in \bar{\phi}(S) \text{ and } xR_j y \,\forall y \in Z^j \,\forall j \in S \backslash i \right\}.$$

So, $(S,x) \in B^i$ if (i) $S \subset N^i$ is connected and S contains i, (ii) x is feasible to S, and (iii) for all other members h, x is as good as any admissible alternative in A^h.

Let $W^i \subseteq B^i$ be the set of coalition-alternative pairs with i's most preferred alternatives in B^i.

$$W^i = \left\{ (S,x) \in B^i \mid xR_i x',\, \forall (S',x') \in B^i \right\}.$$

And, finally,

$$A^i = \left\{ (S,x) \in W^i \mid \nexists (S',x) \in W^i \text{ s.t. } S \subset S', S \neq S' \right\}.$$

A^i consists of pairs $(S,x) \in W^i$ such that S is maximal for x.

The next lemma shows that the admissible set is well-defined in these two cases.

Lemma 2. *In cases 1 and 2, $A^i \neq \varnothing$ for all $i \in N$.*

Proof. In case 1, $A^i \neq \varnothing$ since a maximizer is guaranteed by that R_i is continuous and that $\bar{\phi}(i)$ is non-empty and compact. In case 2, note that $B^i \supseteq \left\{ (\{i\}, x) \mid x \in \bar{\phi}(i) \text{ and } xR_i x', \forall x' \in \bar{\phi}(i) \right\} \neq \varnothing$ since i can always form a one-person coalition. Let $O^i = \left\{ x \in \bar{X} \mid \exists S \text{ s.t. } (S,x) \in B^i \right\}$. Then $W^i \neq \varnothing$ if R_i has a maximizer in O^i. Since R_i is continuous and $O^i \neq \varnothing$ by $B^i \neq \varnothing$, we have to show that O^i is compact. Let $R_i(x) = \left\{ y \in \bar{X} \mid yR_i x \right\}$ be the upper contour set of R_i in $\bar{X}$ at x. Note that

$$O^i = \cup_{S \in \tilde{C}^i} \left(\cap_{x \in Z^h, h \in S \backslash i} R_h(x) \cap \bar{\phi}(S) \right).$$

All $R_i(.)$ are closed, all $\bar{\phi}(S)$ are compact, and all $S \in \tilde{C}^i$ and all $\tilde{C}^i$ are finite. Thus, O^i is compact. So, $W^i \neq \varnothing$. Taking maximal coalitions among $(S,x) \in W^i$, $A^i \neq \varnothing$. ∎

A^i is defined recursively starting from the lowest-level agents in $I\left(\bar{k}+1\right)$ and then move one level up at a time until $I\left(1\right)$. Either Case1 or Case 2 applies for each agent.

In the following, we construct allocations using the admissible sets. Given a collection of admissible pairs $\{(S^i, x^i)\}_{i\in N}$ such that $(S^i, x^i) \in A^i$ for all $i \in N$, we assign coalitions sequentially starting from r. First, S^r forms with alternative x^r. Let $L^0\left(r\right) = \{r\}$, and

$$L^1\left(r\right) = \left\{j \in N\backslash S^r \mid \nexists h \in N\backslash S^r \; s.t. \; N^j \subset N^h\right\}.$$

$L^1\left(r\right)$ is the set of agents without higher-ups in N after deleting coalition S^r. Next, each S^i forms with x^i for all $i \in L^1\left(r\right)$.

Suppose $L^k\left(r\right)$ is determined for $k = 0, ..., m-1$. Let $\hat{S}\left(m-1\right) = \cup_{i\in\cup_{k=0}^{m-1} L^k(r)} S^i$. This is the union of all connected coalitions that have been assigned. Let

$$L^m\left(r\right) = \left\{j \in N\backslash\hat{S}\left(m-1\right) \mid \nexists h \in N^i\backslash\hat{S}\left(m-1\right) \;\; s.t. \; N^j \subset N^h\right\};$$

$L^m\left(r\right)$ is the set of agents without higher-ups in N after deleting all S^i for all $i \in L^k\left(r\right)$ for all $k = 0, ..., m-1$. There is an integer $\bar{l} \leq \bar{k}-1$ such that $L^{\bar{l}+1}\left(r\right) = \varnothing$ since N is finite. Then, each S^i forms with x^i for all $i \in L^m\left(r\right)$. Assign coalitions this way up to $L^{\bar{l}}\left(r\right)$. Note that all coalitions that have formed are connected. Let $\overline{L} = \cup_{k=0,...,\bar{l}} L^k\left(r\right)$. Thus, $\{S^i\}_{i\in\overline{L}}$ is a partition of N. The collection of pairs $\{(S^i, x^i)\}_{i\in\overline{L}}$ constitute an allocation $a^r_{\{(S^i,x^i)\}_{i\in N}}$ (we denote it with a^r for simplicity). Each pair $(S^i, x^i) \in A^i$ is constructed as if i is a benevolent dictator who assures each member a guaranteed welfare level, which is what she enjoys from her admissible sets. This benevolence is carried over to the final allocation a^r: everyone is guaranteed a welfare level no less than what she enjoys from the admissible set.

Lemma 3. *$a^r\left(i\right) R_i x$ for all $x \in Z^i$ for all $i \in N$.*

Proof. For all $i \in N$, either $i \in \overline{L}$ and $a^r\left(i\right) = x^i I_i x$ for all $x \in Z^i$, or $i \notin \overline{L}$ and $i \in S^j$ for some $j \in \overline{L}$ and $a^r\left(i\right) = x^j R_i x$ for all $x \in Z^h$. ∎

Next, we show that the resulting allocation does not involve the added alternative $\underline{z}$.

Lemma 4. *$a^r(i) \neq \underline{z}$ for all $i \in N$.*

Proof. Suppose coalition S consumes $\underline{z}$. Suppose coalition T is adjacent to S and consumes alternative $x \neq \underline{z}$. (Two connected subsets $S, T \subset N$ are *adjacent* on G if there exists an edge ij such that $i \in S$, $j \in T$.) Note that there is only one edge ij such that $i \in S$, $j \in T$ (otherwise, there is a cycle). First, suppose j is of a lower level than i. Thus, j has no higher-ups in T and $T \subset N^j$. By monotonicity, $x \in \bar{\phi}(T \cup i)$. By construction, xR_hy for all $y \in Z^h$ for all $h \in T \subset N^j$. Therefore, $(T \cup i, x) \in B^i$. This implies $x^iR_ixP_i\underline{z} = a^r(i)$; a contradiction to Lemma 3.

Second, suppose j is of a higher level than i. Suppose g is the highest level agent in T (there is a cycle if g is not unique). Thus, $(T, x) = (S^g, x^g) \in A^g$. By monotonicity, $x \in \bar{\phi}(T \cup S)$. Note that by construction xR_hx^h for all $h \in T \backslash g$ and $xP_h\underline{z}R_hx^h$ for all $h \in S$. This implies $(S \cup T, x) \in W^g$ and $(T, x) \notin A^g$; a contradiction.

So, every T adjacent to S must consume $\underline{z}$. Note that every coalition in allocation a^r is adjacent to another. If $a^r(i) = \underline{z}$ for some $i \in N$, then $a^r(i) = \underline{z}$ for all $i \in N$. Note that there exists $S \in 2^N$ such that $\phi(S) \neq \varnothing$. So, there exists $x \in \bar{\phi}(N)$ such that $xP_r\underline{z}$. Moreover, $xP_i\underline{z}R_ix^i$ for all $i \in N \backslash r$. Thus, $(N, x) \in W^r$. This is a contradiction. ∎

Definition 1. The *r-hierarchical benevolence solution* for society $\mathcal{S}$ is the collection of all potential allocations a^r constructed above;

$$\mathcal{H}^r(\mathcal{S}) = \left\{ a^r_{\{(S^i, x^i)\}_{i \in N}} \mid \left(S^i, x^i\right) \in A^i \text{ for all } i \in N \right\}.$$

The *hierarchical benevolence solution* is

$$\mathcal{H}(\mathcal{S}) = \cup_{r \in N} \mathcal{H}^r(\mathcal{S}).$$

Apparently, $\mathcal{H}^r(\mathcal{S}) \neq \varnothing$ by Lemma 2. The following example illustrates the algorithm in a simple society with three individuals and three alternatives.

Example 2. Consider society $\mathcal{S} = (N, X, R, \phi)$ where $N = \{1\ 2\ 3\}$, $X = \{x\ y\ z\}$, $\phi(1) = \{y\}$, $\phi(2) = \{z\}$, $\phi(3) = \{z\}$, $\phi(1\ 2) = \{y\ z\}$, $\phi(2\ 3) = X$, $\phi(1\ 3) = \{y\ z\}$, and $\phi(N) = X$. Individuals are linked according to their labels $1-2-3$ and their preferences are the following:

$$\begin{array}{c} xP_1yP_1z \\ yP_2zP_2x \\ zP_3yP_3x \end{array} .$$

The preference profile has connected support and ϕ is monotonic. Let 1 be the root agent for example. We construct the admissible sets starting at level 3. Agent 3 can only form a one-person coalition, so $A^3 = \{(3, z)\}$. Agent 2 can form $\{2\}$ with z or $\{2\ 3\}$ with z. (x, y are also feasible but to keep 3 as well off as consuming z, 2 cannot choose x or y.) Since $\{2\ 3\}$ is maximal, $A^2 = \{(2\ 3\,, z)\}$. Last, 1 can form $\{1\}$ with y, $\{1\ 2\}$ with y, or $\{1\ 2\ 3\}$ with z. Since y is more preferred, $A^1 = \{(1\ 2, y)\}$. We pick $(S^1, x^1) = (1\ 2, y)$, thus $L(1) = \{3\}$ and $(S^3, x^3) = (3, z)$. The coalitions $\{1\ 2\}, \{3\}$ with alternatives y, z respectively constitute an allocation a^r. This is the only allocation in $\mathcal{H}^1(\mathcal{S})$. $\mathcal{H}^2(\mathcal{S})$ and $\mathcal{H}^3(\mathcal{S})$ can be constructed in the same way. ∎

There are times when single-valued selections are desired. It is, however, difficult to find a natural selection of an admissible pair from A^i. The coalitions in A^i does not have inclusion relationships unless the tree is linear and r is one of the end vertex. Even if we can select a largest coalition in A^i, this coalition may support two alternatives that are indifferent to i. It seems only a predetermined linear order on X can break the ties. The admissible sets can be further refined to

$$\hat{A}^i = \left\{(S, x) \in A^i \mid x > x',\ \forall (S', x') \in A^i\right\},$$

and the solution $\mathcal{H}^r(\mathcal{S})$ is single-valued. Another special case is when there is no indifference, then every $A^i = W^i$ is a singleton.

4. Stability and Equity

Definition 2. A feasible allocation a in society $\mathcal{S}$ is in the *coalition structure core* if there is no coalition $S \in 2^N$ that blocks it. A coalition $S \in 2^N$ *blocks* allocation a if there is an alternative $x \in \phi(S)$ such that $xP_ia(i)$ for all $i \in S$.

Definition 3. A feasible allocation a in society $\mathcal{S}$ is *envy-free* if $a(i) R_i a(j)$ for all $j \neq i$ for all $i, j \in N$.

The notion of envy-free is defined as a version of Tiebout equilibrium in previous literature. It is combined with the coalition structure core as a stronger stability concept. However, this combined stability does not seem consistent: Coalition structure core allows a blocking coalition to form if "every" member can be better off. This means that a coalition can exclude members. One cannot join a coalition if her arrival makes others worse off. On the other hand, Tiebout equilibrium allows an individual to join another coalition freely. It is possible to make existing members worse off when joining a coalition. For example, one can move into a congested community and reduce the welfare of its residents. This means Tiebout equilibrium does not allow coalitions to exclude individual members. These two concepts treat memberships differently; coalitions can exclude members in the former but not in the latter. It is difficult to interpret how coalitions form in the combined notion. Therefore, we differ from the literature and use envy-free, an equity criterion, in accordance with the fair allocation literature (see Foley 1967 and Tadenuma and Thomson 1995).

Theorem 1. *For any society $\mathcal{S}$ satisfying monotonicity and connected support, if an allocation $a \in \mathcal{H}(\mathcal{S})$, then it belongs to the coalition structure core and is envy-free.*

Proof. Suppose allocation $a^r \in \mathcal{H}(\mathcal{S})$ is constructed on G^r with admissible pairs $\{(S^i, x^i)\}_{i \in N}$. The proof is composed of the following three lemmas.

Lemma 5. *If S and T are two adjacent coalitions in allocation a^r and they have a linking edge ij such that $i \in S$ and $j \in T$, then*

$$a^r(i) R_h a^r(j) \text{ for all } h \in M(ij),$$
$$a^r(j) R_h a^r(i) \text{ for all } h \in M(ji),$$

where $M(ij) = \{h \in N \mid ij \notin p(h, i)\}$.

Proof. Without loss of generality, suppose i is of a higher level than j. Note that $T \subset N^i$, $T \cup i \in \tilde{C}^i$, and $a^r(i) R_i x^i$. First, $a^r(j) \in \phi(T \cup i)$ by monotonicity, and $(T \cup i, a^r(j)) \in B^i$. Thus, $a^r(i) R_i a^r(j)$ by construction. Second, suppose $a^r(i) R_j a^r(j)$; then $a^r(i) R_j x^j$ by construction. By monotonicity, $a^r(i) \in \phi(S \cup j)$. Let g be the highest level agent in S. Note that $T \subset N^g$. Therefore, $(S \cup j, a^r(i)) \in W^g$ which means $(S, a^r(i)) \notin A^g$; a contradiction. So, $a^r(j) P_j a^r(i)$. Third, suppose $a^r(j) R_h a^r(i)$ for all $h \in S$. Then, $a^r(j) R_h x^h$ for all $h \in S$. By monotonicity, $a^r(j) \in \phi(S \cup T)$. Therefore, $(S \cup T, a^r(j)) \in W^g$, which means $(T, a^r(j)) \notin A^g$; a contradiction. So, there is $h \in S$ such that $a^r(i) P_h a^r(j)$. Finally, by connected support, there is no $h \in M(ij)$ such that $a^r(j) P_h a^r(i)$, and no $h \in M(ji)$ such that $a^r(i) P_h a^r(j)$. ∎

Lemma 6. *$a^r(i) R_i a^r(j)$ for all $i, j \in N$.*

Proof. Each pair i, j are linked by a unique path that passes through adjacent connected coalitions. Let $p(i,j) = \{i_0 i_1, i_1 i_2, ..., i_{k-1} i_k\}$ and $i_0 = i$, $i_k = j$. For all $m = 1, ..., k$, either i_{m-1} and i_m belong to the same coalition and $a^r(i_{m-1}) = a^r(i_m)$, or $i_{m-1} i_m$ links two adjacent coalitions, $i \in M(i_{m-1} i_m)$ and $a^r(i_{m-1}) R_i a^r(i_m)$ by Lemma 5. So, $a^r(i_0) R_i a^r(i_k)$. ∎

Lemma 7. *There exists no $S \in 2^N$ that blocks a^r.*

Proof. Suppose S blocks a^r with x and S is not connected. Let T be the minimal connected set containing S. That is, $T \supseteq S$, $T \in 2^N$ and there is no connected $T' \in 2^N$ such that $T' \subset T$, $T' \supseteq S$. For any $h \in T \backslash S$, we can find $i, j \in S$ such that h is on the path linking i and j. We have $x P_i a^r(i) R_i a^r(h)$ and $x P_j a^r(j) R_j a^r(h)$ by Lemma 6 and that S blocks a^r. Connected support implies $x P_h a^r(h)$. Hence, $x P_h a^r(h)$ for all $h \in T$, and $x \in \phi(T)$ by monotonicity. This means T blocks a^r with x as well. Therefore, we have $x P_h a^r(h) R_h x^h$ for all $h \in T$. Suppose g is the highest level agent in T, then $T \in N^g$ and $(T, x) \in B^g$. This contradicts with $x P_g x^g$.

If the blocking coalition S is connected, the second half of the proof applies. ∎

Since $\mathcal{H}(\mathcal{S}) \neq \varnothing$, our result means that the intersection of the coalition structure core and envy-free allocations is nonempty for any society satisfying connected support and monotonicity.

5. Conclusion

We study the endogenous formation of coalitions that provide public goods to members. Allocations that are both stable and equitable exist under the following two assumptions. First, coalitions can afford more as new members join. Second, people are linked on a tree graph and those with the same strict preferences over a pair of alternatives are connected. The most common tree graph used in the literature is the linear order. For example, people are ranked by income or taste in most models with public goods. We present an well-defined algorithm to construct allocations that belong to the intersection of the coalition structure core and envy-free allocations. This solution can be further motivated by a noncooperative story of how coalitions form, based on the following observation. Policies are usually advocated by a few leaders. Others, evaluating the proposals, decide whether to follow the leaders or not. This suggests that policy leaders, who announce public alternatives, may serve the roles of benevolent dictators and initiate the formation of coalitions.

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