Social Networks and Interdependent Utilities : Peer Effects, Efficiency and Distributional Conflicts[*]

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Abstract

We present a model where agents are embedded in a network of social relations and experience interdependent utilities: each agent total utility depends on his private utility, but also on the total utilities of the agents with which he is linked. We study the peer effects, meaning global and local externalities, that are generated by these interdependencies, obtaining closed form expressions for any generic situation that rely crucially on the network structure. Next, we study which are the efficient allocations in this setting. We characterize two polar situations: when the Pareto frontier is degenerated (there is a unique efficient allocation), and when the Pareto frontier is regular (all the allocations that exhaust resources are efficient). This characterization is done by means of a sociological concept, prestige, that measures the prominence of an agent derived from his position in the network. Finally, we study ditributional conflicts with interdependent utilities describing how the network structure affects the Nash bargaining solution.

JEL Classification: A13, C78, D11, D13, D62, D64, L14

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“Le cœur a ses raisons que la raison ne connaît point.”
(“The heart has its reasons that reason knows nothing about.”)
Blaise Pascal, *Pensées*

1 Introduction

Sociology and economics have followed different directions in the study of human behavior. Roughly speaking, sociology has centered its attention to collective behavior without taking individual motivations or incentives into account, while methodological individualism has been the principal (and probably the more fruitful) behavioral assumption in economic models. The need for a midpoint between both approaches is expressed in the following opinions of two leading scholars, one of each discipline, George Akerlof and Mark Granovetter:

> “Traditional economics has been based on methodological individualism. Until quite recently, with some rare exceptions, it has not been appreciated that this method can be, or perhaps I should say, should be, extended in describing social decisions to include dependence of individuals utility on the utility or actions of others.” Akerlof (1997, pg.1005)

> “A fruitful analysis of human action requires us to avoid the atomization implicit in the theoretical extremes of under- and oversocialized conceptions. Actors do not behave as atoms outside a social context, nor do they adhere slavishly to a script written for them by the particular intersection of social categories that they happen to occupy. Their attempts at purposive action are instead embedded in concrete, ongoing systems of social relations.” Granovetter (1985, pg.487)

Only recently a mixture between both approaches has arised and new economic models where social interactions take a prominent role have been studied and developed.[1] A especially satisfactory example of this interaction is the application of social networks analysis into the study of a variety of economic phenomena.

Not only *homo economicus* has been critizised for his lack of social relations but also for his persistent selfish behavior. There is also an emerging literature on social preferences that considers human behavior a more complex issue. A person exhibits social preferences if this person not only cares for the material resources allocated to him but also cares for the material resources allocated to other agents in the economy. Altruism, envy, fairness or reciprocity are pervasive kinds of behavior that we can observe in our daily life, and this kind of behavior may have important economic implications.

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[1] See Manski(2000) for a good introduction.

Our work presents a model where agents are socialized, that is, agents are embedded in a social network of relations, and exhibit social preferences: each agent has a private utility over consumption, but his total utility depends also on the total utility of the agents with which he is linked. This dependency on other's utility is linear, and this is what allows us to make a clear relation between social networks and interdependent utilities. In particular, each individual can love or hate with variable intensity any other agent with which he is linked, and if agent i loves (resp. hates) agent j the total utility of agent i will depend positively (resp. negatively) on the total utility of agent j.

Hence, there is a clear interplay between the social network of relations and agents's behavior, because the network structure generates a system of utility interdependencies. The structure of the network can generate recursivities by means of indirect effects between agents that are not directly linked but that are indirectly linked through a chain of links of adjacent agents. We devote the first part of our paper to study the implications the sum of these indirect effects, what we call peer effects, has on the final sign and intensity of the feeling an agent will show for another one. The second part of the paper is devoted to study some important economic implications of our model. The aim is twofold. First, we center our attention on which are the Pareto efficient allocations. Later, we study how the emotional concern people has for each other can change the solution of distributional conflicts.

Social relations are a quite complex issue. Sometimes they are contradictory (the friends of my friends are not necessarily my friends; I can hate or envy them), sometimes they can be specially attractive (if two friends of mine are also friends, this can reinforce our links and make our friendship stronger). As we will see, the cycles and paths of the network of relations generate different kinds of spillovers. Cycles are much more related to sentimental reinforcement while paths may principally generate sentimental contradictions. In fact, we can disentangle the peer effects into two different effects in nature: the social multiplier effect and the externalities effects. The social multiplier effect is a global and homogeneous effect that subsumes the reinforcement effect of cycles. The externalities effects are local effects that subsume the possible spillovers generated by all the possible paths from an agent to another one.

The next step in our work is to study the Pareto efficient allocations in our model. Economists are generally reluctant to deviate from standard economic assumptions, but it is a fact that we live in a social environment where we relate continuously with other people, and this can have an important effect on what is efficient and what is not. An extended idea is that we can treat an economy as a set of isolated selfish agents and that this is a good approximation of reality, but no rationale has been provided for this assertion. In fact, our model predicts that peer effects can have a large impact on what is socially efficent. We characterize two extreme situations: when the Pareto frontier is formed by a unique allocation, what we call the *degenerate case*, and when the Pareto frontier is regular, meaning that all the allocations that exhaust the available resources are efficient, that we call the *regular case*.

An important and suggestive feature of our characterization is that we do it by means of a sociological concept: *prestige*. *The prestige an agent acquires in a social network measures how valuable this agent is for the rest of individuals in the network.* The use of this concept is natural in our setting because valuable has a clear meaning in our model. Consider for example a situation in which an agent is more loved than another one for all the agents in the economy, that is, the first agent is more valuable, more prestigious, than the second for all the individuals in the network. In this case the second agent should not consume anything, because everybody, included himself, would be better off if he transfers his resources to the first agent. In what refers to degenerate situations, we obtain that the unique efficient allocation is that certain agent consumes everything if this agent is more prestigious than any other one and his prestige is positive, while the unique efficient allocation is that nobody consumes if the prestige of all agents is negative. On the other side, the Pareto frontier is regular if all agents are equally prestigious. A particularly striking implication of our model is that, against the extended idea that altruism always exherts positive effects on society, an excess of altruism can be harmful for all agents in the network, sometimes to a dramatic extent.

Prestige proofs to be not only useful for treating efficiency but also to deal with distributional conflicts. The last section of our work adresses the problem of how the utility interdependencies affect the Nash bargaining solution, and again our results are characterized through prestige measures. Since when the Pareto frontier is degenerated (there is only one efficient allocation) the bargaining problem is trivially solved, we center our attention in the case of a regular Pareto frontier, when there is competition for resources between all the agents in the network. We proof that, for certain prestige measure, the agents that consume in the Nash bargaining allocation under interdependent utilities have to be equally prestigious, and the ones that do not consume have to be less prestigious than them. Finally we move to the study of a particular situation, the case when there are only two agents, that connects with the literature on bargaining within marriage. We provide the expression of the Nash bargaining solution in this case and obtain a relation between the strength of the emotional links within the two agents and bargaining power.

There is an increasing amount of experimental evidence that shows large and persistent deviations from pure selfish behavior, and that the social environment can have a large impact on economic problems. An interesting example of it is the work by Henrich et al.(2001), where they experiment with ultimatum, public good and dictator games in 15 different small-scale societies of all around the world. The result is surprising. The canonical economic model is not suported in any society studied and there is high variability of results across groups. The authors conclude that this variability can not be explained by individual or demographic variables. They believe it is principally generated by differences in the social structure of each community. Another example is Levine(1998), where the author studies a model with altruism and spitefulness similar to our one. Using principally results on ultimatum game experiments obtains a distribution of altruism in the population that works quite well in explaining other experimental results. The literature in behavioral economics has been specially imaginative in providing models that try to capture attitudes and behavioral patterns that can not easily be explained through the classical rational choice models. Two excellent surveys of the field of social preferences are Fehr and Schmidt(2002) and Sobel(2004).

Our builds on previous work done by Bergstrom(1999) and Bramoullé(2001). Pollak(1976) is among the first papers to provide a definition of interdependent preferences but Bergstrom was the first on introducing the model of interdependent utilities we use here. Bergstrom rigorously defines what a system of interdependent utilities is, either for a finite or a denumerable number of agents, and devotes the rest of the paper to show that several intergenerational models that appeared previously are in fact systems of interdependent utilities. Bramoullé's work is more close in spirit to our one. The first part of his work addresses the study of some basic features of general interdependence systems not initially studied by Bergstrom. In the second part the author considers, as we do, linear interdependence systems to be able to relate social networks with interdependent utilities. He obtains some results, also relying in well-known concepts in sociology, on the relation between social structure and the final interdependence coefficients, the coefficients that express how the private utility of an agent enters in the total utility of another one. These results are qualitative in nature and are restricted to a certain subclass of networks. On the contrary, we provide closed-form expressions of the final interdependence coefficients and a full characterization of the quantitative effects in arbitrary social networks. We are able to obtain neat expressions for the final interdependence coefficients for (almost) every network. With these expressions we can apply our model to obtain not only features of the interdependent utility systems but also some interesting implications about fundamental economic concepts.

2 Social Networks

People is embedded in networks of social relations. Economists had generally disregarded how these networks, or social structure in general, can affect economic problems, and just in the last times there has been an increasing attention on how this non-market interactions may take a prominent role in, for example, how people can obtain a job (Calvó-Armengol, 2004, Calvó-Armengol and Jackson, 2004), workers are organized within a firm (Radner, 1993, Bolton and Dewatripont, 1994, Van Zandt, 1999), or how people tries to coordinate to accept certain social norms and rules or to go on strike (Chwe, 2000, or Young, 2001).

The principal aim of this paper is to study efficiency and distributional conflicts within groups where emotional links may be quite intense. Since social networks arise naturally in this setting, we introduce in this section some basic terms and notation.

A network is a set of agents, generally called *nodes*, $N = \{1, \ldots, n\}$ and a set of links between them. These links are modelled as a set of values $g = \{g_{ij} \in \mathbf{R};\ i,j \in\ N\ and\ i \neq j\}$. There is a link from agent i to agent j if $g_{ij} \neq 0$. If the number of agents is n, a network is completely defined by these $n(n-1)$ coefficients in g. Hence, an equivalent tool to describe a network is its *adjacency matrix*, denoted by $\boldsymbol{A}(g)$, with entries $a_{ij} = g_{ij}$ if $i \neq j$ and zeros in the diagonal. If there is no possibility of confusion, we generally denote a network simply by g. We denote by G_n the set of all networks with n nodes.

We can provide a graphical representation of any network $g \in G_n$. The nodes are represented by circles, and there is a row from node i to node j if and only if $g_{ij} \neq 0$. This kind of object is called a *directed graph*, or *digraph.* For example, the digraph that represents the network $g \in G_3$[2] that has as adjacency matrix

$$\boldsymbol{A}(g) = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & \delta & 0 \end{pmatrix} \tag{1}$$

where α, β, γ, and δ are different than zero is

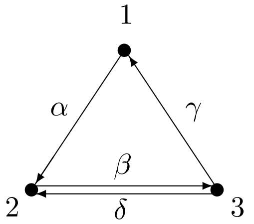

Figure 1.

A network g is *undirected* if $g_{ij} \neq 0$ implies $g_{ji} \neq 0$. A typical situation where an undirected network model arise is the case of the need of mutual consent for the formation of a link. Friendship networks or collaboration networks between firms are examples of it.

A network g is *unweighted* if $g_{ij} \in \{0,1\}$ for every $i,j \in N$. Hence, if a network is unweighted, there is a link from i to j if and only if $g_{ij} = 1$. Generally, in a model with unweighted networks, the important fact is if a link exists or not, not the intensity the link exhibits. This would be the case for example, if we want to study the links between web pages on the internet, where a link means in this case a hyperlink from a webpage to another one.

As we will see in the following section, in our model we allow for unreciprocated relations, meaning that agent i can be linked to agent j while agent j is not linked to agent i, so the networks we study can be directed. Furthermore, our model naturally accepts different intensities for each link, so the networks we study are not only directed but also weighted.

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[2] A part from this example, in the rest of the paper we will simply refer to the network, even if in fact we are refering to its graphical representation

A *path* is an ordered set of agents $p = (i, i_1, \ldots, i_k, j)$ of N, where an agent can appear several times, such that $i \neq j$. We say that the path p belongs to the network g if $g_{ii_1} g_{i_1 i_2} \cdots g_{i_k j} \neq 0$. We say that a path is *simple* if all the nodes of the path are different. In words, a path in g is an indirect connection from agent i to agent j through linked agents in g . We denote by $P_{ij}(g)$ the set of simple paths from i to j in g. Given a path $p \in P_{ij}(g)$ we define its *weight*, that we denote $W(p)$, as the product of weights of the links involved in the paths: $W(p) = g_{ii_1} g_{i_1 i_2} \cdots g_{i_k j}$. A path that do not belong to g has weight equal zero.

Similarly, a *cycle* is defined like a path but with the simple difference that in this case $i = j$. Hence, a cycle is an indirect connection from agent i to himself through linked agents in g. We say that a cycle is a *simple cycle* if the only agent that appears twice in the cycle is the initial-final agent. Observe that a symple path is a path that does not contain any cycle. $C(g)$ denotes the set of simple cycles in g. Like with paths, we define the weight of a cycle $c = (i, i_1, \ldots, i_k, i) \in C(g)$, denoted by $W(c)$, as the product of weights of the links involved in the cycle, that is, $W(c) = g_{ii_1} g_{i_1 i_2} \cdots g_{i_k i}$. We say that a cycle c belongs or is in network g if its weight is different than 0.

For example, consider the following network g

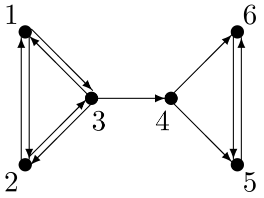
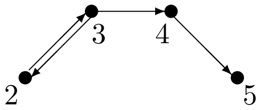


Figure 2.

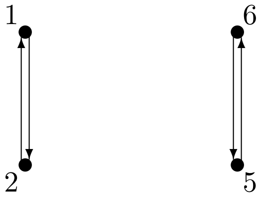
$(1, 2, 3, 4)$ is a simple path in g from 1 to 4, $(1, 2, 1, 2, 3, 4)$ is a path in g that is not simple, and $(5, 4)$ is a path that do not belong to g. Similarly, $(1, 2, 3, 1)$ is simple cycle in g, $(1, 2, 1, 2, 3, 1)$ is a cycle in g that is not simple, and $(3, 4, 3)$ is a cycle that do not belong to g.

Given a network g and a subset $S \subset N$, we denote by $g \backslash S$ the network that is obtained eliminating from g all the nodes in S and all the links that involve a node in S, and this new network $g \backslash S$ has as adjacency matrix the principal submatrix of $A(g)$ obtained by eliminating the rows and columns indexed by S. In particular, given a simple path $p \in P_{ij}(g)$ (resp. cycle $c \in C(g)$), $\{p\}$ (resp. $\{c\}$) denotes the set of nodes involved in this path (resp. cycle), and $g \backslash \{p\}$ (resp. $g \backslash \{c\}$) is the network obtained eliminating this set of nodes.

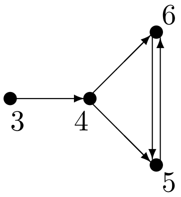
For example, if we consider the previous network g in figure 2, the network obtained by eliminating the nodes 1 and 6, $g \backslash \{1,6\}$, is



the network obtained by eliminating the path $p = (3,4)$ is



and the network obtained by eliminating the cycle $(1,2,1)$ is



3 Interdependent Utilities

Following Bramoullé (2001), we can consider three different types of models of social preferences: (i) agents are concerned for others' allocations, (ii) agents are concerned for others' material or private payoffs, and (iii) agents are concerned for others' social or total payoffs. Obviously the third type of preferences are included in second type, which at the same time are included in the first type. Interdependent utility models are of the third type: agents in a model express interdependent utilities if the total utility of an individual depends not only on his private utility of consumption but also on the total utility of the rest of individuals. In fact, interdependent utilities are the unique kind of social preferences that truly respect the total utilities of other agents. Bergstrom(1999) introduces the first formal definition of interdependent utilities and shows that several economic models that appeared previously are in fact interdependent utility models.[3]

We present in this section a particular model of interdependent utilities. We suppose that the total utility of an agent depends *linearly* on his private utility and on the total utilities of the rest of agents. This linear dependency is crucial in the development of our model. It allows us to provide a natural interpretation of interdependent utilities in terms of social relationships: agents are embedded in a network of social relations and they feel love or hate for the individuals with which they are connected, with different intensities for different agents. This love or hate of agent i versus agent j traduces in a positive or negative sign, respectively, in the coefficient that accompanies the total utility of agent j in the total utility of agent j when defining the interdependencies.

3.1 Linear interdependent utilities

There is a set of n agents, $N = \{1, \ldots, n\}$. As usual, each agent has an increasing *private* utility function on his own consumption, $u_i(c_i)$, but we introduce an extra behavioral assumption: we allow these agents to love or hate. Each agent has also a *total* utility function U_i of the form

$$U_i(c_1, \ldots, c_n) = u_i(c_i) + \sum_{j \neq i} a_{ij} U_j(c_1, \ldots, c_n) \qquad i = 1, \ldots, n \tag{2}$$

that is, each agent may take into account the total utility of each other agent, where a_{ij} is a real number that measures how much agent i cares for agent j. We can imagine that $a_{ij} > 0$ means that agent i loves, apreciates, or takes care of, agent j, while $a_{ij} < 0$ means that agent i hates, or is envious of, agent j. If $a_{ij} = 0$ agent i is indifferent to the situation of agent j. We will call the system of equations in (1) the *interdependence system.*

The interdependence system can be *solved*, meaning that we can express the total utility of each agent in terms of the private utilities of each member of the group, easily. If $\boldsymbol{c} = (c_1, \ldots, c_n)$, $\boldsymbol{U}(\boldsymbol{c}) = (U_1(\boldsymbol{c}), \ldots, U_n(\boldsymbol{c}))$ and $\boldsymbol{u}(\boldsymbol{c}) = (u_1(c_1), \ldots, u_n(c_n))$, we can rewrite the interdependence system as follows

$$\boldsymbol{U}(\boldsymbol{c}) = \boldsymbol{u}(\boldsymbol{c}) + \boldsymbol{A} \cdot \boldsymbol{U}(\boldsymbol{c}) \tag{3}$$

where $\boldsymbol{A}$ is the matrix $\boldsymbol{A} = (a_{ij})_{i,j}$ [4] , that we call the *initial interdependence matrix.* Hence, the unique solution to the interdependence system is, when $(\boldsymbol{I} - \boldsymbol{A})^{-1}$ exists,

$$\boldsymbol{U}(\boldsymbol{c}) = (\boldsymbol{I} - \boldsymbol{A})^{-1} \cdot \boldsymbol{u}(\boldsymbol{c}) \tag{4}$$

Proposition 1 *The interdependence system has a unique solution generically.*

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[3] There is, particularly, a large literature in macroeconomics and in household economics studying intergenerational models that are in fact interdependent utility models with a denumerable number of agents. See Bergstrom(1999) and references therein.

[4] Where $a_{ii} = 0$ for every $i \in N$

Proof: All proofs are in the appendix.

The interdependencies in (1) induce an obvious (weighted and directed) network structure where a link from agent i to agent j exists if $a_{ij} \neq 0$, and the weight of this link is exactly a_{ij}. The network is directed because that agent i cares for agent j, $a_{ij} \neq 0$, does not imply that agent j cares for agent i, $a_{ji} \neq 0$. We call the network associated to an interdependence system the *induced network.* Similarly, each network g generates an *induced interdependence system* that we denote $\boldsymbol{A}(g)$. We call the matrix $\boldsymbol{M} = (\boldsymbol{I} - \boldsymbol{A})^{-1}$ the *final interdependence matrix.* If we want to make explicit the relation from a network g, we will write $\boldsymbol{M}(g) = (\boldsymbol{I} - \boldsymbol{A}(g))^{-1}$.

3.2 Indirect effects and peer effects

In some cases[5] we can express $(\boldsymbol{I} - \boldsymbol{A})^{-1}$ as follows

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{k \geq 0} \boldsymbol{A}^k = \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots \tag{5}$$

Each matrix $\boldsymbol{A}^k$ computes the indirect effects of order k, that is, the entry $\boldsymbol{A}^k_{ij}$ is equal to the sum of weights of all paths of length k that connect agent i to agent j.[6] Hence, the entry $(\boldsymbol{I} - \boldsymbol{A})^{-1}_{ij}$ is equal to the *sum of weights of all the paths of any length* that connect i to j. This sum of indirect effects of any order is what we denote *peer effects.* These peer effects compute how the peers of agent i, and the peers of the peers of agent i, and the peers of the peers of the peers of agent i, and so on, affect the final relations between the total utility of agent i and the private utilities of the rest of agents.

Two components are important for the peer effects: the network structure and the intensities of the links between peers. Furthermore, paths and cycles generate different effects. The weight of a path from agent i to agent j can be very high and of reverse sign compared to the weight of the link from i to j. This generates a tension between the initial feeling that agents i has for agent j and the final concern agent i has for the private utility of agent j. Even if agent i loves agent j, it may be finally the case that the private utility of agent j enters with negative sign in the total utility of agent i. Example 1 shows a simple case where this can happen. On the other hand, cycles have a reinforcement or weakening effect through many different ways. For example, a cycle that involves agent i and that has positive (resp. negative) weight reinforces (resp. weakens) the concern agent i has for his own private utility. This, and other possible effects generated by cycles, are explained in the second example by means of a simple but didactic network.

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[5] Whenever the infinite sum of powers of A expressed below is a well-defined matrix, this matrix is the inverse of $\boldsymbol{A}$. In fact, the necessary and sufficient condition for (4) to hold is that $\boldsymbol{A}$ has to be a contraction, or, what is equivalent, that all the eigenvalues of $\boldsymbol{A}$ have absolute value smaller than 1.

[6] For a proof of this assertion, see Bramoullé(2001).

Example 1. Paths: sentimental contradictions [Bramoullé (2001)]

There are three agents, $N = \{1, 2, 3\}$, and the network g relating them is the following

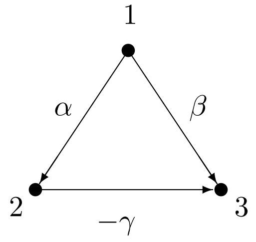

Figure 3.

where α, β and γ are strictly positive parameters. This means that agent 1 loves both agent 2 and agent 3, while agent 2 hates agent 3. Hence, the interdependence system is

$$\begin{aligned} U_1(\boldsymbol{c}) &= u_1(c_1) + \alpha U_2(\boldsymbol{c}) + \beta U_3(\boldsymbol{c}) \\ U_2(\boldsymbol{c}) &= u_2(c_2) - \gamma U_3(\boldsymbol{c}) \\ U_3(\boldsymbol{c}) &= u_3(c_3) \end{aligned} \tag{6}$$

and the initial interdependence matrix $\boldsymbol{A} = \boldsymbol{A}(g)$ is

$$\boldsymbol{A} = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & -\gamma \\ 0 & 0 & 0 \end{pmatrix} \tag{7}$$

In this case we can easily compute the matrices of indirect effects, $\boldsymbol{A}^k$ for $k \geq 2$. The matrix of second order effects is

$$\boldsymbol{A}^2 = \begin{pmatrix} 0 & 0 & -\alpha\gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

while

$$\boldsymbol{A}^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad for\ every\ k \geq 3 \tag{9}$$

We obtain that the final interdependence matrix $\boldsymbol{M} = \boldsymbol{M}(g)$ is

$$\boldsymbol{M} = \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 = \begin{pmatrix} 1 & \alpha & \beta - \alpha\gamma \\ 0 & 1 & -\gamma \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

Hence, the expressions of the total utilities in terms of the private ones are

$$\begin{aligned} U_1(\boldsymbol{c}) &= u_1(c_1) + \alpha u_2(c_2) + (\beta - \alpha\gamma) u_3(c_3) \\ U_2(\boldsymbol{c}) &= u_2(c_2) - \gamma u_3(c_3) \\ U_3(\boldsymbol{c}) &= u_3(c_3) \end{aligned} \tag{11}$$

Observe in (11) that the final coefficient related to the utility of agent 3, $m_{13} = \beta - \alpha\gamma$, depends on the intensities of the interdependence system. If $\alpha\gamma > \beta$ the coefficient is negative, even if in the interdependence system $a_{13} = \beta$ was positive. A tension arise between the affection agent 1 has for the two other agents and the hate that agent 2 shows for agent 3: the hate of agent 2 is internalized in the interdependence system and, if agent 1 cares much more for agent 2 than for agent 3, it makes it possible that finally m_{13} be negative.

A network is called balanced if all the indirect effects are positive, that is, if the weight of any path is positive. The kind of tension that arises in this example can not appear in balanced networks. Sociologists have widely studied balancedness. In fact, there is a huge area of study in social network analysis called structural balance[7] that studied both theoretically and empirically balanced networks. For an early sociological approximation to balanced networks and interpersonal relations see Davis(1963). Harary et al.(1965) provides a good technical introduction to directed graphs, and properties about when a network is balanced. Bramoullé(2001) obtains some particular results about balancedness in linear interdependent utility systems.

Example 2. Cycles: reinforcement and weakening effects.

CASE 1: There are three agents $N = \{1, 2, 3\}$, and the initial situation is the following one

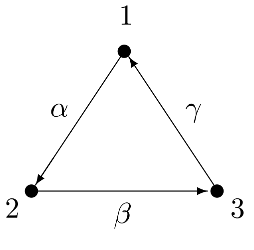

Figure 4.

where $\alpha, \beta, \gamma \in (0, 1)$. This means that agent 1 loves agent 2, agent 2 loves agent 3, and agent 3 loves agent 1. How does this virtuous cycle affects the final interdependence coefficients? Take for example agent 1. m_{11} expresses all the direct and indirect effects that start and finish in agent 1.

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[7] See chapter 6 in Wasserman and Faust(1993).

The unique possibility of indirectly arriving to agent 1 if we start in agent 1 is through the cycle $(1,2,3,1)$, and the weight of this cycle is $g_{12}g_{23}g_{31} = \alpha\beta\gamma$. Hence the sum of indirect effects from agent 1 to himself is

$$\alpha\beta\gamma + (\alpha\beta\gamma)^2 + (\alpha\beta\gamma)^3 + \cdots \tag{12}$$

that is, the sum of the indirect effects of passing one time through the cycle, two times through the cycle, three times, etc. Therefore, m_{11}, which is equal to the sum of the direct effect[8] equal to 1 plus the indirect effects, is

$$m_{11} = 1 + \sum_{k=1}^{\infty} (\alpha\beta\gamma)^k = \frac{1}{1-\alpha\beta\gamma} > 1 \tag{13}$$

The cycle provides a reinforcement of the concern agent 1 has for himself, since, initially loved himself with coefficent 1, but, finally, his private utility enters into his final utility with coefficient $m_{11} > 1$. Obviously, given the symmetry of the network, the same happens with the rest of agents.

But this cycle not only reinforces the concern an agent has for himself. Initially, agent 1 loves agent 2 with coefficient equal to $g_{12} = \alpha$, but the same cycle $c = (1,2,3,1)$ generates indirect effects from agent 1 to agent 2: agent 1 can arrive directly to agent 2 through the link $1 \to 2$, but also indirectly through the paths (1,2,3,1,2), (1,2,3,1,2,3,1,2), etc. The weight of these paths is $\alpha\beta\gamma\alpha = \alpha\,(\alpha\beta\gamma)$, $\alpha\beta\gamma\alpha\beta\gamma\alpha = \alpha\,(\alpha\beta\gamma)^2$, etc. Hence the sum of direct and indirect effects from agent 1 to agent 2, m_{12}, is equal to

$$m_{12} = \alpha \sum_{k=0}^{\infty} (\alpha\beta\gamma)^k = \frac{\alpha}{1-\alpha\beta\gamma} > \alpha \tag{14}$$

The cycle not only provides self-reinforcement, it also reinforces the links between agents in the network. Later we will see that this is a general fact: part of the effect cycles generate is global and homogeneous, it affects in the same way all the relations in the network, reinforcing or weakening all of them in the same way.

CASE 2: The situation is the same than in the previous case but now there is a new link from agent 3 to agent 2, with positive weight equal to δ. Graphically,

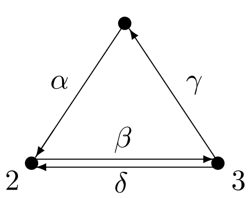

Figure 5.

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[8] In the system of interdependent utilities everybody loves himself with coefficient 1.

To the previous existing relations, we add a new one: now the love agent 2 showed for agent 3 is reciprocated, and agent 3 loves agent 2 with an intensity equal to δ. Does this friendship relation between agent 2 and agent 3 can benefit in any way agent 1? As we will see in just a moment, the answer is yes. This proves that even cycles where an agent is not involved can be beneficial for him. The network structure generates externalities effects: how the rest of people is connected affects the final concerns of an agent. This effects are not always positive. For example, if δ had been negative, the hate agent 3 would profess for agent 2 would indirectly hurt agent 1.

Let's see how this new friendship can benefit agent 1's concern for himself. The reason is clear: this new frindship generates new indirect effects from agent 1 to himself. To the indirect effects derivated from the cycle $(1,2,3,1)$ we have to add the ones that involve the new cycle $(2,3,2)$. For example there is a new indirect effect generated by $(1,2,3,2,3,1)$, that has weight equal $\alpha\beta\delta\beta\gamma > 0$. As a matter of fact, it is obvious that all the new indirect effects have positive weight, since all the initial coefficients, α, β, γ and δ, are positive.

In the following section we obtain expressions for the final interdependence coefficients for any generic network. The expression we obtain for m_{11} is in this case equal to

$$m_{11} = \frac{1-\beta\delta}{1-\alpha\beta\gamma-\beta\delta} \tag{15}$$

Observe that $m_{11} = \frac{1-\beta\delta}{1-\alpha\beta\gamma-\beta\delta} > \frac{1}{1-\alpha\beta\gamma}$ whenever $1-\alpha\beta\gamma-\beta\delta > 0$, but that $m_{11} = \frac{1-\beta\delta}{1-\alpha\beta\gamma-\beta\delta} < 0$ if $1-\alpha\beta\gamma-\beta\delta < 0$. Hence, we are confronted with two opposite situations: if the sum of weights of the cycles $(1,2,3,1)$ and $(2,3,2)$ is smaller than one the situation decribed in this case is still better than the one described in case 1, and hence the new friendship formed is beneficial for agent 1; however, if the sum of weights is larger than one, the situation becomes dramatic. The excessive love intensities generate pernicious effects: agent 1's concern for himself becomes negative! We will later treat some other examples that present similar features and some of its economic implications.

4 Peer Effects. A General Characterization.

We have seen in the examples of the previous section that the indirect effects generated by the network of social relations can have several implications on the magnitude and sign of the final interdependence coefficients. In fact, we have seen that each of this coefficients $m_{ij}(g)$ can, in some cases, be expressed as the sum of all indirect effects from i to j. This interpretation as an infinite sum of indirect effects provides a clear picture of what is going on, and thus is good to have it in mind. However, this sum may be not defined. In this case, we need to provide an alternative formulation to obtain $m_{ij}(g)$, and this is what we do in this section. Using basic tools from combinatorics, principally permutations, we are able to obtain closed form expressions for the entries of the matrix $\boldsymbol{M}(g)$ in any generic situation.

The first part of this section is devoted to introduce the necessary mathematical machinery.

4.1 Grouping people into cycles: permutations

The principal mathematical tool we use in this section are permutations. This subsection presents the basic results on permutations and provides a graph-teoretical representation via subgraphs of this combinatorial tool.

A permutation σ over the set $N = \{1, \ldots, n\}$ is simply a one-to-one function from N to N. We denote the set of permutations over N by S_n. To give a concrete permutation σ, we have to say which are the images of each element in N. An easy way of doing this is writing each element of N in a row and then write below the respective images

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \tag{16}$$

A *cyclic permutation of order* r is a permutation of the form

$$\begin{pmatrix} n_1 & n_2 & \cdots & n_r \\ n_2 & n_3 & \cdots & n_1 \end{pmatrix} \tag{17}$$

where $\{n_1, \ldots, n_r\}$ is a subset of different elements of N.[9] For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tag{18}$$

represents the cyclic permutation where 3 is the image of 1, 2 is the image of 3, and 1 is the image of 2.

It follows from the definition of permutation that the composition of two different permutations σ and τ of S_n is also a permutation of S_n. Generally, the composition is called *product* of permutations, and we denote it by $\sigma\tau$. We have the following result.

Result. Each permutation σ can be decomposed as a product of disjoint cyclic permutations.[10] Furthermore, this decomposition is unique, except for the order in the product.

Given a permutation σ, we denote by $c(\sigma)$ the set of cyclic permutations (of order larger than 1)[11] that belong to the decomposition into cyclic permutations of σ, and by $\sharp\sigma$ the cardinality of this set, that is the number of cyclic permutations in the decomposition of σ. For example, the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 5 & 1 & 6 \end{pmatrix} \tag{19}$$

---

[9] To simplify notation, when defining a permutation σ, it is usual to not express the image of the elements $x \in N$ such that $\sigma(x) = x$. Hence, for example, when defining a cyclic permutation, we suppose that $\sigma(x) = x$ for every $x \in N \backslash \{n_1, \ldots, n_r\}$.

[10] By disjoint we mean that there are no common elements in two different cyclic permutations of the decomposition.

[11] A cyclic permutation of order one lets unaltered the element that belongs to it. Therefore, from now on, when we refere to a cyclic permutation we mean a cyclic permutation of order at least equal to two.

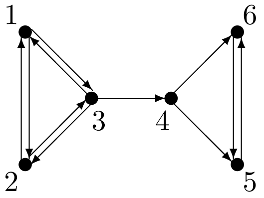
is decomposed into cyclic permutations as follows

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 6 & 5 \end{pmatrix} \tag{20}$$

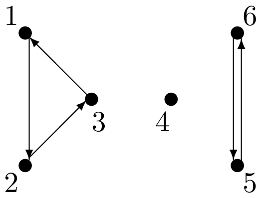
and hence $c(\sigma) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 6 & 5 \end{pmatrix} \right\}$ and $\sharp\sigma = 2$.

Coming back again to networks, there is an obvious one-to-one correspondence between cyclic permutations and network cycles. Hence, each permutation is, in fact, dividing the set N of agents into disjoint cycles.

For example, let's revisit the network in figure 1



The permutation σ in S_6 defined in (19) divides this network into cycles as follows



Consequently, we can define in a natural way the weight of a permutation σ, $W(\sigma)$, as the product of weights of the simple cycles in $c(\sigma)$, i.e.

$$W(\sigma) = \prod_{c \in c(\sigma)} W(c) \tag{21}$$

4.2 Characterization of peer effects

The principal result of this section provides the exact expressions of the final interdependence coefficients. As we will see this expressions depend strongly on the network structure since they depend on the weights of the simple cycles and the direct paths in the network g.

Proposition 2 *Let* $\boldsymbol{A}(g)$ *be a matrix that defines a generic interdependence system, and let* $\boldsymbol{M}(g) = (\boldsymbol{I} - \boldsymbol{A}(g))^{-1}$ *be its associated final interdependence matrix. Then,* $m_{ij}(g) = \frac{1}{m(g)} \bar{m}_{ij}(g)$*, where*

$$m(g) = \sum_{\sigma \in S_n} (-1)^{\sharp\sigma} W(\sigma) \tag{22}$$

and

$$\bar{m}_{ij}(g) = \begin{cases} m(g \backslash \{i\}) & \textit{if } i = j \\ \\ \sum_{p \in P_{ij}} W(p)\, m(g \backslash \{p\}) & \textit{if } i \neq j \end{cases} \tag{23}$$

While this result can be interesting for computational purposes, we believe it is more important because we can *see*, in the expressions we provide, the effects generated by the network structure.[12] These effects can be disentangled into two different ones. As we can see, the coefficients $m_{ij}(g)$ can be decomposed into two terms: $m(g)$ and $\bar{m}_{ij}(g)$. These two terms express two different kinds of effects. The $m_{ij}(g)$'s are specially related to the effects paths have in the final concern agent i has for agent j (the effects we found in example 1 in the previous section) and are local effects, while $m(g)$ is much more related to the reinforcement effects generated by the cycles (the effects in example 2 in the previous section) and it is a global effect. In what follows we extend our explanations about these two different effects.

We have provided in proposition 2 a very compact formulation of $m(g)$ and $\bar{m}_{ij}(g)$. We can provide another formulation as follows: consider the set of simple cycles in g, $C(g)$; then, from the explanation in the previous subsection of permutations and divisions into cycles, we can rewrite $m(g)$ as follows

$$1 - \sum_{c_1 \in C(g)} W(c_1) + \sum_{\substack{c_1, c_2 \in C(g) \\ c_1, c_2 \ disj.}} W(c_1) W(c_2) - \sum_{\substack{c_1, c_2, c_3 \in C(g) \\ c_1, c_2, c_3 \ disj.}} W(c_1) W(c_2) W(c_3) + \cdots \tag{24}$$

The social multiplier effect.

We call $m(g)$ the *social multiplier effect.* It is a *global homogeneous effect* generated by the whole network structure, because it depends on all the simple cycles that the network has. It is a pure reinforcement/weakening effect (that's why it only depends on cycles[13]) that affects in the same way all the agents in the network. The social multiplier effect can be either positive or negative.

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[12] In fact, given any generical interdependence system, defined by $\boldsymbol{A}(g)$, to obtain the matrix $\boldsymbol{M}(g)$ we simply have to find the inverse of the matrix $\boldsymbol{I} - \boldsymbol{A}(g)$, and nowadays any computer can do this almost exactly. The virtue of our result is that we obtain expressions that depend on the network structure that we can interpret, and that we could do comparative statics if we wanted.

[13] Remember from example 2 in the previous section that cycles are the vehicle for reinforcement.

If $m(g) \in (0,1)$, the social multiplier effect will reinforce (increase the weight) the feelings of all agents maintaining the sign, while if $m(g) > 1$ it will weak them. If the social multiplier is negative the reinforcement/weakening effect is the same but it has an important additional effect: it changes the sign of all feelings.

The explanation of why in the expression (24) of $m(g)$ appear not only the weights of the cycles but also the product of combinations of cycles with different signs is that cycles have not a pure effect. The effects of different cycles are entangled and thus a correction term has to be introduced to the pure effects expression $1 - \sum_{c1 \in C(g)} W(c_1)$.[14]

The term social multiplier is specially suggestive in our setting. The social multiplier is generated by the links of the social network in which the agents are embedded and it exerts a global homogeneous effect in all the agents. The term *social multiplier* has arised in some recent works (see Becker and Murphy, 2000, and specially Glaeser et al., 2003) and, even if the definition of the social multiplier each model uses is different, in all cases it is close in spirit to our one.

The externalities effects.
We call the coefficients $\bar{m}_{ij}(g)$ the *externalities effects* because they express how the final coefficients of interdependency of a certain agent depend on how the rest of the group is structured. In this sense, the externalities effects are *local effects* of the network.

Observe that $\bar{m}_{ii}(g)$ is equal to the social multiplier effect of the subnetwork obtained by eliminating agent i, $m(g\backslash\{i\})$. Therefore the final interdependence coefficient of agent i with respect to himself, m_{ii}, is equal to the ratio $\frac{m(g\backslash\{i\})}{m(g)}$. There are several possibilities. For example, if this ratio is positive and the social multiplier effect ot the network obtained removing agent i is larger than the social multiplier effect of the network, then the social structure reinforces the concern agent i has for himself, $m_{ii} > 1$. This comparison between social multipliers can be understood as a cheking how valuable is the rest of the group for agent i. A large social multiplier for the network $g\backslash\{i\}$ compared to the social multiplier of the network g means that the rest of the group exherts a positive externality on agent i.

Similarly, the final coefficient of interdependence of agent i with respect to agent j if $i \neq j$ is

$$m_{ij}(g) = \frac{\bar{m}_{ij}(g)}{m(g)} = \sum_{p \in P_{ij}(g)} W(p) \frac{m(g\backslash\{p\})}{m(g)} \tag{25}$$

It depends on the direct paths from i to j, but the effect of each these paths is not only its weight. The weight of a path p can be reinforced if the agents that not belong to it are valuable, where valuable has a similar meaning than in previous discussion: $m(g\backslash\{p\}$, the social multiplier of the network obtained removing the agents involved in the path, has to be larger than $m(g)$, the social multiplier of the network (and the ratio between both has to be positive).

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[14] To have an analogy, it is much like what happens with the probability of intersection of events in probability theory.

5 Two Examples: Circles and Stars

In this section we study some particular networks that have had a prominent role in the literature: the (directed) circle and the star. The circle represents a very symmetric social structure where each agent plays a similar role in the network. The star representents an asymmetric situation where a central agent is connected to everybody else while the rest of agents are only connected to the central agent.

These two kind of networks, even if they are somewhat unrealistic, represent theoretical representations of two quite different social situations. In fact, they have been succesfully used in some remarkable applied work in economics. The directed circle is the network Glaeser et al.(1996) use to model how criminals relate with each other to explain the high variance of crime rates, while the star is the network Becker(1974) uses to model social interactions within the family in his now classic work, and allows him to state the famous Rotten Kid theorem.

5.1 The directed circle

Each agent, $i \in N = \{1, \ldots, n\}$, cares for the one at his left with value α_i.

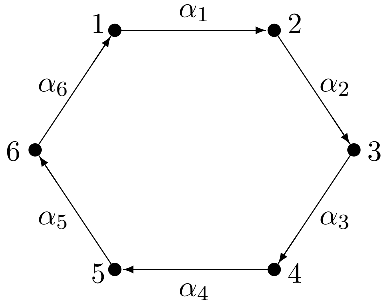

Figure 6.*Directed Circle* $(n = 6)$

The interdependence system is defined by the matrix

$$\boldsymbol{A}(g) = \begin{pmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & 0 & \alpha_2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & \alpha_{n-1} \\ \alpha_n & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{26}$$

The unique cycle in this network is the one that involves all the agents clockwise, and it has weight equal to $\prod_{i \in N} \alpha_i$. Therefore

$$m(g) = 1 - \prod_{i \in N} \alpha_i \tag{27}$$

If we remove agent i from the circle the remaining network is a line that contains no cycles. Since $\bar{m}_{ii}(g) = m(g \backslash \{i\})$, we obtain that

$$\bar{m}_{ii}(g) = 1 \tag{28}$$

The unique direct path from an agent i to another agent $j \neq i$ is the one that goes clockwise from i to j, $i \to (i+1) \to \cdots \to (j-1) \to j$, and if we remove the agents involved in this path the remaining network is a line that contains no cycles. Hence,

$$\bar{m}_{ij}(g) = \prod_{k=0}^{d(i,j)-1} \alpha_{[i+k]} \quad if \ i \neq j \tag{29}$$

where $d(i,j)$ is the distance counting clockwise from i to j, i.e. the number of agents involved in the path from i to j, and $[x]$ is equal to x modulus n (and $\alpha_0 = \alpha_n$).

In particular, if we consider the case of a homogeneous directed circle, with $\alpha_i = \alpha$ for every $i \in N$, the matrix of the interdependence system is

$$\boldsymbol{A}(g) = \begin{pmatrix} 0 & \alpha & 0 & \cdots & 0 \\ \vdots & 0 & \alpha & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & \alpha \\ \alpha & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{30}$$

and we obtain that the final interdependency matrix is equal to

$$\boldsymbol{M}(g) = \frac{1}{1-\alpha^n} \begin{pmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ \alpha^{n-1} & 1 & \alpha & \cdots & \alpha^{n-2} \\ \alpha^{n-2} & \alpha^{n-1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \alpha \\ \alpha & \ldots & \alpha^{n-2} & \alpha^{n-1} & 1 \end{pmatrix} \tag{31}$$

We can observe several features. First, since $m_{ij}(g) = \bar{m}_{ij}(g)/m(g) = a_{ij}/m(g)$, the concern agent i has for agent $i+1$ increases or diminishes directly depending on the social multiplier effect. Second, even if initially each agent is only concerned for the neighbour at his left, given the geometry of the network indirect effects spread among all the individuals in the network, and his total utility finally depends on the private utility of all the agents. Finally, observe that the social multiplier generates a reinforcement effect, that is the social multiplier effect is larger than 1, whenever $\alpha \in (-1,1)$. Hence, the reinforcement effect is not only restricted to the case when agents are altruistic.

5.2 The star

In this example we suppose there are $n+1$ agents, labeled from 0 to n, $N = \{0, 1, \ldots, n\}$. Agent 0 is in the center and cares with certain value α_i for the rest of the n agents that are in the periphery, $i \in N \backslash \{0\}$. Each element in the periphery only cares for the central agent, with value β_i.

The following picture is the graphical representation of an star with $n = 4$

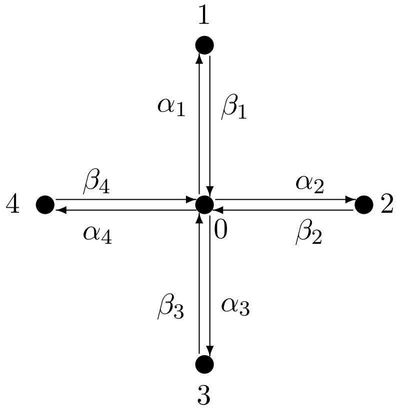

Figure 7.*Star* $(n = 4)$

The matrix that defines the interdependence system is

$$\boldsymbol{A}(g) = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & 0 & \cdots & \cdots & 0 \\ \beta_2 & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \beta_n & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{32}$$

There are no possible two disjoint simple cycles in this network because the central agent is involved in each one. Therefore,

$$m(g) = 1 - \sum_{i \in N} \alpha_i \beta_i \tag{33}$$

Since $\bar{m}_{00} = m(g \backslash \{0\})$ and the network obtained by removing the central agent is a network that involves all the agents in N but in which there is no link between two different agents, and if there is no link it can't be no cycle, we have that

$$\bar{m}_{00}(g) = 1 \tag{34}$$

Similarly, since the unique direct path from the central agent to an agent $j \neq 0$ in the periphery is the direct link from 0 to j , and, again, the network obtained removing agents 0 and j has no links, and therefore no cycles, we obtain that

$$\bar{m}_{0j}(g) = \alpha_j \quad if\ j \neq 0 \tag{35}$$

If $i \neq 0$, since $\bar{m}_{ii}(g) = m(g \backslash \{i\})$, and the network obtained removing agent i from the periphery is still a star, we have that

$$\bar{m}_{ii}(g) = 1 - \sum_{j \in N, j \neq i} \alpha_j \beta_j \tag{36}$$

The unique direct path from i to the central agent is the direct link from i to 0, so

$$\bar{m}_{i0}(g) = \beta_i \tag{37}$$

And, finally, the unique direct path from i to another agent $j \neq i$ in the periphery is the path $i \to 0 \to j$. Thus

$$\bar{m}_{ij}(g) = \alpha_i \beta_j \quad if\ j \neq 0\ and\ j \neq i \tag{38}$$

The final interdependency matrix, $\boldsymbol{M}(g)$, associated to the homogeneous star, the star with $\alpha_i = \alpha$ and $\beta_i = \beta$ for every $i \in N$, is equal to

$$\frac{1}{1 - n\alpha\beta} \begin{pmatrix} 1 & \alpha & \alpha & \cdots & \alpha \\ \beta & 1 - (n-1)\alpha\beta & \alpha\beta & \cdots & \alpha\beta \\ \beta & \alpha\beta & 1 - (n-1)\alpha\beta & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha\beta \\ \beta & \alpha\beta & \ldots & \alpha\beta & 1 - (n-1)\alpha\beta \end{pmatrix} \tag{39}$$

A similar feature to the case of the homogeneous directed circle arise. Again, even if initially each agent, except the central one, cares only for the central agent, finally, each agent's total utility depends on all the agents of the network. The central agent spreads the indirect effects across all agents. Observe that if $\alpha\beta < 0$ the social multiplier effect is always positive while if $\alpha\beta > 0$ the social multiplier becomes negative if n is sufficiently large.

6 Pareto Efficiency: A Question of Prestige

A natural question that arises at this point is which are the Pareto efficient allocations under interdependent utilities. In this section we address this question in one-shot situations. Our work is not the first one exploring the consequences of interdependent utilities on Pareto efficiency. Bernheim and Stark(1988) study the shape of the Pareto frontier in an interdependent utilities situation where two agents live for two periods. They conclude, as we do, that altruism is not necessarily socially beneficial and that, sometimes, agents would like to interact with less altruistic individuals. Another, and, probably, the most famous result on efficiency and altruism, is Becker's Rotten Kid theorem that states that altruism by a member of a group will lead other selfish members to act efficiently from the group viewpoint. The framework in which it was stated for the first time, in Becker(1974), it was a model of interdependent utilities on a star. Bergstrom(1989) is not only a good introduction to the Rotten Kid theorem, but also a clever study of the limited applicability of this result.

Our setting is the following. There is a quantity π of a certain good that has to be distributed within a group of n agents. We suppose that the private utility over consumption of each agent is $u_i(c_i) = c_i$, and therefore the utility possibility set under private utilities is the standard n-dimensional simplex of size π, denoted by $\Delta_n(\pi)$,

$$\Delta_n(\pi) = \{(c_1, \ldots, c_n) \in \mathbf{R}^n; c_i \geq 0 \quad \forall i \quad and \quad c_1 + \cdots + c_n \leq \pi\} \tag{40}$$

We assume that the agents are risk neutral for simplicity and because we don't want that risk-aversion effects interfere with the effects that are generated by the network structure, that are the effects in which we really want to concentrate.
First of all, let's remember some definitions related to efficiency.

Definition Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. We say that $\boldsymbol{x}$ *Pareto dominates* $\boldsymbol{y}$ if $x_i \geq y_i$ for every $i \in N$ and the inequality is strict for at least one i. Given a set $X \subseteq \mathbf{R}^n$, we say that $\boldsymbol{x} \in X$ is *Pareto undominated* if there does not exist any $\boldsymbol{y} \in X$ that Pareto dominates $\boldsymbol{x}$. We call the set of Pareto undominated elements of X the *Pareto frontier* of X.

Denote by $\boldsymbol{m}^{(1)}, \ldots, \boldsymbol{m}^{(n)}$ the columns of the matrix $\boldsymbol{M}(g) = (\boldsymbol{I} - \boldsymbol{A}(g))^{-1}$, which are n-dimensional vectors, and let $\mathbf{0}$ denote the zero vector of $\mathbf{R}^n$. Let $I(g) = \{\boldsymbol{m}^{(1)}, \ldots, \boldsymbol{m}^{(n)}, \mathbf{0}\}$. $UPS(g;\pi)$ denotes the utility possibility set under interdependent utilities, and $PF(g;\pi)$ denotes its Pareto frontier. Given a finite set $X \subset \mathbf{R}^n$, $conv(X)$ denotes the convex hull of X, i.e. the set of convex combinations of elements of X. The following result characterizes $UPS(g;\pi)$.

Proposition 3 $UPS(g;\pi) = \pi \cdot conv(I(g))$.

Thus, we obtain that the utility possibility set under interdependent utilities is also a n-dimensional simplex.[15]. It is simply a deformation of the standard n-dimensional simplex by the linear application given by the matrix $\boldsymbol{M}(g)$. What is not true is that the Pareto frontier of $UPS(g;\pi)$ is equal to the image by $\boldsymbol{M}(g)$ of the Pareto frontier of $\Delta_n(\pi)$. The shape of the Pareto frontier of $UPS(g;\pi)$ relies heavily on the structure of the interdependence system, as we now explore.

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[15] This is so because the n columns of $\boldsymbol{M}(g)$ are linearly independent in a generic interdependence system: $\boldsymbol{M}(g) = (\boldsymbol{I} - \boldsymbol{A}(g))^{-1}$ is a non-singular matrix if and only if $\boldsymbol{I} - \boldsymbol{A}(g)$ is non-singular.

6.1 Prestige in social networks

Not all the agents in a social network are necessarily equally important, whatever important may mean. There are several variables that can determine the importance or prominence of an actor in the network. Furthermore, the definition of prominence may depend on the setting we are studying. It is not the same if we deal with directed or undirected networks, or with weighted or unweighted networks. Hence, there is not in the social networks analysis literature a unique standard definition of prominence.

Sociologists have defined importance or prominence in networks mainly through two different concepts: *centrality* and *prestige*. Both concepts are related to *connectivity*: centrality of an agent is related to the paths that start in this agent, while prestige is related to the paths that finish in this agent. It depends on the particular setting, but, roughly speaking, centrality is related to what you give and prestige is related to what you receive. For example, an agent can be important in the structure of a firm because he can transmit easily information to many other workers; in this case the relevant notion to study would be centrality. However, in our case an agent is important depending on how much he is loved, that is, on how much love or affection he receives. Hence prestige is the relevant concept we have to deal with. There is a huge literature in sociology about prominence in networks and several centrality and prestige measures have been defined.[16] Our work is not the first economic model relying in concepts related to prominence in social networks. For an example of an application of network theory to economics in which centrality plays a key role see Ballester et al.(2004).

As we have previously stated, in our model an agent is more important than another one if he is more loved than the other: the private utilities of agents j_1 and j_2 enter into the total utility of agent i with coefficients m_{ij_1} and m_{ij_2}, respectively; given that agents are risk-neutral, agent j_1 is more important than agent j_2 for agent i if and only if $m_{ij_1} > m_{ij_2}$. Each column of $\boldsymbol{M}(g)$ contains in its entries the basic information about prestige of each agent. To obtain an aggregate value of prestige we do it in the simplest possible way. First, we define what is a system of weights.

Definition A *system of weights* is a n-dimensional vector $\boldsymbol{\mu}$ such that $\mu_i \geq 0$ for every $i \in N$ and $\sum_{i=1}^{n} \mu_i = 1$. We say that a system of weights $\boldsymbol{\mu}$ is *strict* if $\mu_i > 0$ for every $i \in N$.

We define the prestige of an agent as follows.

Definition Given a network $g \in G_n$ and a system of weights $\boldsymbol{\mu}$ we define the *prestige measure* or, simply, the prestige of agent i as $\boldsymbol{\mu} \cdot m^{(i)}(g)$.

Hence, the prestige of agent i is simply a weighted sum of the entries of the i-th column of

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[16] See chapter 5 in Wasserman and Faust(1994) for a survey of this literature and references.

$\boldsymbol{M}(g)$. In the case of an strict system of weights, the concern of each agent versus agent i is taken into account.

6.2 The Pareto frontier. Degenerate case.

In this subsection we address with situations in which the Pareto frontier is a unique allocation. The following result provides the characterization of these cases in terms of prestige.

Proposition 4 *The unique Pareto efficient allocation under interdependent utilities is that agent i consumes everything if and only if for every strict system of weights $\boldsymbol{\mu}$ we have that $\boldsymbol{\mu}\cdot\boldsymbol{m}^{(i)} > \boldsymbol{\mu}\cdot\boldsymbol{m}^{(j)}$ for every $j \neq i$, and $\boldsymbol{\mu}\cdot\boldsymbol{m}^{(i)} > 0$. The unique Pareto efficient allocation is that nobody consumes if and only if for every strict system of weights $\boldsymbol{\mu}$ we have that $\boldsymbol{\mu}\cdot\boldsymbol{m}^{(i)} < 0$ for every $i \in N$.*

This result has a clear and suggestive interpretation. It states that the Pareto frontier is completely degenerated when either there is a privileged agent in terms of prestige that, given any system of weights, is always more prestigious than the rest and his prestige is positive, or when every agent, no matter which system of weights we consider, has negative prestige. If there is a unique important agent, he has to be the one that consumes everything. If all agents are humbleness, the best they can do is to not consume at all.

The intuition behind the result is the following. First of all, observe that given the structure of the utility possibility set under interdependent utilities, the associated vector of (total) utilities of a Pareto efficient allocation can only be expressed as a a convex combination of Pareto undominated elements of $I(g)$. Hence, $PF(g)$ is formed by a unique point if and only if the set of undominated elements of $I(g)$ has a unique element. There are two options: this element can either be a column $\boldsymbol{m}^{(i)}$ or the zero vector, $\mathbf{0}$. Proposition 5 is this last statement written in terms of prestige measures.[17]

Suppose column i Pareto dominates the rest of elements in $I(g)$. This means that finally everybody loves (the feeling has to be positive because column i Pareto dominates the zero vector) more agent i than any other agent, included himself. If it is the zero vector that Pareto dominates each column of $\boldsymbol{M}(g)$, this means that, finally, everybody hates everybody, included himself. In this case, the fact that somebody consumes a portion of π produces a negative effect in everybody, included the consumer. That's why each agent best option in this situation is to consume nothing. This situation may arise even if the initial interdependence coefficients are all positive, as we can see in the following example.

Example 3. True lovers hate spaghetties.[Bergstrom (1999)]

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[17] See lemma 1 in the appendix.

There are two agents, $N = \{1, 2\}$. We call agent 1 "Romeo" and agent 2 "Juliet". Romeo loves Juliet so much ($a_{12} = \alpha > 1$) and Juliet loves Romeo so much ($a_{21} = \beta > 1$) that when they have to share the spaghetties they have prepared for dinner, they prefer to eat nothing, because they both feel better if the other eats all the spaghetties. A coordination problem arise because the coefficients of interdependency are excessively large.
The initial interdependence matrix is

$$\boldsymbol{A} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \tag{41}$$

and therefore the final interdependence matrix is

$$\boldsymbol{M} = (\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{1}{1 - \alpha\beta} \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \tag{42}$$

Since both α and β are larger than 1, the social multiplier effect $m(g) = 1 - \alpha\beta$ is negative, meaning that the reinforcement effect generated by the love each one feels for the other is in this case negative. Hence, each entry in M is negative. This implies that each column of M is dominated by the zero vector. The following picture provide a graphical representation of the situation.

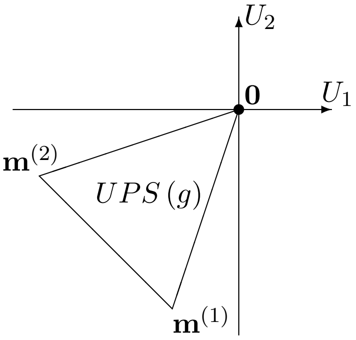

Figure. $\mathbf{UPS}\,(\mathbf{g})$

The utility possibility set is the triangle with vertices the columns of $\boldsymbol{M}(g)$ and the zero vector. Observe that the network in this case is a directed circle with two agents. As a corollary of proposition 4, we can generalize the situation just described to any other purely altruistic directed circle, meaning that all agents are altruist. (The proof is analogous to the case $n = 2$ in the example).

Corollary *If a directed circle with n agents is such that $\alpha_i > 0$ for every $i \in N$ and $\prod_{i=1}^{n} \alpha_i > 1$, then the unique Pareto efficient allocation is that nobody consumes.*

This proves that pure altruism can have harmful effects. Being alone, any agent would obtain positive utility from his private consumption, but the high level of altruism of the network in which he is embedded generates undesirable effects. This indicates that it may not be always a good policy to incentivate the altruism of agents of the network: this policy can generate positive spillovers sometimes, but in some other cases it may be extremely pernicious.

Example 4. The homogeneous star.

We can expect that in some cases the privileged situation of the central agent in the star in terms of connectivity traduces also in a privileged situation in terms of efficiency. Consider for example the case in which $\alpha \in (0,1)$ and $\beta \in \left(\frac{1}{n\alpha+1-\alpha}, \frac{1}{n\alpha}\right)$. Then, the social multiplier effect $m(g) = 1 - n\alpha\beta$ is positive.[18] Furthermore, we have that $\beta > 1 - (n-1)\alpha\beta$.[19] From these two facts we obtain that the first column of the matrix $\boldsymbol{M}(g)$ in (39) Pareto dominates the rest of columns of the matrix and also the zero vector, since all the components of the column are strictly positive. We can apply propostion 4 to obtain that in this case the unique Pareto efficent allocation is that the central agent consumes everything.

However, like in the "true lovers hate spaghetties" example, there are situations in the star in which even if the initial coefficients are all positive, all the final coefficients are negative. If $\alpha > 0$ and $\beta > 0$ are such that $\frac{1}{n} < \alpha\beta < \frac{1}{n-1}$, the social multiplier effect $m(g) = 1 - n\alpha\beta$ is negative, while all the externalities effects are positive, since $\alpha > 0$, $\beta > 0$, $\alpha\beta > 0$ and $1 - (n-1)\alpha\beta > 0$. Therefore, all the elements of the matrix $\boldsymbol{M}(g)$ are negative. Hence, each column of $\boldsymbol{M}(g)$ is Pareto dominated by the zero vector and by proposition 4 we can conclude that the unique Pareto efficient allocation is that nobody consumes.

6.3 The Pareto frontier. Regular case.

In this subsection we address a completely opposed situation compared to the degenerate case. Now we want to characterize the situations where all the allocations that exhaust the available resources are efficient. We refer to this situations as the regular ones, and we say in this cases that the Pareto frontier is regular. Hence, the efficient allocations in a regular case are the same that in a situation where agents are isolated and selfish (only care for their private utility). The difference between both situations relies on the utility each agent obtains. It may be that the social structure enhances the total utility an agent can obtain in an interdependent setting with respect to his private utility, as well as it may be that it finally decreases this total utility, obtaining a lower level of total utility than his private utility of consumption.

The following result characterizes all the possible situations in which the Pareto frontier is regular, again in terms of prestige.

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[18] Observe that, if α and β are positive, $1 - n\alpha\beta > 0$ if and only if $\frac{1}{n\alpha} > \beta$

[19] Since, if α and β are positive, $\beta > 1 - (n-1)\alpha\beta$ if and only if $\beta > \frac{1}{n\alpha+1-\alpha}$

Proposition 5 *The Pareto frontier is* $PF(g) = \pi \cdot conv\left(\boldsymbol{m}^{(1)}, \ldots, \boldsymbol{m}^{(n)}\right)$ *if and only if there exist a strict system of weights* $\boldsymbol{\mu}$ *and a positive constant* $r > 0$ *such that* $\boldsymbol{\mu} \cdot \boldsymbol{m}^{(i)} = r$ *for every* $i \in N$.

In words, if all the columns of $\boldsymbol{M}(g)$ are Pareto undominated, the necessary and sufficient condition to obtain that the Pareto frontier is the convex hull of these columns is that there is a system of weights such that the prestige of all agents is equal and positive. This fact expresses that to obtain a regular Pareto frontier we need homogeneity in prestige among all the agents in the network.

We proceed now to explain how to obtain the system of weights $\boldsymbol{\mu}$ stated in proposition 5. Observe that the condition that $\boldsymbol{\mu} \cdot \boldsymbol{m}^{(i)} = r$ for every $i \in N$ is equivalent to the following linear system

$$\boldsymbol{M}(g)^T \cdot \boldsymbol{\mu} = r \cdot \mathbf{1}_n \tag{43}$$

Let's solve first the following system (where $\boldsymbol{\rho}$ can be any n-dimensional vector)

$$\boldsymbol{M}(g)^T \cdot \boldsymbol{\rho} = \mathbf{1}_n \tag{44}$$

The solution to this system is

$$\boldsymbol{\rho} = (\boldsymbol{I} - \boldsymbol{A}(g))^T \cdot \mathbf{1}_n \tag{45}$$

that is

$$\rho_i = 1 - \sum_{j=1, j\neq i}^{n} a_{ji} \qquad i = 1, \ldots, n \tag{46}$$

Let $\rho = \sum_{i=1}^n \rho_i = n - \sum_{i \neq j} a_{ij}$ and normalize $\boldsymbol{\rho}$ by ρ to obtain our candidate for $\boldsymbol{\mu}$

$$\boldsymbol{\mu} = \frac{1}{\rho}\boldsymbol{\rho} \tag{47}$$

Observe that for $\boldsymbol{\mu}$ to be a strict system of weights, we need that either $\rho_i > 0$ for every i or that $\rho_i < 0$ for every i. In the first case we obtain that

$$\boldsymbol{M}(g)^T \cdot \boldsymbol{\mu} = \frac{1}{\rho}\mathbf{1}_n \tag{48}$$

and since $\rho > 0$ we obtain the system of weights $\boldsymbol{\mu} = \frac{1}{\rho}\boldsymbol{\rho}$ and the constant $r = \frac{1}{\rho} > 0$ desired. In the second case we obtain the same equality but $\frac{1}{\rho} < 0$, so the conditions in proposition 5 can not be satisfied. Hence, we have not only showed how to obtain $\boldsymbol{\mu}$ and r in proposition 5 but also we have, through this process, proved the following result.

Proposition 6 A necessary condition to satify the conditions in proposition 5 is that $\rho_i =$

$1-\sum_{j=1,j\neq i}^{n} a_{ji} > 0$ for every $i \in N$. If all the columns of $M\left(g\right)$ are undominated in $I\left(g\right)$, the condition is sufficient.

This results provides us a very simple way to check if the Pareto frontier is regular or not: first, we have to check that all the columns of $\boldsymbol{M}$ are undominated in $I\left(g\right)$; then, we have to check if the sum of entries of each column of the matrix $\boldsymbol{I}-\boldsymbol{A}$ is strictly positive. If the answer in both cases is afirmative, then the Pareto frontier is regular. Otherwise it is not. The interpretation of the condition $1-\sum_{j=1,j\neq i}^{n} a_{ji} > 0$ is that the sum of affection in the initial situation of the rest of the group versus agent i can not be larger than how much agent i loves himself. If an agent is initially excessively loved, the Pareto frontier can not be regular.

We present now some examples where we can apply propositions 5 and 6.

Example 5. The homogeneous directed circle again

In this case it is easy to see that under some conditions on α the Pareto frontier is regular. First of all observe in (31) that whenever $\alpha \in (-1,1)$ all the columns of $\boldsymbol{M}\left(g\right)$ are undominated in $I\left(g\right)$ since the elements in the diagonal are strictly positive and larger than any other entry in each row. Next, the sum of elements of each column of the matrix $\boldsymbol{M}\left(g\right)$ is

$$\frac{1}{1-\alpha^n}\left(1+\alpha+\alpha^2+\cdots+\alpha^{n-1}\right)=\frac{1}{1-\alpha^n}\frac{1-\alpha^n}{1-\alpha}=\frac{1}{1-\alpha} \tag{49}$$

Now, consider the system of weights $\boldsymbol{\mu}=\left(\frac{1}{n},\ldots,\frac{1}{n}\right)$ and the constant $r=\frac{1}{n}\frac{1}{1-\alpha}$. Then, for any $\alpha<1$ we have that

$$\boldsymbol{\mu}\cdot\boldsymbol{m}^{(i)}=r>0 \quad for\ every\ i=1,\ldots,n \tag{50}$$

Hence, we can apply proposition 5 to conclude that the Pareto frontier of the homogeneous directed circle is regular for every $\alpha \in (-1,1)$.

Example 6. The homogeneous star again

Let $\alpha \in \left(0,\frac{1}{n}\right)$ and $\beta \in \left(0,\frac{1}{n\alpha+1-\alpha}\right)$. Since $\beta < \frac{1}{n\alpha+1-\alpha} \leq 1$ we have that $\alpha\beta < \frac{1}{n}$, that implies that the social multiplier effect $(g)=1-n\alpha\beta$ is positive and that $1-(n-1)\alpha\beta > \beta > \alpha\beta$. Therefore, none of the columns of $\boldsymbol{M}\left(g\right)$ in (39)is Pareto dominated in $I\left(g\right)$. Moreover,

$$1-\sum_{j=1}^{n} a_{j0}=1-n\beta>0 \tag{51}$$

since $\beta<\frac{1}{n\alpha+1-\alpha}\leq\frac{1}{n}$, and

$$1-\sum_{j=1}^{n} a_{ji}=1-\alpha>0 \qquad for\ every\ i\in\{1,\ldots,n\} \tag{52}$$

since $\alpha < 1$. Therefore, we can apply proposition 6 to conclude that the homogeneous star has a regular Pareto frontier in this case.

7 Bargaining with Interdependent Utilities

In this section we study how the introduction of interdependent utilities affects into bargaining problems. In particular we focus our attention in the Nash bargaining solution, introduced by Nash(1950) in his seminal paper. We maintain throughout this section that the resources available are limited to a quantity π of a certain good, and that agents are risk-neutral, i.e. the private utility of each agent is $u_i\left(c_i\right) = c_i$.

First, we have to justify that we can apply the Nash bargaining solution in our setting of interdependent utilities. The classical definition of a *n-person bargaining problem* is given by a pair $\langle X, \boldsymbol{d}\rangle$, where X is a convex and compact subset of $\mathbf{R}^n$, and $\boldsymbol{d} \in X$ is a n-dimensional vector such that there exists another vector $\boldsymbol{v} \in X$ such that $v_i > d_i$ for every $i = 1, \ldots, n$. This is an abstract type of object. The general interpretation of it is that X is the set of possible achievable utilities, the *utility possibility set*, and that $\boldsymbol{d}$, called the *disagreement point*, is the vector of utilities agents obtain if they are not able to reach an agreement.

Let $\boldsymbol{\gamma}$ be a system of weights. The asymmetric Nash bargaining solution with vector of bargaining powers $\boldsymbol{\gamma}$ of a bargaining problem $\langle X, \boldsymbol{d}\rangle$ is the allocation that solves the following maximization problem

$$\max_{\boldsymbol{x}\in X} \prod_{i=1}^{n} \left(x_i - d_i\right)^{\gamma_i} \tag{53}$$

We call the case when everybody has the same bargaining power, i.e. if $\gamma_i = 1/n$ for every i, the (symmetric) Nash bargaining solution. The solution to the maximization problem (53) is

$$\boldsymbol{x} = \boldsymbol{d} + (\pi - d)\boldsymbol{\gamma} \tag{54}$$

that is, everybody obtains his disagreement utility and a fraction γ_i of the remaining resources.

Even if we are dealing with private or with total utilities, we have proved in the previous section that the utility possibility set in both situation is a simplex, and therefore convex and compact. Hence, if we provide adequate disagreement points, we can define a bargaining problem in both situations. A bargaining problem with private utilities is of the form $\langle \Delta_n\left(\pi\right), \boldsymbol{d}\rangle$, while a bargaining problem with interdependent utilities, for a given network g, is given by a pair of the form $\langle UPS\left(g;\pi\right), \boldsymbol{D}\rangle$.

Since there is a one-to-one correspondence between points in $\Delta_n\left(\pi\right)$ and points in $UPS\left(g;\pi\right)$, we can interpret $\boldsymbol{D}$, the disagreement point under interdependent utilities in two different ways: directly, as the minimum total utility each agent wants to obtain, or indirectly, as the disagreement point in total utilities derived from a disagreement point for private utilities. In the second interpretation each agent wants to obtain at least a private utility equal to d_i, but this implies that the total utility he would obtain in case of disagreement would be equal to $D_i = U_i(d_1, \ldots, d_n)$. In vectorial terms this is equal to $\boldsymbol{D} = \boldsymbol{U}(\boldsymbol{d}) = \boldsymbol{M}(g) \cdot \boldsymbol{d}$, implying that the associated private disagreement point $\boldsymbol{d}$ associated to the interdependence disagreement point $\boldsymbol{D}$ is $\boldsymbol{d} = (\boldsymbol{I} - \boldsymbol{A}(g)) \cdot \boldsymbol{D}$.

Since if the Pareto frontier with interdependent utilities is degenerated the Nash bargaining problem is trivially solved, because the Nash bargaining solution is Pareto efficient and in this situation there is a unique Pareto efficent allocation, we center our results in the cases where the Pareto frontier is regular. In these cases there is competition between all the agents in the network for the available resources. The Nash bargaining solution provides the solution allocation for the distributional conflict. We want to determine if the network structure affects this solution, and, if it does, in which sense.

7.1 Bargaining and prestige

The first result we obtain states a relation between consumption in the Nash bargaining solution and prestige.

Proposition 7 *Suppose we are in a case where the Pareto frontier is regular. Then, if* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *is the allocation solution of the Nash bargaining problem under interdependent utilities, we have that there exists a strict system of weights* $\boldsymbol{\eta}$ *and a constant* $\lambda > 0$ *such that*

$$\boldsymbol{\eta} \cdot \boldsymbol{m}^{(i)} \leq \lambda \quad if\ x_i = 0 \tag{55}$$

$$\boldsymbol{\eta} \cdot \boldsymbol{m}^{(i)} = \lambda \quad if\ x_i > 0 \tag{56}$$

Furthermore, the utility each agent obtains with the Nash bargaining solution is

$$U_i^{NBS} = D_i + K \cdot \frac{1}{\eta_i} \tag{57}$$

where K *is a strictly positive constant.*

In words, this result expresses that the agents that consume a strictly positive quantity in the Nash bargaining solution have to be equally prestigious, and more prestigious than the rest, for certain system of weights. This system of weights selects the prominent actors for the Nash bargaining solution. Moreover, the inverses of the coefficients of the system of weights provide the utilities.

Let $\boldsymbol{\varphi}$ be the n-dimensional vector with components $\varphi_i = \frac{(\boldsymbol{I}-\boldsymbol{A}(g))_{(i)} \cdot \left(\frac{\mathbf{1}}{\boldsymbol{\mu}}\right)}{\sum_{j=1}^{n} (\boldsymbol{I}-\boldsymbol{A}(g))_{(j)} \cdot \left(\frac{\mathbf{1}}{\boldsymbol{\mu}}\right)}$, where $(\boldsymbol{I} - \boldsymbol{A}(g))_{(i)}$ is the i-th row of the matrix $\boldsymbol{I} - \boldsymbol{A}$, $\boldsymbol{\mu}$ is the system of weights from proposition 5, and $\left(\frac{\mathbf{1}}{\boldsymbol{\mu}}\right)$ is equal to the vector with each entry being the inverse of the respective entry of $\boldsymbol{\mu}$, i.e. $\left(\frac{\mathbf{1}}{\boldsymbol{\mu}}\right)_i = \frac{1}{\mu_i} = \frac{1}{1 - \sum_{j=1, j \neq i}^{n} a_{ji}}$. The following result provides the necessary and sufficient conditions to obtain an interior (in which everybody consumes a strictly positive quantity) Nash bargaining solution.

Proposition 8 *The solution allocation to the Nash bargaining problem under interdependent utilities is interior, meaning that* $x_i > 0$ *for every* $i \in N$*, if and only it is the solution of the linear system*

$$M\left(g\right)^T \cdot \boldsymbol{\eta} = \lambda \cdot \mathbf{1}_n \tag{58}$$

and the solution is equal to

$$\boldsymbol{x} = \boldsymbol{d} + \left(\pi - d\right) \cdot \boldsymbol{\varphi} \tag{59}$$

A carefull observation of the vector $\boldsymbol{\varphi}$ helps us to clarify the effects of the network in this solution. Observe that now the result depends not only to prestige (through the vector $\frac{\mathbf{1}}{\boldsymbol{\mu}}$) but also to the initial coefficents of the matrix $\boldsymbol{I} - \boldsymbol{A}$, and, particularly, to the sum of the entries of each row, that express the difference between how much an agent loves himself and how much he loves the rest of individuals. Hence, this final result depends both to the prestige an agent acquires (how much he is finally loved) and what we could denote as *initial centrality* (how much he initially loves).[20] What an agent obtains in the Nash bargaining solution depends both on what he gives (the love or hate he initially provides) and what he receives (the love or hate he finally receives), and the expresion of $\boldsymbol{\varphi}$ provides us the exact interlinkage, the trade-off, between both variables. Hence, prestige is not sufficient to attack totally distributional conflicts, and a more complex relation with the network structure and its geometry arises.

Observe, that in particular, proposition 8 establishes a connection between the structure of the network in bargaining problems with interdependent utilities and bargaining power in bargaining problems with private utilities if all the entries of the vector $\boldsymbol{\varphi}$ are positive. The Nash bargaining solution with interdependent utilities in these cases is equivalent to the solution of an assymetric Nash bargaining solution with private utilities, and the vector $\boldsymbol{\varphi}$ "traduces" the effects of the network structure into assymetric bargaining powers.

7.2 An application. Bargaining within marriage

The economic study of marriage and household behavior is a complex affair. Specially since the seminal works by Becker(1974,1981), several lines of research have arised trying to explain how resources are distributed within the family (or any other linked group). We can separate the existing works mainly into four different strands of research: altruistic models la Becker that study similar situations to the Rotten Kid theorem (Becker, 1974, 1981; Bruce and Waldman, 1990); cooperative

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[20] Remember from the section on prestige in social networks the difference between centrality and prestige.

bargaining models, pioneered by Manser and Brown(1980) and McElroy and Horney(1981), principally applying the Nash bargaining solution; non-cooperative bargaining models (see for example Lundberg and Pollak, 1994); and collective models, that the only hypothesis that make is that the allocation achieved is efficient (see Bourguignon and Chiappori, 1992, or Chiappori and Browning, 1998). See Lundberg and Pollak(1996) or Bergstrom(1997) for good surveys of this heterogeneous literature that is not exempt of critique, as Nelson(1994) or Pollak(1994) discuss.

We can understand our approach to the Nash bargaining solution with interdependent utilities as being in a midpoint between the first two kind of models: it is a cooperative bargaing model with agents that show altruism or envy. As a matter of fact, in the Rotten Kid situation there is no bargaining. The distributional conflict is solved by the total bargaining power the head implicitly has, since the head decides how to distribute the available resources. We are addressing the following question: what happens if, instead of allowing an altrusitic agent to become a dictator, we give bargaining power to each agent in the network? This subsection wants to exemplify in the simple case of two agents the answers we have obtained above.

We characterize in the following proposition the Nash bargaining solution with interdependent utilities when $n = 2$, and supposing that the Pareto frontier is regular.

Proposition 9 *Let*

$$\phi = \left[\frac{1}{2} + \frac{\beta - \alpha}{2(1-\alpha)(1-\beta)}\right]\pi + \frac{d_1 + \alpha d_2}{2(1-\alpha)} - \frac{d_2 + \beta d_1}{2(1-\beta)}$$

Then the NBS under interdependent utilities is equal to:

1. $c_1 = \phi$ and $c_2 = \pi - \phi$ if $\phi \in (0, \pi)$
2. $c_1 = \pi$ and $c_2 = 0$ if $\phi \geq \pi$
3. $c_1 = 0$ and $c_2 = \pi$ if $\phi \leq 0$

In some cases, we can traduce this solution as the solution of an assymetric Nash bargaining problem with private utilities, as the following result states.

Proposition 10 *The NBS under interdependent utilities is equivalent to the Assymetric NBS under private utilities with bargaining power for agent 1 equal to* $\gamma = \frac{1}{2} + \frac{\beta-\alpha}{2(1-\alpha)(1-\beta)}$ *and for agent 2 equal to* $1 - \gamma = \frac{1}{2} + \frac{\alpha-\beta}{2(1-\alpha)(1-\beta)}$, *whenever* $\gamma \in [0, 1]$.

The interdependency generates a change in bargaining power related to the weight of the interdependency coefficients. Observe that this bargaining power not only depends on the difference between both coefficients but also on the intensities of this coefficients. It is not the same the case when $\alpha = 0.1$ and $\beta = 0.2$ than when $\alpha = 0.8$ and $\beta = 0.9$.

8 Conclusion

We have considered a model with two elements: (i) individuals are concerned for others' total utility (interdependent utilities), (ii) each individual is concerned for a different group of agents and with different intensity for each agent (social networks). The intensity of each link can be positive, negative or zero, allowing for altruism, envy, of selfishness. The interdependence system or, alternatively, the social network, is determined by a set of $n(n-1)$ real values, the intensities with which each agent cares for each other. But, even if the model is very rich, we have been able to obtain closed-form expressions of the utility possibility set under interdependent utilities for any generic situation. Moreover, we have obtained several results about the Pareto frontier under interdependent utilities thanks to n values, the prestige measures of each of the n agents, that aggregate all the information the $n(n-1)$ initial values contain. Hence, to study efficiency, we reduce the dimension of complexity of the problem from the initial dimension of the interdependence system, which is of order n^2, to the number of agents in the economy, n.

We have concentrated in this work in the study of situations where there are no risk-aversion effects to concentrate on the study of network effects. While the assumption of risk-neutrality is restrictive, we believe our conclusions are valuable also in the case of risk-averse agents. Risk-aversion introduces a new dimension to the problem that interfers with the network effects. With risk-neutrality we can measure how valuable is an agent for the rest of the group simply by means of prestige, but if agents are risk-averse an agent is not only valuable by his position in the network: an individual will be more valuable the more prestigious he is, but also the less risk-averse he shows to be. This is because a risk-averse agent has a concave utility function, and an increment in his own consumption can produce an smaller effect than the same increment in a risk-neutral agent. If a risk-averse agent is very prestigious it can be optimal to transfer resources to him, but maybe he is not prestigious that much, and in this case it can be possibly a better option to transfer resources to a slightly less prestigious, but risk-neutral, agent.

From our study of bargaining problems it seems that the measures of prestige of each agent are not sufficient to fully characterize tha Nash bargaining solution. A more complete study of the effects of the geometry of the network in distributional conflicts would be a good advance. Moreover, this could allow us to define and study an interesting game of strategic network formation[21] as follows: if the total available resources are distributed using the Nash bargaining solution and before the distribution agents can strategically choose with who they want to be linked and with which intensity, which are the equilibria of this game? It is only a conjecture, but it seems that the Nash equilibria of this game should internalize prestige, centrality or other notions related to the geometry of the network, in the strategy each agent would play.

Our methodology seems sufficiently general to be applied in other situations where the effects generated by interdependent utilities can be preponderant, such as public goods games. The utility

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[21] See Jackson(2003) for a survey of this literature.

interdependencies can have significant effects on individual contributions. Altruistic agents can be disposed to increment their contributions while envious agents maybe prefer to decrease their ones. A conjecture might be that, since in this case it seems that the behavior of each agent may depend not only on the affection he receives but also on the one he provides, both centrality and prestige can affect the equilibria of the game.

Finally, we present the opinion of one of the leading scholars in the field of networks in economics. As Jackson (2003) expresses:

> "There are also some substantial challenges for the future literature on networks that is coming out of economics and game theory. One challenge is bridging to the sociology ("social networks") literature. That literature is well-established, very large, and full of interesting questions, insights, data sets, and knowledge of network structure and what influences it. The main challenge comes in the differences in terminology, the points of view, and the techniques of analysis. As the literatures continue to grow, the cross fertilization which is just beginning now should become more and more natural."

We hope this work can be seen as an step in the construction of this bridge.

9 Appendix: Proofs.

Proposition 1

The determinant of the matrix $I-A(g)$ is a polynomial in $n(n-1)$ variables. The set of points of $R^{n(n-1)}$ in which this polynomial vanishes forms an algebraic variety of dymension $n(n-1)-1$, and hence it is a set with Lebesgue measure equal to zero in $R^{n(n-1)}$.

Proposition 2

Let $\bar{M}$ be the matrix that has by entries the $\bar{m}_{ij}$'s defined in the statement of the theorem. We have to prove that

$$(\boldsymbol{I}-\boldsymbol{A})\,\bar{\boldsymbol{M}} = m\boldsymbol{I} \tag{60}$$

First, let's consider first the entries (i,i) of the matrix $(\boldsymbol{I}-\boldsymbol{A})\,\bar{\boldsymbol{M}}$, that are equal to

$$\sum_{k=1}^{n}(I-A)_{ik}\,\bar{m}_{ki} = m\left(g\backslash\{i\}\right) - \sum_{j=1,j\neq i}^{n} a_{ij}\sum_{p\in P_{ji}} W(p)\,m\left(g\backslash\{p\}\right) \tag{61}$$

Observe that the weight of any simple cycle $c=(i,i_1,\ldots,i_k,i)\in C(g)$ is equal to $a_{ii_1}a_{i_1i_2}\cdots a_{i_ki} = a_{ii_1}W(p)$ where p is the simple path equal to $(i_1,\ldots,i_k,i)$. Since the elements that form the cycle c are the same that ones that form the path p, we can rewrite the second term of the right-hand side of (61) as

$$\sum_{j=1,j\neq i}^{n} a_{ij}\sum_{p\in P_{ji}} W(p)\,m\left(g\backslash\{p\}\right) = \sum_{\substack{c\in C(g)\\ i\in c}} W(c)\,m\left(g\backslash\{c\}\right) \tag{62}$$

Thus, using the formulation of $m(g)$ in (24) to the network $g\backslash\{p\}$, we have that the right hand side of (61) is equal to

$$\begin{aligned}&\left(1-\sum_{c_1\in C(g\backslash\{i\})} W(c_1)+\sum_{\substack{c_1,c_2\in C(g\backslash\{i\})\\ c_1,c_2\ disj.}} W(c_1)\,W(c_2)-\cdots\right)-\\ &-\sum_{\substack{c\in C(g)\\ i\in c}} W(c)\left(1-\sum_{\substack{c_1\in C(g)\\ c,c_1 disj.}} W(c_1)+\sum_{\substack{c_1,c_2\in C(g)\\ c,c_1,c_2 disj.}} W(c_1)\,W(c_2)-\cdots\right)\end{aligned} \tag{63}$$

but we can reorder the terms in this expression as follows

$$1-\left(\sum_{c_1\in C(g\backslash\{i\})}W(c_1)+\sum_{\substack{c\in C(g)\\ i\in c}}W(c)\right)+\left(\sum_{\substack{c_1,c_2\in C(g\backslash\{i\})\\ c_1,c_2\ disj.}}W(c_1)W(c_2)+\sum_{\substack{c,c_1\in C(g),i\in c\\ c,c_1\ disj.}}W(c)W(c_1)\right)-$$

$$-\left(\sum_{\substack{c_1,c_2,c_3\in C(g\backslash\{i\})\\ c_1,c_2,c_3\ disj.}}W(c_1)W(c_2)W(c_3)+\sum_{\substack{c,c_1,c_2\in C(g),i\in c\\ c,c_1,c_2\ disj.}}W(c)W(c_1)W(c_2)\right)+\cdots \quad (64)$$

and this expression is equal to

$$1-\left(\sum_{c_1\in C(g)}W(c_1)\right)+\left(\sum_{\substack{c_1,c_2\in C(g)\\ c_1,c_2\ disj.}}W(c_1)W(c_2)\right)-\left(\sum_{\substack{c_1,c_2,c_3\in C(g)\\ c_1,c_2,c_3\ disj.}}W(c_1)W(c_2)W(c_3)\right)+\cdots \quad (65)$$

which is exactly equal to (24) and therefore equal to $m(g)$.

If we resume what we have obtained from equations (61) to (65), we have established that for any network g and for any agent $i\in N$ we have that

$$m(g)-m(g\backslash\{i\})=-\sum_{j=1,j\neq i}^{n}a_{ij}\sum_{p\in P_{ji}}W(p)\,m(g\backslash\{p\})=-\sum_{\substack{c\in C(g)\\ i\in c}}W(c)\,m(g\backslash\{c\}) \quad (66)$$

Now we have to prove that the entries (i,j) with $i\neq j$ of the matrix $(\boldsymbol{I}-\boldsymbol{A})\bar{\boldsymbol{M}}$ are equal to 0. The entry (i,j) of this product is equal to

$$\sum_{k=1}^{n}(I-A)_{ik}\,\bar{m}_{kj}=\underbrace{\bar{m}_{ij}}_{[A]}-\underbrace{a_{ij}m(g\backslash\{j\})}_{[B]}-\underbrace{\sum_{k=1;k\neq i,j}^{n}a_{ik}\bar{m}_{kj}}_{[C]} \quad (67)$$

Let's develope each term in the right hand side of (67) individually.

$[A]$:

$$\bar{m}_{ij}=\sum_{p\in P_{ij}(g)}W(p)\,m(g\backslash\{p\}) \quad (68)$$

$[B]$:

$$a_{ij}m(g\backslash\{j\})=\underbrace{a_{ij}m(g\backslash\{i,j\})}_{[B1]}+\underbrace{a_{ij}\left[m(g\backslash\{j\})-m(g\backslash\{i,j\})\right]}_{[B2]} \quad (69)$$

$[C]$:

$$\sum_{\substack{k=1\\k\neq i,j}}^{n} a_{ik}\bar{m}_{kj} = \underbrace{\sum_{\substack{k=1\\k\neq i,j}}^{n} a_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} W(p)\, m\left(g\backslash\{p\}\right)}_{[C1]} + \underbrace{\sum_{\substack{k=1\\k\neq i,j}}^{n} a_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\in p}} W(p)\, m\left(g\backslash\{p\}\right)}_{[C2]} \tag{70}$$

Furthermore, $[C1]$ is equal to

$$\underbrace{\sum_{\substack{k=1\\k\neq i,j}}^{n} a_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} W(p)\, m\left(g\backslash\{p,i\}\right)}_{[C1.a]} + \underbrace{\sum_{\substack{k=1\\k\neq i,j}}^{n} a_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} W(p)\left[m\left(g\backslash\{p\}\right) - m\left(g\backslash\{p,i\}\right)\right]}_{[C1.b]} \tag{71}$$

where $g\backslash\{p,i\}$ means the network obtained removing not only all agents in the path p but also agent i. Observe that $[C1.a]$ can be rewrited as follows

$$\sum_{\substack{k=1\\k\neq i,j}}^{n} a_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} W(p)\, m\left(g\backslash\{p,i\}\right) = \sum_{\substack{p\in P_{ij}(g)\\p\neq(i,j)}} W(p)\, m\left(g\backslash\{p\}\right) \tag{72}$$

It follows from this last expression of $[C1.a]$ that $[C1.a]+[B1]=[A]$. Therefore, to be able to conclude that $[A]+[B]+[C]=0$, we still have to prove that $[B2]+[C1.b]+[C2]=0$. First, observe that, if $k\neq i,j$ and $p=(k,i_1,\ldots,i_l,i,i_{l+1},\ldots,i_{l+s},j)\in P_{kj}(g)$ such that $i\in p$, we have that

$$a_{ik}W(p) = a_{ik}\left(a_{ki_1}\cdots a_{i_l i}a_{ii_{l+1}}\cdots a_{i_{l+s}j}\right) = \left(a_{ik}a_{ki_1}\cdots a_{i_l i}\right)a_{ii_{l+1}}\cdots a_{i_{l+s}j} = W(c)\,W(\bar{p}) \tag{73}$$

where $\bar{p}=(i,i_{l+1},\ldots,i_{l+s},j)\in P_{ij}(g)$, and $c=(i,k,i_1,\ldots,i_l,i)\in C\left(g\backslash\left(\{\bar{p}\}\setminus\{i\}\right)\right)$ is such that $i\in c$. Hence, taking into account that with the objects defined just above $\{p\}=\{\bar{p},c\}$, i.e. the elements in p are the same that the union of elements of $\bar{p}$ and c, we can rewrite $[C2]$ as follows:

$$\sum_{\substack{k=1\\k\neq i,j}}^{n} a_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\in p}} W(p)\, m\left(g\backslash\{p\}\right) = \sum_{\bar{p}\in P_{ij}(g)} W(\bar{p}) \sum_{\substack{c\in C(g\backslash(\{\bar{p}\}\setminus\{i\}))\\i\in c}} W(c)\, m\left(g\backslash\{\bar{p},c\}\right) \tag{74}$$

Now, we rewrite $[C1.b]$ and $[B2]$ applying the equation (66) to the network $g\backslash\{p\}$. $[C1.b]$ is equal to

$$\sum_{\substack{k=1\\k\neq i,j}} a_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} W(p)\left[m\left(g\backslash\{p\}\right) - m\left(g\backslash\{p,i\}\right)\right] = \sum_{\substack{\bar{p}\in P_{ij}(g)\\\bar{p}\neq(i,j)}} \left[-\sum_{\substack{c\in C(g\backslash(\{\bar{p}\}\setminus\{i\}))\\i\in c}} W(c)\, m\left(g\backslash\{\bar{p},c\}\right)\right] \tag{75}$$

while, similarly, $[B2]$ is equal to

$$a_{ij}\left[m\left(g\backslash\left\{j\right\}\right)-m\left(g\backslash\left\{i,j\right\}\right)\right]=-a_{ij}\sum_{\substack{c\in C(g\backslash\{j\})\\ i\in c}}W\left(c\right)m\left(g\backslash\left\{i,j\right\}\right) \tag{76}$$

Hence, we obtain that

$$[C1.b]+[B2]=-\sum_{\bar{p}\in P_{ij}(g)}W\left(\bar{p}\right)\left[\sum_{\substack{c\in C(g\backslash(\{\bar{p}\}\backslash\{i\}))\\ i\in c}}W\left(c\right)m\left(g\backslash\left\{\bar{p},c\right\}\right)\right]=-\left[C2\right] \tag{77}$$

that is what we wanted to prove.

Proposition 3

First of all, observe that $\Delta_n\left(\pi\right)=\pi\cdot conv\left(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n,\boldsymbol{0}\right)$. The image of the vector $\pi\cdot\boldsymbol{e}_i$ by the linear application with associated matrix $\boldsymbol{M}\left(g\right)$ is $\pi\cdot\left(\boldsymbol{M}\left(g\right)\cdot\boldsymbol{e}_i\right)=\pi\cdot\boldsymbol{m}^{(i)}$, and the image of $\boldsymbol{0}$ is $\boldsymbol{0}$. The set of points that generate a convex hull are called its extreme points. The result follows from the fact that the image of a convex hull C by a linear application is the convex hull generated by the image of this application of the extreme points of C.

Lemma 1 *Let $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$. $\boldsymbol{x}$ Pareto dominates $\boldsymbol{y}$ if and only if $\boldsymbol{\mu}\cdot\boldsymbol{x}>\boldsymbol{\mu}\cdot\boldsymbol{y}$ for every strict system of weights $\boldsymbol{\mu}$.*

Proof:

The fact that if $\boldsymbol{x}$ Pareto dominates $\boldsymbol{y}$ then $\boldsymbol{\mu}\cdot\boldsymbol{x}>\boldsymbol{\mu}\cdot\boldsymbol{y}$ for every strict system of weights is trivial. We prove the other direction. Let $i\in N$ and let $\boldsymbol{\mu}^{(i)}\left(k\right)$ be the n-dimensional vector that has components $\mu_j^{(i)}\left(k\right)=\frac{1}{k}$ if $j\neq i$ and $\mu_i^{(i)}\left(k\right)=1-\left(n-1\right)\frac{1}{k}$. Observe that if $k>n$ this vector is a strict system of weights. Thus, by assumption, we have that if $k>n$

$$\boldsymbol{\mu}^{(i)}\left(k\right)\cdot\boldsymbol{x}>\boldsymbol{\mu}^{(i)}\left(k\right)\cdot\boldsymbol{y} \tag{78}$$

If we make k tend to $+\infty$ we obtain, by continuity, that

$$x_i\geq y_i \tag{79}$$

Hence, repeating this reasoning for every $i\in N$ we obtain that

$$x_i\geq y_i\qquad for\ every\ i\in N \tag{80}$$

If we had equality for all $i\in N$, this would imply that $\boldsymbol{\mu}\cdot\boldsymbol{x}=\boldsymbol{\mu}\cdot\boldsymbol{y}$ for every strict system of weights, contradicting our hypothesis. Therefore, it must be that $x_i>y_i$ for at least one i, meaning that $\boldsymbol{x}$ Pareto dominates $\boldsymbol{y}$.

Proposition 4
In text (pg.25) and applying the previous lemma.

Proposition 5
The result follows from the fact that the convex hull of the columns of $M(g)$, that we denote here by C, is the Pareto frontier if and only if it is a subset of a hyperplane H in R^n such that the components of a normal vector to H are all strictly positive or all strictly negative, and the zero vector lyes below H. Given any such normal vector $\boldsymbol{q}$ to H, we can normalize it so that we obtain a strict system of weights $\boldsymbol{\mu}$. We only have to define $\boldsymbol{\mu} = \frac{1}{q} \cdot \boldsymbol{q}$, where $q = q_1 + \cdots + q_n$. The zero vector lyes below the hyperplane H if and only if there exists a positive constant r such that the equation of H is

$$\boldsymbol{\mu} \cdot \boldsymbol{x} = r \tag{81}$$

And, finally, C is a subset of H if and only if

$$\boldsymbol{\mu} \cdot \boldsymbol{m}^{(i)} = r \qquad for\ every\ i \in N \tag{82}$$

Proposition 6
In text.

Proposition 7
If the Pareto frontier is regular, the Nash bargaining problem under interdependent utilities is

$$\max_{x_1,\cdots,x_n} \prod_{n}^{i=1} \left(U_i\left(x_1, \ldots, x_n\right) - D_i\right)$$

subject to:

$$x_i \geq 0 \qquad i = 1, \ldots, n \tag{83}$$

$$x_1 + \cdots + x_n = \pi \tag{84}$$

Observe that restriction (84) is related to the regularity of the Pareto frontier.
Since $U_i\left(x_1, \ldots, x_n\right) = m_{i1}x_1 + \cdots + m_{in}x_n$, an equivalent formulation if we transform the objective function through the logarithm function is

$$\max_{x_1,\cdots,x_n} \ \ln\left(m_{11}x_1 + \cdots + m_{1n}x_n - D_1\right) + \cdots + \ln\left(m_{n1}x_1 + \cdots + m_{nn}x_n - D_n\right)$$

subject to (83) and (84). If ψ is the multiplier associated to the last restriction, and $\boldsymbol{\rho}$ is the n-dimensional vector with components $\rho_i = \frac{1}{m_{i1}x_1 + \cdots + m_{in}x_n - D_i}$ for $i = 1, \ldots, n$, we have that the

Kuhn-Tucker conditions are

$$\rho_1 m_{11} + \cdots + \rho_n m_{n1} \leq \psi \qquad with\ equality\ if\ x_1 > 0 \tag{85}$$

$$\vdots$$

$$\rho_1 m_{1n} + \cdots + \rho_n m_{nn} \leq \psi \qquad with\ equality\ if\ x_n > 0 \tag{86}$$

Observe that $U_i(x_1, \ldots, x_n) - D_i = m_{i1}x_1 + \cdots + m_{in}x_n - D_i$ must be strictly positive for any $i \in N$. If not, the value function would be equal to $-\infty$. This implies that $\rho_i > 0$ for every $i \in N$. Also, ψ is strictly positive since it is equal to the derivative of the value function with respect to π and, since we are by assumption in a case where the Pareto frontier is regular, the value function is strictly increasing in π.

Let $\rho = \sum_{i=1}^{n} \rho_i$, $\boldsymbol{\eta} = \frac{1}{\rho}\boldsymbol{\rho}$, and $\lambda = \frac{\psi}{\rho}$. From the comments just made above, we have that $\boldsymbol{\eta}$ is a strict system of weights and that $\lambda > 0$. The Kuhn-Tucker conditions can be rewrited as follows

$$\boldsymbol{\eta} \cdot \boldsymbol{m^{(i)}} \leq \lambda \qquad with\ equality\ if\ x_i > 0 \quad i = 1, \ldots, n \tag{87}$$

which are just the conditions stated in the theorem.

To obtain the utilities under the Nash bargaining solution just observe that for any $i \in N$ we have

$$\eta_i = \frac{1}{\rho\left(U_i^{NBS} - D_i\right)} \Rightarrow \frac{1}{\eta_i} = \rho\left(U_i^{NBS} - D_i\right)$$

Hence, if $K = \frac{1}{\rho}$ we obtain the desired equations

$$U_i^{NBS} = D_i + K \cdot \frac{1}{\eta_i} \tag{88}$$

Proposition 8

If the solution is interior, i.e. $x_i > 0$ for every $i \in N$, the Kuhn-Tucker conditions are

$$\boldsymbol{\eta} \cdot \boldsymbol{m}^{(i)} = \lambda \qquad i = 1, \ldots, n$$

or, what is the same

$$(\boldsymbol{M}(g))^T \cdot \boldsymbol{\eta} = \lambda \cdot \mathbf{1}_n$$

where $\boldsymbol{\eta}$ and λ are the system of weights and the positive constant obtained in the previous proof. Since $M(g)$ is non-singular *it must be that* $\boldsymbol{\eta} = \boldsymbol{\mu}$ *and* $\lambda = r$, where $\boldsymbol{\mu}$ and r are the system of weights and the constant from proposition 5. Hence, $\eta_i = 1 - \sum_{j=1, j\neq i}^{n} a_{ji}$

Since $\boldsymbol{U}^{NBS} = \left(U_1^{NBS}, \ldots, U_n^{NBS}\right) = \boldsymbol{M}(g) \cdot \boldsymbol{x}$ and $\boldsymbol{D} = (D_1, \ldots, D_n) = \boldsymbol{M}(g) \cdot \boldsymbol{d}$, we have, from (88), that

$$\boldsymbol{M}(g) \cdot \boldsymbol{x} = \boldsymbol{M}(g) \cdot \boldsymbol{d} + K\left(\frac{\mathbf{1}}{\boldsymbol{\eta}}\right) \tag{89}$$

where $\left(\frac{\mathbf{1}}{\boldsymbol{\eta}}\right)$ represents the n-dimensional vector that has for components the inverses of the components of $\boldsymbol{\eta}$. Hence, since $(M\,(g))^{-1} = I - A\,(g)$, and since $K = \frac{1}{\rho} = \frac{\lambda}{\psi} = \frac{c}{\psi}$, we have that

$$\boldsymbol{x} = \boldsymbol{d} + \frac{c}{\psi} \cdot (\boldsymbol{I} - \boldsymbol{A}\,(g)) \cdot \left(\frac{\mathbf{1}}{\boldsymbol{\eta}}\right) \tag{90}$$

If $\boldsymbol{\theta} = (\boldsymbol{I} - \boldsymbol{A}\,(g)) \cdot \left(\frac{\mathbf{1}}{\boldsymbol{\eta}}\right)$, $\theta = \sum_{i=1}^{n} \theta_i$ and $d = \sum_{i=1}^{n} d_i$ we obtain adding all the components of the vectors in (90), and taking into account the restriction (84) in the Nash bargaining problem, that

$$c = \frac{\psi}{\theta}\,(\pi - d) \tag{91}$$

Therefore

$$\boldsymbol{x} = \boldsymbol{d} + (\pi - d) \cdot \boldsymbol{\varphi} \tag{92}$$

where $\boldsymbol{\varphi} = \frac{1}{\theta}\boldsymbol{\theta}$. Since $\boldsymbol{\eta} = \boldsymbol{\mu}$, each component of $\boldsymbol{\varphi}$ is equal to

$$\varphi_i = \frac{(\boldsymbol{I} - \boldsymbol{A}\,(g))_{(i)} \cdot \left(\frac{\mathbf{1}}{\boldsymbol{\mu}}\right)}{\sum_{j=1}^{n} (\boldsymbol{I} - \boldsymbol{A}\,(g))_{(j)} \cdot \left(\frac{\mathbf{1}}{\boldsymbol{\mu}}\right)} \qquad i = 1, \ldots, n \tag{93}$$

where $(\boldsymbol{I} - \boldsymbol{A}\,(g))_{(i)}$ is equal to the i-th row of the matrix $I - A\,(g)$.
The *if and only if* statement follows from the fact that the Kuhn-Tucker conditions are necessary and sufficient conditions when the objective function is strictly concave and the restrictions are linear and define a convex set.(See, for example, the mathematical appendix in Mas-Colell et al., 1995).

Proposition 9 It can easily be obtained that an equivalent formulation of the Nash bargaining problem in this case is

$$\max_{x_1} \left(x_1 + \alpha\,(\pi - x_1) - (d_1 + \alpha d_2)\right)\left(\pi - x_1 + \beta x_1 - (d_2 + \beta d_1)\right)$$

subject to $\pi \in [0, \pi]$.
The result follows from the fact that the objective function is strictly concave and has a maximum in the term ϕ defined in the statement of the proposition.

Proposition 10
It is a direct corollary of proposition 8. Some straightforward algebra leads to the result.

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