Dissemination of Spillovers in Cost-Reducing Alliances

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Abstract

Firms raise cost-reducing alliances before competing with each other, but cannot fully appropriate the shared knowledge. When spillovers disseminate through the network of alliances, a link generates positive externalities for third parties, as well as can be a conduit for spillovers emanating from other links. This results in the following tradeoff: firms want to shorten their distance to firms forming links in order to capture more spillovers, but by doing so they become intermediary in the spreading of spillovers to other firms. This leads to the emergence of networks characterized by a moderate level of asymmetry in the number of partners.

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1 Introduction

This article studies the impact of spillovers on firms' incentives to engage R&D alliances. We address the point by assuming that the knowledge generated by a technological alliance benefits third parties. More specifically, we postulate that spillovers propagate through the network of existing alliances, and with possible decay. At least two reasons motivate this hypothesis. First, the mobility on labor market of inventors may explain why knowledge flows between firms (see the notion of cross-firm inventor in Breschi and Lissoni [2005]);

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we guess (without providing empirical evidence) that the probability that a cross-firm inventor moves from firm A to firm B is higher when the firms have form a R&D alliance. Second, the return of alliance formation may be affected by the whole stock of knowledge of the partners; hence, the knowledge produced as an output of some alliance may be an input for the return of another alliance. Hence, the R&D network may reasonably approximate the social network of inventors through which knowledge flows.

The major implication of this hypothesis is that the formation of alliances may entail (positive) asymmetric shocks on the marginal costs of the partners, as well as those of other firms. Particularly, the incentives of link formation take into account the positive externalities provided to third parties as well as those emanated by other links. This results in the following tradeoff: firms want to shorten their distance to agents forming other links, but by doing so they become intermediary in the conduct of spillovers. Our main conclusion is that, despite negligible cost of link formation, the asymmetries of spillover flows can be sufficient to deteriorate the incentive of link formation and can lead to the emergence of networks characterized by a moderate level of asymmetry in the number of partners.

We sum up the results in more details. As a preliminary remark, we observe that in linear Cournot competition, when the formation of an alliance benefits the whole population of firms (what we call *global* spillovers), individual incentives to form alliances are not destroyed. Thus the complete network is both uniquely stable and uniquely efficient. Then we explore the possibility of spillovers' dissemination through the network of alliance. In this context, we first stress that the complete network is both stable and uniquely efficient. Next, we examine under what conditions non complete (and thus non efficient) networks may emerge.

The incentives of link formation can be decomposed in three factors: the new link generates value to the partners, further it induces the spreading of spillovers to third parties, last its allows the partners to capture additional spillovers from other links. First, we focus on our most general setting, consisting in Cournot oligopoly constrained by assumptions 1-3 as defined in Yi (1998). Assuming that spillovers spread through the network of alliances with possible decay, we obtain that if a symmetric network is stable, then it contains a unique component; if a network is both stable and not symmetric, two distinct components necessarily have distinct architectures; if spillovers disseminate with convex decay, the unique symmetric and stable network is the complete network. In the linear Cournot setting, we also find that if the spillovers' decay is convex, two symmetric agents in any same stable component (whether symmetric or not) form an alliance.

Second, we explore more deeply two polar specifications of decay under linear Cournot oligopoly. Both models entail the formation of networks with moderate asymmetry and under-connection with regard to social welfare. In the first model, spillovers only benefit the direct partners of the allied firms (we shall say that spillovers are *partner-restricted*). The possibly stable and non complete architectures have no more than two components, and if it is the case one is an isolated agent. The greatest component is either complete or its diameter is equal to 2, in which case it does not contain some agent connected to all other agents in the component. Moreover, stable incomplete components satisfy two additional constraints: if two firms in the component have the same number of partners, they form a link; if two firms in the component do not form a link, then all partners of the firm having the smallest number of alliances are also partners of the other firm. In the second model, spillovers disseminate without friction (except at order one) toward the end of the component of the allied firms (what we call *component-restricted* spillovers). The stable networks are unions of components of distinct size, and possibly contain small components (typically of size less than half the population). Also, our analysis indicates that the incentives of link formation between distinct components can be non monotonic with respect to the size of the partner's component.

To sum up, the analysis suggests that the presence of spillovers may qualify the incentives to enter R&D partnerships in a context that is intrinsically favorable to link formation, which would result in the existence of underconnected networks with regard to efficiency issue. Hence, from a policy point of view, the analysis suggest that a policy maker should sponsor specific links. What is decisive for deteriorating the individual incentives of alliance formation is not the spillovers' intensity or the number of firms benefiting from spillovers induced by a given alliance, but rather to what extent the new partnership is a bridge for dissemination of spillovers emanated from other alliances. In that respect, when the potential partners occupy asymmetric structural positions in the alliance network, this can induce asymmetric flows of spillovers transiting through them, deterring one of the two firms from allying. A last lesson is that the very mode of dissemination of spillovers through the alliance network affects drastically the nature of stable networks.

From an empirical point of view, evidences about knowledge spillovers are not plethoric. The pioneering work of Jaffe *et al.* (1993) documents the existence of geographic localization of knowledge spillovers as evidenced by patent citations. Further studies refine the issue by taking an explicit account of the social network of inventors as a key determinant of patent citation (Breschi and Lissoni [2005], Singh [2005]). These studies stress the role of interpersonal contacts necessary for knowledge dissemination, as well as they confer to "cross-firm inventors" a key role. The results generally confirm that the probability of knowledge flow is a decreasing function of the social distance (in terms of the shortest path) between individuals and that the decrease is rather convex and rapid. With respect to these findings, our central hypothesis views the alliance network as approximating the social networks of inventors.

From a theoretical perspective, this paper can be inserted in the literature on strategic cooperation. The closest paper is Goyal and Joshi (2003), the model of which we extend to spillovers' dissemination. With respect to this work, our context entails specific incentives to form links. One consequence is that when considering the incentive of link formation between firms belonging to components of distinct size, the less incited can be the firm in the largest components, since more spillovers come from her own component toward the other component (actually, the less incited firm lies in the component combining both a large number of links and short distances between that agent and links of her own component). In contrast, the model of Goyal and Joshi exhibits the opposite tendency: because of the increasing return property, responsible for the emergence of dominant groups, the less connected agents, the less the incentive to form links.

The paper also echoes Bloch (1995), notably concerning the case in which spillovers affect the whole component of the partners: the stability of asymmetric and complete components shall be compared with the stability of two asymmetric coalitions in Bloch (1995). Indeed, a rather similar mechanism applies: the incitation to refuse a connection increases with the difference between own and the partner's component size, which can result in the destruction of the incentive to form a link. In the coalitional context of the author, the incitation for the members of a coalition to exclude a newcomer increases linearly with the size of the current coalition, which results in the stability of two asymmetric coalitions. In contrast, our context enables the coexistence of small components (whose cardinal is less than half the population): indeed, spillovers flows are proportional to the number of links in each components. When the component is complete, the flows are quadratic (and not linear) with respect to the size of the component they come from.

Let us also note that our paper departs from Goyal and Joshi (2005). The authors propose a general model that relates the return of link formation to topological variables. They obtain results as considering monotonic responses to these variables. In contrast, in our model the incentives depend on the distance from agents to links, which originates non monotonic returns.

In our paper, being in intermediary position may discourage from forming a link, since 'bridging' the network may decrease significantly rival costs. In this respect, the role of rivalry is prominent and is similar to Yi (1998) in spirit: firms may not form links to impede rivals to reduce costs. This is in contrast with social networks environment, in which agents add a lot in occupying an intermediary position (Burt [1992], Goyal and Vega-Redondo [2004]). The absence of rivalry in that latter context is crucial for explaining the differences.

To finish, an original feature of the model is that the existence of spillovers induces a situation where firms are rewarded according to the shortest path to links. Therefore, relaxing rivalry would produce a specific model with positive externalities, distinct from standard communication models like the *connections model* of Jackson and Wolinsky (1996). Indeed, in a communication network the sources of benefits are the firms themselves (so they are exogenous), and the aim of each firm is to access the others through costly direct links and non costly indirect links. By contrast, the sources of benefit in the context of spillovers' dissemination are the links established in the network, which are therefore *endogenous* to the choices of firms. This leads to significant differences in stability issue. In the connections model with decay, the star network is stable for low (but not negligible) cost of link formation, since two peripheral firms may not be incited to form a link as soon as indirect benefit is greater than direct benefit less the cost of link formation; in our context, the unique stable network under low cost of link formation would be the complete network, given that forming a link generates additional value. From this perspective, another implication of our paper is that introducing rivalry modifies significatively the shaping of stable architectures.

Section 2 presents the model, section 3 provides the results under the assumption that spillovers transit through the network of alliances with possible decay, section 4 examines two polar cases on decay function. Section 5 concludes. The last section gives all proofs.

2 The model

We consider an industry containing a set $N = \{1, ..., n\}$ of firms. We set up a standard two-periods game in which the first is devoted to the formation of cost-reducing alliances and the second to competition. For this purpose, we propose a spillover-augmented version of the linear Cournot game initiated in Goyal and Joshi (2003).

Graphs. We denote $[\cdot]$ as the floor operator and $|\cdot|$ means the cardinal of a set. A non directed graph represents the set N of firms (the nodes) plus the set $L(N)$ of bilateral alliances between the firms (the edges between nodes). We denote by G the set of all non directed graphs with n nodes. We shall abuse the notation by writing that some link $ij \in g$ whatever $g \in G$. We denote by $N_i(g)$ the set of firms with whom firm i forms a link in the graph g (firm i is not included in the set by convention) and $\mu_i(g)$ represents the cardinal of this set. We define $v_{ij}(g) = |N_i(g) \cap N_j(g)|$, $v_{ij}(g) \in \{0, \cdots, n-2\}$ as the number of common partners of firms i and j in the graph g. Symbol $g - ij$ (resp. $g + ij$) denotes the graph g less (resp. plus) the link ij. A *subgraph* of a graph g is a pair $(A, L(A))$ with $A \subset N$ and $L(A) = \{ij \in L(g)/i \in A, j \in A\}$ (The set A can be a singleton). A *complete subgraph* is a subgraph such that every pair of firms in the subgraph forms a link. A *path* in the graph g is a sequence of firms $\{a_0, \cdots, a_p\}$ such that $a_i a_{i+1} \in g$ for all $i \in \{0, \ldots, p-1\}$. A *component* $C(g)$ in the graph g is a subgraph such that there is a path between any pair of firms in the component, and there is no link in the graph g between any firm inside the component and any firm outside the component. We denote $i \in C(g)$ when firm i belongs to component $C(g)$. Symbol $L(C(g))$ will represent the set of links in the component $C(g)$ and $\zeta_i(g)$ will represent the number of agents in the component of agent i in the graph g. For clarity and when there are no confusions, we shall omit the argument g from the main symbols: μ, C, A, ζ. The *architecture* associated to the graph g is the set of graphs resulting from permutations in strategies of agents keeping unchanged the structure of individual payoffs. Two agents are *symmetric* in the graph g if permuting their strategies does not change the architecture of the graph. A symmetric graph is such that all pairs of agents are symmetric.

We define $d(i, j; g)$ as the smallest number of arcs between firm i and firm j in the graph g, *i.e.*, the (geodesic) distance between agents i and j in the graph g. The diameter of a graph g is the maximal distance between two firms in the graph, and it is denoted $d(g)$ (it can be infinite in our convention). Second, we define the application $f(i, jk; g)$, taking two arguments, some agent i in the industry as the first one, some link jk in the network of alliances as the second one, and producing the positive integer $f(i, jk; g) = \min[d(i, k; g), d(i, l; g)]$. This value represents the length of the shortest path between firm i and either agent j of agent k in the graph g. We shall abuse the language by saying that this value represents the 'distance' between agent i and arc jk. We denote by $F_i(p; g) = |\{kl \in g / f(i, kl; g) = p\}|$ the number of arcs at distance p from firm i in the graph g.

Network architectures. The empty (resp. complete, denoted g^c) network is the graph such that no pair (resp. every pair) of firms forms a link. The class g_k, with $k \in \{\frac{n}{2} + 1, \cdots, n-1\}$ if n is even and with $k \in \{\frac{n+1}{2}, \cdots, n-1\}$ if n is odd, denotes a network containing two complete components, the greatest being of size k. Other architectures will be defined further in the paper.

The industry. We assume positive marginal costs and no fixed cost. In the empty network, all marginal costs are identically fixed at a level $c_0 > 0$. A new alliance induces a fixed positive decrease γ_0 in the partners' marginal costs, whatever the number of already existing alliances[1]. Involuntary spillovers may arise in the industry. When two firms i and j form an R&D alliance, other firms benefit from marginal cost reduction, by an amount that does not exceed the cost reduction of the allied (we will precise further which firm is beneficial, when presenting our scenarios on spillovers' dissemination). We denote by $\pi_i(g)$ the profit made by firm i in the network g. Each firm produces some homogenous good, sold at price p_i and in quantity q_i. The linear inverse demand schedule is given by $p_i = \alpha - q_i - \sum_{j \neq i} q_j$ in the region where the price is positive, with $\alpha > 0$ measuring the absolute size of the market.

The two-stage game. In the first stage, firms simultaneously raise collaborative links. Throughout the rest of the paper, we assume that forming links is non costly. We apply

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[1] In Deroian (2004) an alliance may entail different cost reductions for the partners.

the standard stability criterion of *pairwise stability*, adapted from Jackson and Wolinsky (1996): (i) for $ij \in g$, $\pi_i(g) > \pi_i(g - ij)$ and $\pi_j(g) > \pi_j(g - ij)$, (ii) for $ij \notin g$, if $\pi_i(g + ij) > \pi_i(g)$, then $\pi_j(g + ij) \leq \pi_j(g)$. Once alliances are formed, firms compete with each other *à la* Cournot. We also define the set of *efficient* networks as the collection of networks which maximize the sum of consumer surplus and aggregate profits.

3 Results - part I

3.1 Preliminary

The results of the paper are established on the basis of a simple argument: when two firms form an alliance, and whatever the spillovers' specification, the technological profile of the industry, as represented by that of marginal costs, evolves toward a new profile in which certain firms benefit from favorable shocks. The next claim states that in linear Cournot oligopoly, whatever the initial profile of marginal costs, a symmetric shock on any subset of firms is beneficial to them.

Claim 1 *Consider a linear Cournot oligopoly and suppose that the cost of link formation is negligible. Consider an incomplete network* g *with the link* $ij \notin g$ *without loss. Consider also the profile* $\{c_1(g), ..., c_n(g)\}$ *of marginal costs on the network* g*. Then, the formation of the link* ij *is profitable to agent* i *if and only if it results in a profile* $\{c_1(g) - \gamma_1, ..., c_n(g) - \gamma_n\}$ *with* $\gamma_i > \frac{\sum_{j \neq i} \gamma_j}{n}$.

The claim stems directly from the fact that when the cost of link formation is negligible, the incentives of link formation are given by quantity variations, which are themselves directly related to the evolution of marginal costs profile.

The second claim gives the price to generalizing the competitive environment: Yi (1998, lemma 3) states that under weak restrictions on demand and technology one can ensure that whatever the initial *symmetric* profile of marginal costs, a symmetric shock on any subset of firms is beneficial to them. Hence, the following claim applies:

Claim 2 *Consider the assumptions 1-3 as stated in Yi (1998). Suppose that the cost of link formation is negligible. Consider an incomplete network* g *with say the link* $ij \notin g$ *without loss. Suppose also that* $c_i(g) = c_j(g) = c$ *for all* i, j *in* N*. Then, the formation of* *the link ij is profitable to both agents i and j if it results in a profile $\{c-\gamma_1,...,c-\gamma_n\}$ with $\gamma_i=\gamma_j\geq\gamma_k$ for all $k\neq i,j$.*

Indeed, Yi (1998, lemma 3) provide reasonable conditions under which a symmetric favorable shock to a subset of symmetric firms will increase their net profits. These conditions concern the general demand (downward-sloping, outputs are strategic substitutes) and the technology of the industry (marginal cost is convex in own output, total and marginal cost are strictly decreasing with the number of links). The claim states that, when marginal costs are identical in the industry, two firms will benefit from a positive shock on certain marginal costs as soon as they both benefit from the largest decrease.

For convenience, we define by *moderate competition* the environment described by assumptions 1-3 in Yi (1998).

3.2 Global spillovers

Before introducing the possibility of spillovers' dissemination through the network of alliances, we begin with the benchmark case in which the formation of an alliance benefits all other firms.

We denote by global spillovers the situation in which, whatever the current network of alliances, if two firms form a new alliance all other firms benefit from spillovers. When spillovers are global, the incentives for firm i to form the link ij in the graph g are given by two factors: first, the marginal cost of firm i decreases, which is positive for firm i; second, the marginal cost of firm j decreases as well as that of the rest of the firms, which is negative for firm i. It follows directly that

Proposition 1 *Suppose that spillovers affect all the firms in the industry. Assume a given spillovers' intensity. Under moderate competition, if a stable network is symmetric, it is the complete network. In linear Cournot oligopoly, the complete network is uniquely stable and uniquely efficient.*

Under moderate competition, we can ensure that no symmetric network distinct from the complete network is stable, without being able to state that the complete network is stable (since the complete network less one link is not symmetric).

Under the more restrictive environment of linear Cournot oligopoly, individual incentives to cooperate are not destroyed even when the content of joint R&D becomes a public good and benefits the whole industry, and this is a good news with regard to social welfare. Let us note that this situation is encompassed by Goyal and Joshi (2005), in their section on 'field games': the Cournot framework satisfies the properties inducing that the unique stable networks, whatever the cost of link formation, are reduced to dominant groups (including the complete network and the empty network).

3.3 Spillovers' dissemination through the network of alliances

We suppose here that the formation of an alliance generates the dissemination of spillovers through the network of alliances. For that purpose, we will use the application $f(i, jk; g)$ which gives the shortest path between agent i and either agent j or agent k in the graph g, or what we call abusively the 'distance' between agent i and the link jk in the graph g. Precisely, let us denote by $\gamma(p)$, $p \geq 1$, the reduction of marginal cost that a firm benefits from being at distance p of a given alliance. For instance, $\gamma(1)$ is the marginal cost reduction received by all partners of two collaborating firms. We will suppose thereafter that spillovers do not increase with distance, *i.e.* the function $\gamma(.)$ is not increasing in its argument and $\gamma(1) \leq \gamma_0$. The marginal cost of firm i in the graph g is written:

$$c_i(g) = c_0 - \gamma_0 \cdot \mu_i(g) - \sum_{p\geq 1} \gamma(p) \cdot F_i(p; g).$$

Assuming negligible cost of link formation, examining the impact on equilibrium quantities of link formation is sufficient to evaluate profit variations. Denoting by $\Delta q_i^{ij}(g) = q_i(g + ij) - q_i(g)$ and defining $\Delta F_k^{ij}(p; g) = F_k(p; g + ij) - F_k(p; g)$, we can compute the variation of firm i's equilibrium quantity when passing from the graph g to the graph $g + ij$. More precisely, whatever graph g in the linear Cournot case, and whenever the link $ij \notin g$, we obtain that

$$(n+1).\Delta q_i^{ij}(g) = (n-1)\gamma_0 + \sum_{p=1}^{d(C_i \cup C_j)} \gamma(p). \left[n.\Delta F_i^{ij}(p; g) - \sum_{k \in C_i \setminus \{i\} \cup C_j} \Delta F_k^{ij}(p; g) \right] \qquad (1)$$

The RHS of the expression can be decomposed into three effects:

$$\underbrace{n\big[\gamma_0 + \sum_{p\geq 1} \gamma(p).\Delta F_i^{ij}(p; g)\big]}_{(E1)} - \underbrace{\big[\gamma_0 + \sum_{p\geq 1} \gamma(p).\Delta F_j^{ij}(p; g)\big]}_{(E2)} - \underbrace{\sum_{p\geq 1} \gamma(p) \sum_{k \in C_i \cup C_j \setminus \{i,j\}} \Delta F_k^{ij}(p; g)}_{(E3)}$$

The effect $(E1)$ represents the marginal cost variation of agent i; it captures the intrinsic return that agent i benefits from the formation of the link ij and also the spillovers' flows issuing from other links, as a consequence of the formation of the link ij (since the formation of the link ij may decrease distances between agent i and links). The effect $(E2)$ represents the marginal cost variation of agent j; it contains the return from the link ij plus an amount of additional spillovers captured by agent j from links distinct from the link ij. The effect $(E3)$ measures the additional spillovers that the other agents access, including those provided by the link ij.
The following proposition tackles the stability and efficiency of complete network in linear Cournot case:

Proposition 2 *Consider a non increasing function $\gamma(.)$. In linear Cournot oligopoly, the complete network is pairwise stable and uniquely efficient.*

In linear Cournot case, efficiency comes from the sum of costs and is proved in Goyal and Joshi (2003, 2004). Stability is easily grasped: when forming the link ij in the graph $g^c - ij$, the effects $(E2)$ and $(E3)$ do not exist. Further, effect $(E4)$ reduces to spillovers emanating from the link ij. The resulting net effect is positive for agent i. As agents i and j are symmetric, the link is profitable to both partners. Hence, the set of efficient networks is included in the set of stable networks whatever spillovers' intensity and decay.

We turn now to stability analysis of other architectures. Under moderate competition, we can state the following results:

Proposition 3 *Consider the moderate competition environment and a non increasing function $\gamma(.)$. If a stable network is symmetric, then it contains a unique component. If a stable graph is not symmetric, two distinct components have distinct architectures. Moreover, if the function $\gamma(.)$ is convex, a symmetric and stable network is the complete network.*

In a symmetric network, if two agents in two distinct component form a link, they receive by symmetry the same amount of spillovers, and third parties do not receive more spillovers than them. Furthermore, if the function $\gamma(.)$ is convex, two symmetric agents in a component also capture a larger amount of spillovers than third parties: indeed, in a given incomplete component, we can see that when a link, say ij, is formed, if the distance from agent i to a given link kl is shortened, the variation of distance is at least equal to the variation of distance from any agent q to that link kl. Then, convexity (we need the function to be convex for distance equal to or greater than 1-and not 0-) is sufficient to guarantee that when agents i and j are in a symmetric position, the amount of spillovers captured by other agents does not exceed the amount obtained by agents i and j. Since we start with a symmetric network, the claim 2 applies. Finally, one can say more under moderate competition without assuming an initial symmetric cost profile: note first that a link between two firms belonging to distinct components may affect the cost of the sole firms in those components. Second, if two distinct components have the same architecture, *there exists* one firm i in a component which minimizes the marginal cost among firms of *her* component, and a symmetric firm j exists in the other component (w.r.t. Yi (1998)'s paper, firm i has the greatest productive asset among all firms in her component). Then, the formation of the link ij is more favorable to both than the case where all firms in the two components have the same costs, a situation such that the link formation would be profitable by application of the claim 2.

Under linear Cournot competition, we also obtain that two distinct components have distinct architectures. The difference with the more general setting is that now *every* pair of symmetric firms i and j in the respective components have an incentive to form a link. Furthermore, in a given incomplete component, the same argument as before entails that two symmetric agents find profitable to form a link. But now, the analysis can be led for non homogenous cost profiles, so the point concerns any component, whether symmetric or not. The next corollary sums up the findings, mostly issued from the previous proposition:

Corollary 3 *Consider a linear Cournot oligopoly, a non increasing function* $\gamma(.)$ *and a stable network. Then, two distinct components have distinct architectures. Moreover, if the function* $\gamma(.)$ *is convex, two symmetric agents in a component form a link.*

Therefore, when the decay is convex in the linear Cournot oligopoly, stable networks are moderately asymmetric. First, the absence of symmetric components ensures asymmetry and stems from the fact that, when forming an alliance, symmetric firms in symmetric components capture more spillovers than third parties. Second, in a given component two symmetric firms form a link, which banishes the presence of two agents forming a unique connection with a same agent, like peripherals in a star network.

We sum up our findings for all non increasing functions $\gamma(.)$: we can ensure under moderate competition that a symmetric and stable network contains a unique component; further, under convex decay no network distinct from the complete network is stable, without being able to state that the complete network is stable (as in the case of global spillovers). Also, whether decay is convex or not, no network containing two components with identical architecture is stable, since we can always find two firms, one each component, who would find profitable to form an alliance. In the linear Cournot setting, we can say more: we check easily that the complete network is pairwise stable and uniquely efficient, and we obtain that if the decay is convex, two symmetric firms necessarily form an alliance in a stable graph.

4 Results - Part II: Two polar scenarios

We pursue the study in the context of linear Cournot competition under two extreme scenarios, corresponding each to a specific function $\gamma(.)$. These polar cases will enable us to give more precision on the existence and architectures of pairwise stable networks. Let us consider two real numbers ρ and γ such that $0 < \rho \leq \gamma$, and denote by τ the ratio $\frac{\rho}{\gamma}$. We examine the following two scenarios:

Partner-restricted spillovers: $\gamma_0 = \gamma$, $\gamma(1) = \rho$, $\gamma(p) = 0$ for all integer $p \geq 2$.

Component-restricted spillovers: $\gamma_0 = \gamma$, $\gamma(p) = \rho$ for all finite integer $p \geq 1$.

Note that in both scenarios, the function $\gamma(.)$ is convex.

4.1 Partner-restricted spillovers

In this case, the marginal cost of firm i in the graph g can be written:

$$c_i(g) = c_0 - \gamma\mu_i(g) - \rho F_i(1; g).$$

In this case, the effects $(E1)$, $(E2)$ and $(E3)$ influencing the incentives for agent i to form the link ij take the following forms:

$(E1)$ is written $n\big[\gamma + \rho(\mu_j - v_{ij})\big]$, as agent i benefits from intrinsic return of the link ij plus additional spillovers from the links between agent j and the partners of agent j who are not partners of agent i,

$(E2)$ is written $\gamma + \rho(\mu_i - v_{ij})$, as agent j benefits from intrinsic return of the link ij plus additional spillovers from the links between agent i and the partners of agent i who are not partners of agent j,

$(E3)$ is written $\rho(\mu_i + \mu_j - v_{ij})$, as the partners of both agents i and j receive spillover issued from the formation of the link ij.

We know from proposition 3.1 that the complete network is stable for any $\tau \in [0, 1]$. Certain incomplete networks can be stable. Our main finding is that they have a moderate level of asymmetry. The next proposition sums up our findings:

Proposition 4 *Suppose that spillovers are partner-restricted. Then, the complete network is stable for all values of* τ*, the network* g_{n-1} *for any* $\tau \in \left[\frac{n-1}{2(n-2)}, 1\right]$*. The other possibly stable architectures have no more than two components, and if so one is an isolated agent. The greatest component satisfies the following properties:*

(i) its diameter is equal to 2*,*

(ii) whatever two agents i *and* j *in the component such that* $\mu_i = \mu_j$*, the link* $ij \in g$*,*

(iii) whatever two agents i *and* j *in the component such that* $\mu_i > \mu_j$ *and the link* $ij \notin g$*, then* $v_{ij} = \mu_j$ *and* $\mu_i > (n-1)/2$*,*

(iv) its does not contain any agent connected to all other agents in the component.

By point (ii), if two firms do not form a connection they do not have the same number of partners. The reason is that the conditions for applying the claim 1 are valid when the firms have the same number of links. The point (iii) states that if two firms do not form a link in a stable component, all the partners of the firm having the smallest number of partners are also partners of the other. The point (ii) favors asymmetric networks, the second ensures some minimal overlapping between asymmetrically positioned firms. the point (iv) forbids the presence of central agents, which indicates that the level of asymmetry with respect to the number of partners is moderate. This last point entails that stable components are not inter-linked stars, as defined in Goyal and Joshi (2005).

In the case $n = 3$, the unique non complete and stable network is the one-link network and it is stable when $\tau = 1$. If $n \in \{4, 5\}$, $S = \{g^c\}$ for any $\tau \in [0, 1]$. If $n = 6$, denote $\tilde{g}$ as the graph depicted in the figure 1. Then $S = \{g^c\}$ for any $\tau \in [0, 1[$ and $S = \{g^c, \tilde{g}\}$ as $\tau = 1$.

We present a class of graphs that are stable for appropriate values of parameter τ. We first note that a network g can always be decomposed as follows: partition g into a set of disjoint complete subgraphs $A_1(g), \cdots, A_p(g)$ such that $\sum_{i \leq p} |A_i| = n$. This set is said *minimal* when there is no partition with less elements. Then we can build the graph g as considering the collection $\{A_i\}_{i \in \{1, \cdots, p\}}$ and completing the residual links between the sets. When possible and for convenience, we shall abuse the notation by writing $g = \{A_i\}_{i \in \{1, \cdots, p\}}$ (the notation gives no precision about the links between firms in distinct subgraphs).

Given two integers $\alpha \geq 2$ and $q \geq 2$, we denote by $\Gamma(q, \alpha)$ the component architecture such that (i) it contains $(\alpha+1)q$ firms, (ii) the component is minimally partitioned into two distinct complete subgraphs $\{A_1, A_2\}$ with $|A_2| = q$, (iii) $\mu_k = (\alpha+1)q - 2$ for any $k \in A_1$ (firms in A_1 have $q-1$ partners in A_2), (iv) $\mu_i = (\alpha+1)(q-1)$ for any $i \in A_2$ (firms in A_2 have $\alpha(q-1)$ partners in A_1). Hence, the component $\Gamma(q, \alpha)$ contains two complete subgraphs of distinct size and every firm in the greatest complete subgraph forms connections with all firms less one in the smallest complete subgraph. Further, the organization of links between the two subgraphs is such that any two firms in a given complete subgraph have the same number of partners (see figure 1). We extend this class to $r \geq 2$ complete subgraphs as follows: we denote by the class $\Gamma(q, \alpha_1, \cdots, \alpha_{r-1})$ the component architecture, containing $(\sum_{i=1}^{r-1} \alpha_i + 1)q$ firms, minimally partitioned into r complete subgraphs $\{A_i\}_{i=1,\cdots,r}$, with $|A_r| = q$, $|A_i| = \alpha_i |A_{i+1}|$, $\alpha_i \geq 2$ being an integer, and such that:

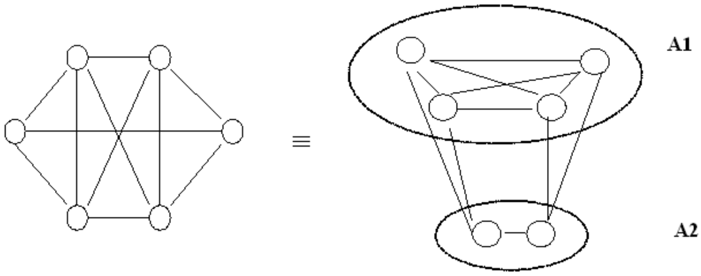


Figure 1: $n = 6$; the network architecture $\Gamma(2,2)$

$$\begin{cases} \text{for all } j \in \{1, \cdots, r\}, \text{ for all } i, i' \in A_j, \ \mu_i = \mu_{i'} \\ \text{for all } i, j, i < j, \ \forall k \in A_i, \ |N_k(g) \cap A_j| = |A_j| - 1 \\ \text{for all } i, j \text{ such that } \mu_j < \mu_i \text{ and } ij \notin g, \text{ then } v_{ij} = \mu_j \end{cases}$$

The first line says that all firms in a given subgraph have the same number of partners. The second line indicates that when considering two firms in two distinct complete subgraphs, the firm in the largest subgraph forms a link with all firms less one in the other subgraph. The third line expresses that for any pair of firms with different number of partners and not forming a link, the set of partners of the less connected firm belongs to the set of partners of the other firm. To finish, we denote by $\Gamma'(q, \alpha_1, \cdots, \alpha_{r-1})$ the class of graphs with $n = (1 + \sum_{i=1}^{r-1} \alpha_i)q + 1$ firms and consisting in the union of one isolated firm and one component $\Gamma(q, \alpha_1, \cdots, \alpha_{r-1})$.

Networks in the classes $\Gamma(q, \alpha_1, \cdots, \alpha_{r-1})$ and $\Gamma'(q, \alpha_1, \cdots, \alpha_{r-1})$ are pairwise stable for appropriate values of spillovers intensity. Let us present the conditions under which the class $\Gamma(\alpha_1, \alpha_2)$ (which contains three minimally complete subgraphs) is pairwise stable: $g = \{A_1, A_2, A_3\}$, with $|A_1| = \alpha_1|A_2| = \alpha_1\alpha_2|A_3|$ ($\alpha_1, \alpha_2 \geq 2$ and integers), $|A_3| \geq 2$. By definition, $\mu_k = n - 3$ for all $k \in A_1$, $\mu_j = n - \alpha_1 - 2$ for all $j \in A_2$, $\mu_i = n - (\alpha_1 + 1)\alpha_2 - 1$ for all $i \in A_3$. The graph is stable if $\tau \geq \max\left[\frac{n-1}{n-4+\alpha_1}, \frac{n-1}{n-1+(\alpha_2-2)(\alpha_1+1)}\right]$. Hence, if $\alpha_1 = 2$, the graph is always instable; if $\alpha_1 = 3$, the graph is stable for $\tau = 1$; if $\alpha_1 \geq 4$, the graph is stable for any $\tau \geq \frac{n-1}{n-4+\alpha_1}$ as soon as $\alpha_2 \geq \frac{3\alpha_1-1}{\alpha_1+1}$.

4.2 Component-restricted spillovers

In this case, the marginal cost of firm i in the graph g writes:

$$c_i(g) = c_0 - \gamma\mu_i(g) - \rho\big[|L\big(C_i(g)\big)| - \mu_i(g)\big].$$

The factors driving the incentives for agent i to form the link ij take the following forms:

($E1$) is equal to $n\big[\gamma + \rho(L(C_j))\big]$ if $C_i \neq C_j$, $n\gamma$ otherwise; that is, agent i captures additional spillovers from all the the links in agent j's component only if agents i and j do not belong the the same component (otherwise agent i already accessed those links though paths of finite length),

($E2$) is written $\gamma + \rho(L(C_i))$ if $C_i \neq C_j$, γ otherwise,

($E3$) is equal to $\rho[(\zeta_i - 1)(1 + L(C_j)) + (\zeta_j - 1)(1 + L(C_i))]$ if $C_i \neq C_j$, $\rho(\zeta_i - 2)$ otherwise. Indeed, in the former case agents in agent i's (resp. agent j's) component receive spillovers from the link ij plus from all the links in agent j's (resp. agent i's) component. In the latter they receive spillovers from the sole link ij.

The following proposition strongly restricts the stable architectures:

Proposition 5 *When spillovers are component-restricted, stable networks are the union of complete components of distinct size. Furthermore, the incentives of link formation between components are possibly non monotonic (decreasing and then increasing) with respect to the size of the components.*

The reason why components are complete is that when two agents in an incomplete component form a link, the conditions of the claim 1 always apply. In an incomplete component, when two firms form an alliance, they do not obtain additional spillovers from the component, since shortening the distances between agents and links has no impact on the amount of spillovers; this is as well for other firms in the component: they receive the sole additional spillover generated by the new link ($(E4)$ writes $\rho(\zeta_i - 2)$). That is, the link formation induces a positive shock on the marginal costs of a subset of firms in the industry, and the two partners receive at least the same amount as other firms.
The fact that components have not an equal size can be derived from proposition 3.4 (i). Basically, in that situation, if two agents i and j in distinct components form an alliance, they receive the same positive shock on marginal cost, and the firms in the respective components receive no greater shock. We note that this property strongly depends on the absence of friction in spillovers' dissemination: integrating friction would change the spillovers' flows originated from other links (than the link ij) in the component, resulting in possibly asymmetric shocks in the cost reduction and eventually destroying the incentives to link formation.

We turn to non monotonicity result. Consider two firms in two distinct complete components, the following two important properties apply:

Property (P1) If the firm in the greatest component has an incentive to form the link, this is also true for the firm in the smallest component.
Indeed, the respective incentives only differ through effects $(E2)$ and $(E3)$. And the former dominates when the size of the component of agent i is smaller than that of firm j.

Property (P2) Suppose that a firm is incited to form a link with a firm in another component of size s. This is also the case when the size of the partners component increases if and only if s exceeds a minimal value.
From (P1) we deduce that, to examine pairwise stability of such a link, it is sufficient to consider the incentive of the agent in the greatest component. Property (P2) stems

from the fact that, considering such an agent, when her potential partner is in a very small component, her incentive to form a link decreases with the size of the partner's component since more firms shall receive spillovers from her own component. On the opposite, when the size of the partner's component is sufficiently large, she can expect receiving a more substantial amount of spillovers from the other component, which finishes to dominate the negative incentive.

Let us examine the stability of graphs containing two distinct components. The figure 2 illustrates the non monotonicity of the result in the case $n = 7$, which is the smallest value of n such that the class g_{n-k} contains three distinct architectures (distinct from the empty and the complete network). Hence, this is the smallest value of n such that we may observe the non monotonicity property. The curve depicts for each network in the class g_{n-k} the critical value τ_k such that the graph g_{n-k} is stable iff $\tau \geq \tau_k$. The figure shows that, for a given value of spillovers intensity, an increase in the size of the smallest component (and thus a decrease in the size of the largest one) decreases and then increases the profitability of link formation of the firm in the greatest component. More precisely, we see that when $\tau \in]0.27, 0.30[$, the networks g_7 and g_6 are not stable, while the network g_5 is stable.

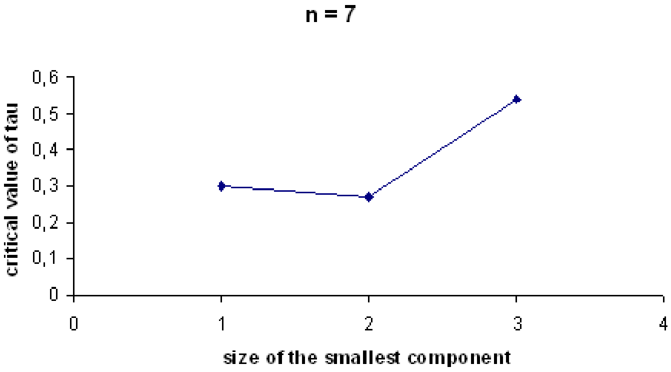


Figure 2: A consequence of non monotonicity; X-axis=k, Y-axis=τ_k

The next proposition sums up the findings. Consider the three following values: $\tau^0 = \frac{8(2n-1)(n-1)}{(n^2-n+1)^2+8(n-2)(2n-1)}$, $\tau^1 = \frac{2(n-1)}{(n-2)(n+1)}$ and $\tau^2 = \frac{8(n-1)}{n^2-6n+32}$. Then, for $n > 5$, $0 < \tau^0 < \tau^1 < \tau^2 \leq 1$ and the following proposition applies:

Proposition 6 *Suppose* $n > 5$. *When* $\tau \in [0, \tau^0[$, $S = \{g^c\}$. *When* $\tau \in [\tau^0, \tau^1[$, $S = \{g_{k-},, g_{k+}, g^c\}$ *with* $\frac{n}{2} < k- < k+ < n-1$. *When* $\tau \in [\tau^1, \tau^2[$, $S = \{g_{k-},, g^c\}$ *with* $k'- < k-$. *When* $\tau \in [\tau^2, 1[$, $S = \{g_q,, g^c\}$, *with* $q = \frac{n+2}{2}$ *if* n *is even,* $q = \frac{n+1}{2}$ *otherwise.*

The reason for non monotonicity lies in the relative intensity of effect $(E2)$: this effect, $n\frac{k(k-1)}{2}$, is quadratic in k. While negligible for small values of k, it becomes prominent for larger values, as amplified by the factor n. Hence, for little size of the smallest component, an increase in that size is not beneficial to the agent in the greatest component (effects $(E3)$ and $(E4)$ are prevailing); when the size of the smallest component is more substantial, the effect $(E2)$ becomes decisive, and an increase in the size of the smallest component reinforces the effect $(E2)$. We note that the value τ^0 obtains for a component size $k^0(n) = \frac{n^2-n+1}{2(2n-1)} \simeq \frac{n}{4}$.

Let us give some examples of stable networks containing more than two complete components: the smallest population size entailing stability of three components is $n = 10$ and the triplet $(1, 4, 5)$ is stable when $\tau = 1$ (each number in the triplet denotes the size of a complete component). To find a stable network with four components we must reach the size $n = 55$, and we find $(1, 10, 19, 25)$ as being stable. To obtain a stable network with five complete components we need to reach a network size around $n = 2000$. For instance, the network $(1, 63, 324, 659, 953)$ is easily seen to be stable (see the end of the appendix A for an illustration of how checking that a network containing complete components is stable).

As a final remark, we would like to stress that two components of size less than $\frac{n}{2}$ may not coalesce (as illustrated by the stability of the three components of size 1, 4 and 5 in the case $n = 10$). This result contrasts with Bloch (1995), who finds in a coalition formation approach that two coalitions of size less than half the population have always an incentive to merge. The reason for the difference is that in Bloch's paper the marginal cost reduction from being in a coalition is proportional to the size of the coalition, whereas in our context, when a link is formed between two distinct components, the amount of spillovers' flow received by members of a component is proportional to the square of the size of the component, since spillovers emanate from the *links* of the component, and not the firms themselves. Hence, the asymmetries in the returns from link formation are stronger than what arises in Bloch's context, so that the maximal component size below which members of the big component find profitable to form a link with agents in a smaller component is smaller. Of course, this phenomenon is restricted to the hypothesis that spillovers spread without decay (except at order 1), otherwise the stable components may not be complete.

5 Conclusion

This paper has analyzed the impact of spillovers on rival firms' incentives to form R&D alliances. Particularly, we have studied the case where spillovers propagate with possible decay through the network of existing alliances, in a environment, favorable to alliance formation, where the cost of link formation is negligible. An original feature of the model is that firms are rewarded according to their 'distance' to links (*i.e.*, the shortest path between them and one agent involved in the link). The major implication of this hypothesis is that the formation of alliances shall entail positive asymmetric shocks on the marginal costs of the partners, as well as other firms. Two main conclusions can be given: first, those asymmetries can be sufficient to deter link formation, despite the cost of link formation is negligible; second, incomplete networks are asymmetric and we provide situations in which a moderate level of asymmetry in the number of partners is likely to emerge. The reason for the latter conclusion is the existence of the following tradeoff: agents form links in order to shorten the paths between them and links, but by doing so they can improve their intermediary position, which provides more externalities to third parties.

Some deficiencies of the study have to be noted. First, due to the lack of tractability, and particularly because in the model the returns of link formation are non monotonic with respect to topological variables, it has not been possible to integrate large costs of link formation. Second, in terms of results, the price for generalizing from linear case to moderate competition is high. Third, our model does not relate the individual returns of link formation to investments spent by the partners. Last but not least, one may make knowledge transfers endogenous, by considering firms' abilities to control spillover flows, *i.e.* by considering firms not only as bridge but also as gatekeepers.

References

Bloch, F., 1995, Endogenous structures of association in oligopolies, *Rand Journal of Economics*, **26**, 537-556.

Breschi, S. and F. Lissoni, 2005, "Cross-firm inventors and social networks: localised knowledge spillovers revisited, forthcoming in *Annales d'Economie et Statistiques.*

Deroian, F., 2004, A Note on cost-reducing alliances in vertically differentiated oligopoly, *Economics Bulletin*, **12**, 1-6.

Goyal, S. and S. Joshi, 2003, Networks of collaboration in oligopoly, *Games and Economic Behavior*, **43**, 57-85.

Goyal, S. and S. Joshi, 2004, Erratum to "Networks of collaboration in oligopoly" [Games Econ. Behav. 43 (1) (2003) 57-85], *Games and Economic Behavior*, **46**, 219.

Goyal, S. and S. Joshi, 2005, Unequal connections, mimeo Erasmus University.

Goyal, S. and Vega-Redondo, 2004, Structural holes in social networks, mimeo.

Jackson, M. and A. Wolinsky, 1996, A Strategic model of economic and social networks, *Journal of Economic Theory*, **71**, 44-74.

Jaffe, A., M. Trajtenberg and R. Henderson, 1993, Geographic localization of spillovers as evidenced by patent citations, *Quaterly Journal of Economics*, **108**, 577-598.

Singh, J., 2005, Collaborative networks as determinants of knowledge diffusion patterns, *Management Science*, **51**, 756-770.

Yi, S., 1998, Endogenous formation of joint ventures with efficiency gains, *Rand Journal of Economics*, **29**, 610-631.

6 Appendix: Proofs

Proof of proposition 3.1. Denote by ρ the amount of spillovers given to third parties. In the linear Cournot setting, we check it easily by remarking that in the case spillovers are global,

$$q_i(g+ij) - q_i(g) = \frac{(n-1)\gamma_0 - (n-2)\rho}{n+1}$$

which is a positive quantity. Further, the result established in the more general setting is a direct implication of the claim 2. □

Proof of proposition 3.2. The proof on stability is omitted as directly resulting from Yi (1998). For efficiency issue, the proof is a direct application of Goyal and Joshi (2003, 2004). Indeed, $c_i(g) > c_i(g^c)$ whenever $g \neq g^c$. □

Proof of proposition 3.3. The first part of the proposition is straightforward. Note that we can compute quantity variations only on symmetric networks. Since the network consisting of the complete network less one link is not symmetric, we cannot discuss stability of the complete network in great generality.

First, there is a unique component otherwise two agents in distinct components would find more spillovers than third parties when forming an alliance.

Second, consider an asymmetric network. Note that if two firms in two distinct components form a link, the costs of the sole firms in those components may be affected. In a component, suppose that all costs benefit from a symmetric positive shock. Rank therefore the firms by decreasing value of profit variation. For sure, firm 1 (typically one having initially the smallest marginal cost) receives a positive profit variation from the shock: indeed, either they all costs are symmetric and we are done, or not and the firms with smaller cost benefit more. Now, if she forms a link with a symmetric firm in the other component, both extract the same amount of spillovers and this amount is not smaller than that of third parties. Hence, they are at least as well as if all firms of the two components receive a symmetric positive shock.

Third, suppose that the function $\gamma(.)$ is decreasing and convex. Consider two symmetric agents i and j belonging to the same component. Then the convexity of the function $\gamma(.)$ is a sufficient condition for guaranteeing that the amount of spillovers captured by two symmetric agents who form an alliance exceeds that of third parties. Indeed, in that case, suppose that agent i captures additional spillovers from a link kl when the link ij is formed, *i.e.*, $f(i,kl;g+ij) < f(i,kl;g)$. Now, whatever agent q in the component, we will see that $f(q,kl;g) - f(q,kl;g+ij) \leq f(i,kl;g) - f(i,kl;g+ij)$. Indeed, if the distance from agent q to link kl is shortened by the addition of the link ij, then agent i belongs to the (unique) shortest path in the graph $g+ij$, so that we have both $f(q,kl;g+ij) \geq f(i,kl;g+ij)$ and $f(q,kl;g) > f(i,kl;g)$. Two cases may arise: either agent i lies on one shortest path from agent q to the link kl in the graph g, in which case $f(q,kl;g) - f(q,kl;g+ij) = f(i,kl;g) - f(i,kl;g+ij)$ and we are done, or not and the following obtains: denote by a (resp. b) the shortest path from agent m (resp. i) to the link kl in the graph g and denote by c the shortest path from agent m to agent i in the graph g. Finally, denote by d the distance from the agent i to the link kl in the graph $g+ij$. Then $a < c+b$ means that in the graph g agent i does not lies on the shortest path between agent m and the link kl.

Suppose now that $b - d < a - (c + d)$, *i.e.*, the variation of distance from agent m to the link kl exceeds that from agent i to link kl in the graph $g + ij$. Then we have $b < a - c$, which contradicts the preceding inequality. Hence, as the function $\gamma(.)$ is decreasing and convex, $\gamma(f(i, kl; g + ij)) - \gamma(f(i, kl; g)) \geq \gamma(f(q, kl; g + ij)) - \gamma(f(q, kl; g))$. This means that the amount of spillovers received by third parties when the link ij is formed is not greater than that captured by both agent i and agent j. Consequently, starting from a symmetric network, the claim 2 applies. ■

Proof of proposition 4.1.

Profitability of link formation: in homogenous Cournot oligopoly with linear demand, equilibrium quantity of firm i writes $q_i = \frac{\alpha - nc_i + \sum_{j \neq i} c_j}{n+1}$. Consider a non complete graph g and one link $ij \notin g$. Then, the equilibrium quantity in the graph $g + ij$ writes:

$$q_i(g{+}ij) = \frac{\alpha - n(c_i - \gamma - \rho(\mu_j - v_{ij})) + c_j - \gamma - \rho(\mu_i - v_{ij}) + \sum_{k \neq i,j} c_k(g + ij) - \rho(\mu_i + \mu_j - v_{ij})}{n+1}$$

(with $c_i = c_i(g), c_j = c_j(g), \mu_i = \mu_i(g), \mu_j = \mu_j(g), v_{ij} = v_{ij}(g)$), so that one obtains

$$(n+1)\Delta q_i^{ij}(g) = (n-1)\gamma + \rho\Big[n(\mu_j - v_{ij}) - (\mu_i - v_{ij}) - (\mu_i + \mu_j - v_{ij})\Big].$$

Hence,

$$\Delta q_i^{ij}(g) > 0 \text{ iff } \mu_i < \frac{n-1}{2\tau} + \frac{n-2}{2}(\mu_j - v_{ij}) + \frac{\mu_j}{2}.$$

Given that profit variation follows quantity variation in linear Cournot competition and recalling that the cost of link formation is negligible, the link is formed if the above inequality applies to both firms i and j (substituting labels i and j in this above inequality). Denote $f_\tau(\mu_j, v_{ij}) = \frac{n-1}{2\tau} + \frac{n-2}{2}(\mu_j - v_{ij}) + \frac{\mu_j}{2}$. We note that f_τ is decreasing *w.r.t.* both parameter τ and argument v_{ij}. Hence, the condition under which the link formation is profitable is more restrictive when τ and v_{ij} attain their maximum value.
Thus, the stability of the network g^{n-1} is checked by direct inspection.

Lemma 4 *A stable network contains either one component or two components with one being an isolated firm. The greatest component has diameter* 2.

Proof. Suppose that a stable network g contains two components. Consider two firms i and j in each. Then $v_{ij} = 0$. The link ij is profitable for firm i if $\mu_i < \frac{n-1}{2}\left(\frac{1}{\tau} + \mu_j\right)$. Note

that if $\mu_j \geq 1$ the condition is automatically satisfied. Then a stable network with two components contains at most one isolated firm. To finish, check that two distinct isolated firms have an incentive to form a link. $\square$
The lemma strongly restricts the set of stable architectures. Second, we focus on stable networks with incomplete components and we remark two properties:

Lemma 5 *Suppose that a stable graph g contains one incomplete component C. Then for every pair of firms (i, j) in the component C such that the link $ij \notin g$, $\mu_i \neq \mu_j$.*

Proof. Suppose that $\mu_i(g) = \mu_j(g) = \mu(g)$. With respect to equation (1), the specification of the function $\gamma(.)$ induces

$$(n+1)\Delta q_i^{ij}(g) = (n-1)\gamma - \mu\rho + (n-3)(\mu(g) - v_{ij}(g))\rho$$

which is a positive quantity, implying that the link ij is profitable to both partners. $\square$

Lemma 6 *If component contains two agents i and k with $ik \notin g$ and $\mu_k > \mu_i$, then $v_{ik} = \mu_i$.*

Proof. With respect to equation (1), the specification of the function $\gamma(.)$ induces

$$(n+1)\Delta q_k^{ik}(g) = (n-1)\gamma + \rho\Big[(n-1)\mu_i - 2\mu_k - (n-2)v_{ik}\Big].$$

entailing that

$$\Delta q_k^{ik}(g) < 0 \text{ iff } \mu_k > \frac{n-1}{2\tau} + \frac{n-2}{2}(\mu_i - v_{ik}) + \frac{\mu_i}{2}.$$

Thus, if $\mu_i > v_{ik}$, the inequality is not compatible with $\mu_k \leq n-2$. $\square$

Lemma 7 *In a stable network, no incomplete component contains an agent linked to all the other agents of the component.*

Proof. Consider an incomplete component C. Denote by $\mu_{min} = \min_{i \in C} \mu_i$. We need both $\tau < \frac{|L(C)|-1}{2(|L(C)|-2)-(\mu_{min}-1)}$ and $\tau \geq \max_{\substack{(j,k),\\ \mu_j < \mu_k,\\ jk \notin g}} \frac{|L(C)|-1}{2\mu_k - \mu_j}$, which is impossible. $\square$ ■

Proof of proposition 4.2.

In a stable graph, components are complete: for every triplet of firms i, j, k in a component such that $ij \notin g$, $\sum_{p=1}^{d(C_i)} \Delta F_k^{ij}(p, g) = 0$ (firm k may not be distinct from firm i). Hence, recalling that in this specification $\gamma(p) = \rho$ for any finite integer $p \geq 1$, we deduce from equation (1) that the link is profitable.

Profitability of link formation between two distinct complete components:
Consider a network g containing two distinct complete components, and two firms i and j taken from two distinct components. Let us denote the size of (resp. the number of links in) firm i's component as ζ_i (resp. L_i). We compute the equilibrium quantity variation of firm i when the alliance ij is formed as follows:

$$\Delta q_i^{ij}(g) = \frac{n(\gamma + \rho L_j) - (\gamma + \rho L_i) - (\zeta_i - 1)\rho(L_j + 1) - (\zeta_j - 1)\rho(L_i + 1)}{n+1},$$

so that

$$\Delta q_i^{ij}(g) > 0 \text{ iff } \frac{n-1}{\tau} + L_j(n - \zeta_i + 1) - L_i\zeta_j - \zeta_i - \zeta_j + 2 > 0$$

Replacing L by $\frac{\zeta(\zeta-1)}{2}$, one obtains after rearrangement:

$$\Delta q_i^{ij}(g) > 0 \text{ iff } \zeta_i^2 + \left(\frac{2}{\zeta_j} + \zeta_j - 2\right)\zeta_i - \left[\frac{2(n-1)}{\tau\zeta_j} + (n+1)(\zeta_j - 1) - 2\frac{\zeta_j - 2}{\zeta_j}\right] < 0$$

This order-2 polynomial admits two roots of opposite sign. Hence, it is profitable for both firms i and j to form an alliance with each other whenever $\zeta_i < f(\zeta_j)$ and $\zeta_j < f(\zeta_i)$, with

$$f_\tau(x) = 1 - \frac{x}{2} - \frac{1}{x} + \frac{1}{2}\sqrt{\left(x - 2 + \frac{2}{x}\right)^2 + 4(n+1)(x-1) + \frac{8(n-1)}{\tau x} - \frac{8(x-2)}{x}}$$

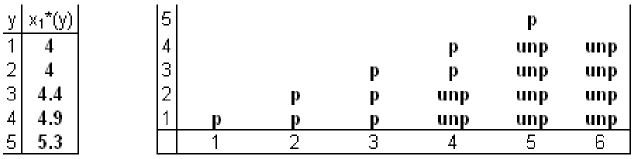
In a stable graph, two components cannot have equal size:
We know from the above analysis that in stable networks components are complete. From proposition 3.4 (i) we deduce the point.

non monotonicity of incentives to link formation with respect to the size of the components:
Let us define the function $h_\tau(x, y) = \frac{2(n-1)}{\tau} + (n+1)y(y-1) - (xy+2)(x+y-2)$, with $x \in \{1, \cdots, n-1\}$ and $y \in \{1, \cdots, n-x\}$. Then $h_\tau(\zeta_i, \zeta_j) = (n+1)\Delta q_i^{ij}(g)$. For any $\tau \in [0, 1]$, note first that if $1 \leq y < x$, $h_\tau(x, y) < h_\tau(y, x)$: then, if the firm in the greatest complete component has an incentive to form a link with the other, the

latter has also an incentive. Therefore, to examine pairwise stability of such a link, it is sufficient to consider the incentive of the agent in the greatest component. Second $\forall x > 0$, $h_\tau(x+1, y) < h_\tau(x, y)$: then, if a firm has an incentive to form a link with some firm in another component, this would also be the case if the size of the component of the former was smaller (and the size of the later constant). Third $h_\tau(x, y+1) - h_\tau(x, y) > 0$ iff $y > \frac{x^2-x+2}{2(n+1-x)}$: then, if some agent finds profitable to form a link with some agent in another component, this would also be the case if the size of the partner's component was greater iff that latter size exceeds a minimal value. ■

Proof of proposition 4.3. The values are easily computed: recalling that τ_k is such that $\Delta q_i^{ij}(g) = 0$ if agent i is in a component of size $n-k$ and agent j in a component of size $k < n-k$, we obtain $\tau_k = \frac{2(n-1)}{k^2(1-2n)+k(n^2-n+1)+2(n-1)}$. Then $\tau^1 = \tau_1$ and $\tau^2 = \tau_q$ with $q = \frac{n+2}{2}$ if n is even, $q = \frac{n+1}{2}$ otherwise. To finish, τ^0 is the minimal value of the set $\{\tau_k\}$, with $k = 1, ..., \frac{n-2}{2}$ if n is even and $k = 1, ..., \frac{n-1}{2}$ otherwise. This value is attained for a component size $k^0 = \frac{n^2-n+1}{2(2n-1)}$, and we find $\tau^0 = \frac{8(2n-1)(n-1))}{(n^2-n+1)^2+8(n-2)(2n-1)}$. ■

We give a simple illustration of how checking that a network composed of complete components is or is not stable, in the case $n = 10$ and $\tau = 1$ (this is the minimum network size generating a stable union of three complete components):



y	x₁*(y)
1	4
2	4
3	4.4
4	4.9
5	5.3

5					p	
4				p	unp	unp
3			p	p	unp	unp
2		p	p	unp	unp	unp
1	p	p	p	unp	unp	unp
	1	2	3	4	5	6

Table 1: $n = 10$, $\tau = 1$

The left-hand table represents, for two components of size y and x, $y \leq x$, the maximum size $x_1^*(y)$ below which two firms belonging to the respective components find profitable to form a link. The condition examines the link formation profitability of the firm in the greatest component -if profitable, this is also the case for her partner-; for instance the line $y = 3, x_1^*(3) = 4.4$ should be interpreted as 'a firm in a component of size 4 (resp. 5 or more) finds profitable (resp. not profitable) to form a link with some firm in a component of size 3'. This makes possible to determine which potential link would be profitable for both parties, as summarized in the right-hand table: coordinates represent component sizes, with the convention that the size x is on the X-axis and the size y ($\leq x$) on the Y-axis; when the link is profitable (resp. not profitable) to both partners, the symbol 'p' (resp. 'unp') is used. We check that $S = \{(1, 4, 5), g_6, g_7, g_8, g_9, g^c\}$, where $(1, 4, 5)$ denotes the network composed of three complete components of size 1, 4 and 5.