On Heterogeneous Size of Stable Jurisdictions

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Abstract

This paper examines a model of multi-jurisdictional formation considered by Alesina and Spolaore (1997) and Le Breton and Weber (2003), where individuals' characteristics are uniformly distributed over the finite interval. Every jurisdiction chooses a location of a public good and equally shares the cost of production among its residents. We consider various notions of stability: Nash-stability, local stability, ε-stability and group-stability, and examine the existence and characterization of stable partitions. The main feature of our analysis is the *heterogeneity gap* in jurisdiction sizes that may emerge for *every* stability notion mentioned above. These results may explain the creation of cities of different size even though the individuals' characteristics are represented by the uniform distribution.

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1 Introduction

Consider a problem where an urban population has to decide how many horizontally[1] differentiated public projects (Mas-Colell (1983) (say, libraries) to be built in the city, where to locate these libraries, how to assign each individual to a library, and, finally, how to split the burden of financing the libraries among the residents.

The group choice of the locational problem described here consists of three items:

• - *jurisdictional structure*, which is a partition of the set I into subsets of individuals, *jurisdictions*, assigned to the same library;

• - *libraries locations* in each jurisdiction, and

• - *sharing rule*, that is a choice of a contribution scheme in order to cover the total cost of libraries in all jurisdictions.

In this paper we focus on the first part of the problem, namely, a search for a partition of the entire population into several jurisdictions. As far as a library location within each jurisdiction and sharing the libraries cost, we impose two following principles, *efficiency* and *equal share*.

The efficiency requires that each jurisdiction chooses a location of the library in such a way as to minimize the total transportation cost of its residents. Under our assumption of linear transportation costs, the efficiency is

[1]For the analysis of vertically differentiated products, see Guesnerie (1995), Guesnerie and Oddou (1981, 1988), Greenberg and Weber (1986), Jehiel and Scotchmer (2001), Weber and Zamir (1985), Westhoff (1973), Wooders (1978,1980).

equivalent to the majority voting requirement, and each jurisdiction should place the library at the location of its *median* resident. Following Alesina and Spolaore (1997) and Haimanko et. al (2004a), we also impose the assumption of equal share where all members of the same jurisdiction make equal contributions towards the financing of the public project.[2]

We then introduce several notions of stability. The first is the standard Nash stable partition which is immune against individual moves from jurisdiction to jurisdiction. The alternative notions of stability is *local* and *ε-stability* where small group of individuals are allowed to migrate to different jurisdiction while taking into account the median location of the public project and the equal share cost mechanism in a newly created jurisdiction. Finally, we turn to *group* or *coalitional* stability, where the migration possibilities are unrestricted and every group of individuals is allowed to migrate or even create a new jurisdiction.[3]

We then derive the conditions for the existence of each of these stability notions and characterize stable partitions. The special attention will be devoted to the so-called *heterogeneity gap* in sizes of jurisdictions in stable partitions. We show that, unlike in Alesina and Spolaore (1997), each of our stability notions may yield a stable partition with the sharply distinct jurisdictional sizes. Thus, even the uniform distribution of individuals' char-

[2]See Haimanko et. al (2004b) and Le Breton et. al (2004) for alternative approach to cost sharing mechanisms.

[3]The latter concept is obviously linked to the core of the associated cooperative game.

acteristics in an environment with horizontally differentiated projects, does not guarantee the uniformity of jurisdictional sizes.

2 The Model

We assume that individuals are located over the interval $I = [0, 1]$. The distribution of their locations is assumed to be uniform, in order to keep things as simple as possible, while stressing the stability issues. The analysis of the general case is more involved, though main ideas and notions are highlighted even in the simplest possible case of the uniformly distributed population.

The total mass of I is equal to 1 and we denote by λ the associated (Lebesgue) probability measure. The cost of every library is given by a positive parameter g. The transportation cost incurred by individual t[4], assigned to a library located at point p, is given by the cost function $d(t, p) = |t - p|$.

The efficiency requires that each jurisdiction chooses a location of the library in such a way as to minimize the total transportation cost of its residents. The linearity of transportation costs, the efficiency is equivalent to the majority voting requirement, and each jurisdiction should place the library at the location of its *median* resident. Formally, for each measurable

[4]We will not distinguish between individual t and an individual located at point t.

subset S of I, denote by $M(S)$ the set of *median locations* defined by:

$$M(S) = \left\{ p \in I : \lambda\left(\{t \in S : t \leq p\}\right) = \lambda\left(\{t \in S : t \geq p\}\right) = \frac{1}{2}\lambda(S) \right\}.$$

Note that $M(S)$ is always an interval, denoted by $[\underline{m}(S), \overline{m}(S)]$. Since the minimization of the aggregate transportation cost of jurisdiction S:

$$D(S) = \min_{p \in I} \int_S d(t,p)dt$$

is attained when $p \in M(S)$, it follows that, whenever jurisdiction S forms, it selects a library location p from the set $M(S)$. If the jurisdiction S is an interval itself, it has a uniquely defined median, $m(S)$ that coincides with $\underline{m}(S)$ and $\overline{m}(S)$. However, if S is not an interval, the set of its median locations could be multi-valued. In this case we select the unique location $m(S)$ as follows:

$$m(S) \equiv \frac{\underline{m}(S) + \overline{m}(S)}{2}.$$

A sharing rule describes the monetary contribution of each individual t towards the cost of the libraries. We assume that each jurisdiction is self-supportive and covers the cost of its own library. A measurable function x defined for members of jurisdiction S is called a sharing rule (for S) if it satisfies the budget constraint:

$$\int_S x(t)dt = g.$$

Whenever jurisdiction S forms, the total cost of individual $t \in S$, given selected library location p and cost allocation x, is

$$|t - p| + x(t).$$

We assume that whenever jurisdiction S forms, it distributes the cost equally among its residents, i.e.,

$$x(t) = \frac{g}{\lambda(S)}.$$

Under efficiency, specified location rule and equal-share, our model turns *hedonic*, where the identity of a jurisdiction is sufficient to determine the total cost, transportation and contribution, of each of its members. Thus, our choice would be to select a jurisdictional structure under various notions of stability. We will consider finite partitions of I into measurable sets of positive measure[5]. In fact, we can confine ourselves with the family of jurisdictions which are just unions of finite number of intervals.

In order to avoid pathological situations, we assume that the median points $m(S)$ are different for all jurisdictions of the partition; it is stupid to has two or more jurisdictions with one and the same library location, for their pooling together would not change the location, at the same time decreasing costs of everyone involved. In the next section we turn to the examination of various notions of stability.

3 Various notions of stability: definitions

In what follows, we introduce a concept of n-partition for an arbitrary positive integer n, where n-partition consists of n sets with a positive measure (for

[5]If one allows for null-set jurisdictions, then its members would experience infinitely high costs; at the same time, if such a jurisdiction joins any of jurisdictions of positive measure, neither hurts from the latter one since the location of a new, pooled jurisdiction does not change.

simplicity, these sets are all just unions of finite number of intervals). To begin with, we introduce a notion that will prove to be useful.

Definition: An n-partition $P = (S_i)_{1 \leq i \leq n}$ is called *stratified* or *consecutive*[6] if every set $S_i \in P$ is an interval.

A stratified n-partition could be identified with $n-1$ points $0 < x_1 < \ldots < x_{n-1} < 1$ such that $S_i = [x_{i-1}, x_i] fori = 1, \ldots, n$, where, by definition, $x_0 = 0$, and $x_n = 1$. In what follows, we will simply write $P = (x_1, \ldots, x_{n-1})$ to denote a stratified partition.

Nash Stability

We will denote by $v_t\,(S)$ the payoff of individual $t \in S$, when jurisdiction S chooses $m(S)$ as its library location:

$$v_t\,(S) = -|m(S) - t| - \frac{g}{\lambda(S)}.$$

Definition: A jurisdictional structure $P = \{S_1, \ldots, S_n\}$ is *Nash stable* if the following is true:

$$v_t(S^t) \geq v_t(S_i)$$

for all $t \in I$ and all $S_i \in P$, where S^t denotes the jurisdiction in P that contains t.

[6]See Greenberg and Weber (1986) and Demange (1994)

Note that, due to excluding jurisdictional structures in which different jurisdictions could choose same library locations, the set of Nash stable jurisdictional structures is the set of pure Nash equilibria of the non-cooperative game where each agent announces his "address," and all the agents with same addresses form a coalition.

Alternatively, we could see a Nash stable jurisdictional structure as a *free individual mobility equilibrium*: no individual has an incentive to move from his/her current location to another jurisdiction[7].

Locally stable Nash equilibrium

Since no single individual has an impact on the library location selected by a jurisdiction, the impact of his migration from one jurisdiction to another on other individuals is ignored. However, if an individual t finds it beneficial to migrate to another jurisdiction, then other individuals, close to t, would contemplate the same migrational choice. Then the group of individuals willing to migrate has a positive measure and its relocation would affect both the new jurisdiction and that they left behind. To introduce a possibility of a migration by a small group with a positive measure, while preserving the spirit of the Nash paradigm, we adopt the stability concept of "free mobility equilibrium" of Jehiel and Scotchmer (2001):

Definition: A jurisdictional structure $P = \{S_1, \ldots, S_n\}$ is said to be *stable*

[7]We should emphasize here that there is no physical movement of agents' addresses, t; agents are allowed only to choose their *jurisdictions*.

Nash equilibrium if there exists $\varepsilon > 0$ such that for every set-valued *matrix of transitions* $\{S_{ij}\}_{i,j=1,\ldots,n}$ with $S = \cup_{i,j} S_{ij}$ and $\lambda(S) \leq \varepsilon$, there exists $t \in S$ such that

$$v_t\left(S'^t\right) < v_t\left(S^t\right),$$

where S^t and S'^t are the groups that contain t in the initial partition and in the perturbed partition, respectively. Here, $S_{ij} \subset S_i \ \forall i, j$.

This definition surely needs to be clarified. By a *matrix of transitions* we mean that S_{ij} is the set of agents from jurisdiction S_i that try to move to jurisdiction S_j, during the "test for local stability".

Fortunately, for stratified partitions, this kind of stability is in fact implied by easily tractable requirement that no pair of small intervals of equal length located at the two borders outside one and the same jurisdiction is willing to move to adjacent jurisdiction.

Definition: A stratified partition $P = (x_1, \ldots, x_{n-1})$ is called *locally stable Nash equilibrium* if there exists $\varepsilon > 0$ such that for every $\mu < \varepsilon$ and for every $i = 1, \ldots, n-1$

$$v_{x_i-\mu}\left([x_i - \mu, x_{i+1}]\right) < v_{x_i-\mu}\left(S_i\right), \tag{1}$$

and

$$v_{x_i+\mu}\left([x_{i-1}, x_i + \mu]\right) < v_{x_i+\mu}\left(S_{i+1}\right). \tag{2}$$

Sometimes we will address these as *locally stable* partitions, oppose to *stable Nash equilibrium* introduced above; the latter notion is weaker, and requires only that local (or connected) small groups could move. This notion could be sometimes more appropriate than that allowing for far-distant small group to change their jurisdictions.

Group Stability

Definition: Let $P = \{S_1, \ldots, S_n\}$ be an n-partition of I. We say that a coalition $S \subset N$ *blocks* P if

$$v_t(S) > v_t(S^t)$$

$\forall t \in S$, where S^t is a group containing t. A partition P is called *group stable* (or simply, *stable*) if there is no coalition S that blocks P.

In this definition, blocking or deviation by a coalition occurs whenever all members of the deviating coalition strictly prefer the new jurisdiction to that they were members before the move.

It is important to observe that on logical grounds, the notions of stability and Nash stability are independent. On one hand, one can consider Nash stability as a weak form of stability which limits the set of feasible coalitions. However, the migration of an individual from a jurisdiction to another jurisdiction is not contingent on some sort of approval by the members of the jurisdiction that he joins.

On the other hand, one could have been extended the set of moving coalitions to all the measurable sets, still requiring an unanimous approval by the current members of the jurisdiction towards which migration is directed.[8] This is a weaker and slightly modified notion of group stability — because only specific group formation is considered, but strictly increasing payoffs for incumbents are not required, just nondecreasing.

One could also consider a more general set of migration moves described by some given a priori rules that could be interpreted as constitutional requirements governing migration with respect to the current jurisdictional structure. Alesina and Spolaore 1997 consider such a form of stability; their notion of *B-stability* requires that any change from the current jurisdictional structure is approved by a majority of residents in any of the jurisdictions affected by the change.

In our examination we distinguish between two types of jurisdiction structures, *homogeneous*, where all jurisdictions are of the same size, and *heterogeneous*, that consist of jurisdictions of different size. Our prime goal is to demonstrate the existence of heterogenous stable partitions, for each of stability notions discussed above, as well as to estimate the *heterogeneity gap*, that is, to what extent the jurisdictions in a stable partition could differ from each other in their size. We begin with the analysis of Nash stability.

[8]This notion of coalitional stability, called *individual Nash stability* (see Bogomolnaia and Jackson 2002) will not be discussed in this paper.

4 Nash stable partitions

Obviously, the set I itself is trivially a Nash stable 1-partition. We shall show that the set of Nash stable n-partitions is nonempty for every n and provide a complete characterization of this set.

The following observation will be useful in our discussion:

Proposition 4.1: Every Nash stable jurisdictional structure is stratified.

Proof: By contradiction, suppose that we have a non-stratified Nash stable jurisdictional structure P; hence, there exist $S, T \in P$, and $s_1, s_2 \in S,\ t \in T$ such that $s_1 < t < s_2$. Define $\varphi_S(x)$ and $\varphi_T(x)$ as follows:

$$\begin{array}{l} \varphi_S(x) = |x - m(S)| + \frac{g}{\lambda(S)}, \\ \varphi_T(x) = |x - m(T)| + \frac{g}{\lambda(T)}. \end{array} \tag{3}$$

That is, $\varphi_S(x)$ equals to the total cost incurred by agent x if he belongs to jurisdiction S, and $\varphi_T(x)$ equals to his costs as the member of T. Now, three cases are distinct:

Case 1. $\forall x \in I \quad \varphi_T(x) \leq \varphi_S(x)$. In this case, a partition P is by no means Nash stable, since each agent from S would be better off in the jurisdiction T[9].

Case 2. $\forall x \in I \quad \varphi_S(x) \leq \varphi_T(x)$. Again, a partition P could not be Nash stable, for just the same reasons.

[9]See figure 1 for details. Some technicalities arise when the inequality is not strict everywhere.

Case 3. Neither of functions φ_S, φ_T dominates the other one. WLOG, assume that $m(S) < m(T)$ (in the opposite case, just revert the real line). Then, the difference $\varphi_S - \varphi_T$ is a nondecreasing function, see figure 1.

Hence, if $\varphi_S(t) > \varphi_T(t)$ then we have $\varphi_S(s_2) > \varphi_T(s_2)$ too, hence, s_2 will move to the jurisdiction T; if $\varphi_S(t) < \varphi_T(t)$ then t will move to S. Again, some technicalities present when $\varphi_S(t) = \varphi_T(t)$, though everything becomes clear from figure 1. □

We now state and prove the following result:

Proposition 4.2: For every integer $n > 0$ there is a Nash stable n-partition.

Indeed, note first of all that a stratified n-partition is Nash stable if and only if every individual located at $x_1, \dots, x_{n-1}$ is indifferent between joining two adjacent jurisdictions. In other words, the equation

$$x_k - m(x_{k-1}, x_k) + \frac{g}{x_k - x_{k-1}} = m(x_k, x_{k+1}) - x_k + \frac{g}{x_{k+1} - x_k} \qquad (4)$$

holds for all $k = 1, \dots, n-1$, where for every $0 \le a \le b \le 1$, $m(a, b)$ denotes the median of the interval $[a, b]$. Indeed, violation of any of these equalities induce even positive mass of agents to move to the adjacent jurisdiction; if all these equations hold, no individuals located at the endpoints $x_1, \dots, x_{n-1}$, would be willing to move to another jurisdiction in the same partition. But then no other individual would be willing to migrate as well.

Knowing that $m(a, b) = \frac{a+b}{2}$, rewrite conditions (4) in the form of

$$\Psi(x_k - x_{k-1}) = \Psi(x_{k+1} - x_k), \qquad (5)$$

for all $k = 1, \ldots, n-1$, where

$$\Psi(s) \equiv \frac{g}{s} + \frac{s}{2} \tag{6}$$

is the total cost incurred by the peripheral individual who is located at the border of a jurisdiction of size s (this function is being widely used in what follows). In tact, $\Psi(s) = \varphi_{[0,s]}(s)$.

And now we can see that, at least when every jurisdiction is of the same size equal to $1/n$, conditions (5) are trivially satisfied.□

By (5), if we introduce $s_k = x_k - x_{k-1}$, in any Nash stable n-jurisdictional structure $P = \{S_1, \ldots, S_n\}$, we have:

$$\Psi(s_k) = \Psi(s_{k+1}) \text{ for all } k = 1, \ldots, n-1$$

The function Ψ, whose minimum is attained at $\sqrt{2g}$, is strictly convex, and thus, there could be two values of s that satisfy the above equation. It follows, therefore, that there could be heterogeneous Nash stable jurisdiction structures. Moreover, we are able to prove that if the cost value g is small enough then there exists a heterogenous Nash stable n-partition. Let

$$g^n = \frac{1}{8(n-1)}.$$

Proposition 4.3: If $g < g^n$, then there exists a Nash stable heterogenous n-jurisdictional structure. It contains jurisdictions of two different sizes, $s < s'$, which satisfy the following equation:

$$\Psi(s) = \Psi(s'). \tag{7}$$

Moreover, a Nash stable heterogenous n-jurisdictional structure exists if and only if $g \leq g^n$ for $n > 2$, and $g < g^n$ for $n = 2$.

Proposition 4.4 - Heterogeneity gap: Let $g < g^n$ and denote by $\underline{s}(g)$ and $\overline{s}(g)$, respectively, the maximal size of the smaller jurisdiction and the minimal size of a larger jurisdiction over all heterogenous Nash stable partitions. Then

$$\lim_{g \to 0} \frac{\overline{s}(g)}{\underline{s}(g)} = \infty.$$

In contrast, the A-stability criterion of Alesina and Spolaore requires that the size s of a jurisdiction satisfies:

$$\Psi(s + \varepsilon) > \Psi(s - \varepsilon) \text{ for all } \varepsilon > 0.$$

This means that if all individuals located in the interval, say, $[s, s + \varepsilon]$ contemplate to join the jurisdiction $[0, s]$, then the individual located in $s + \varepsilon$ prefers to stay in his previous (and smaller) jurisdiction $[s + \varepsilon, 2s]$. This condition is not an equilibrium criterion as the benchmark to test the profitability of the deviation is not the status quo before the deviation takes place. If the individual located at $s + \varepsilon$ does find the deviation beneficial why wouldn't he assume that his immediate could share the same belief? If this is the case, no movement would take place and A-stability criterion is not very useful.[10]

[10] This criterion is close in spirit to a concept of CPNE.

Instead, we elaborate on the notion of local stability introduced above. We will ask how the reluts would change if we elaborate on alternative notions of stability introduced above.

5 Locally stable and group stable partitions

Proposition 5.1: Let P be a locally stable n-jurisdictional structure. Then for every $S \in P$, $s \geq \sqrt{\frac{2g}{3}}$.

Proposition 5.2: Let P be a stable Nash equilibrium n-jurisdictional structure. Then for every $S \in P$, $s \geq \sqrt{g}$.

Proposition 5.3: Let g be given. Then, homogeneous n-jurisdictional structure is locally stable for $n = 1$ or $n \leq \sqrt{\frac{3}{2g}}$; and it will be a stable Nash equilibrium if, in addition, $n = 2$ or $n \leq \sqrt{\frac{1}{g}}$.

Proposition 5.4: In every heterogenous locally stable partition, the heterogenous gap

$$\frac{\overline{s}(g)}{\underline{s}(g)} \leq 3;$$

In every heterogenous stable Nash equilibrium partition, the heterogenous gap

$$\frac{\overline{s}(g)}{\underline{s}(g)} \leq 2.$$

Proposition 5.5: Let $g < g^n$ and n is sufficiently large (say, $n \geq 15$). Then, there exists a heterogenous stratified n-partition, which is locally stable and, even more, it is a stable Nash equilibrium.

Now, we turn to group stability:

Proposition 5.6: There always exists a stratified stable partition. The heterogenity gap for stable partitions is unbounded above:

$$\lim_{g \to 0} \frac{\overline{s}(g)}{\underline{s}(g)} = \infty.$$

Proposition 5.7: A stratified n-partition, which consists of intervals of equal length s, is stable if and only if $\sqrt{g} \leq s \leq \sqrt{g}\left(2 + \sqrt{2}\right)$.

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