A strategic model of club formation; existence and characterization of equilibrium

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Abstract. We introduce a new model of a club economy as a two stage game. Players derive utility from consumption of private good, consumption of public good, and the profile of crowding characteristics –those characteristics of a player that directly affect other players – of members of the same club. In the first stage of the game, players choose amounts to consume of an endowment of private good. The crowding characteristics acquired by a player are determined by this choice of consumption level, as is the amount of private good 'left over' to contribute to the production of the club good in the second stage of the game. In the second stage of the game, given the profile of crowding characteristics of the total player set, club memberships are endogenously determined by competition, modelled as a price-taking Tiebout equilibrium. We establish conditions for the existence of a subgame perfect equilibrium and some characterization results on the similarity of players within clubs.

1 Introduction

The essential idea underlying models of Tiebout economies or, in other words, economies with multiple clubs is that the benefits of forming large clubs or jurisdictions are eventually offset or almost offset by negative externalities due to congestion or other problems associated with the organization of large groups of players. Conley and Wooders (1996,1997) introduced the idea of separating the crowding effects of a player – those characteristics of a player that directly affect other individuals in the same club – from other characteristics, such as tastes, which are presumed to have no direct effects on other members of the same club. In Conley and Wooders (1997) the crowding type of each individual player is exogenously given and decentralization of the core with anonymous prices, that is, prices depending solely on crowding characteristics, is demonstrated. In Conley and Wooders (1996, 2001) individuals pay costs for the acquisition of crowding types, such as educational level. Thus, the authors can show that in addition to the more traditional roles of generating efficient provision of club or public goods and allocating players over jurisdictions, anonymous competitive prices can also be used to induce players to make optimal educational investment and labor market choices.

In this model, we assume that players acquire crowding characteristics through consumption of a private good. Following Conley and Wooders (2001) we also consider situations where players may have different genetic abilities so that the same pattern of consumption by two different players does not necessarily lead to the two players acquiring the same crowding characteristics. The main point of departure of our approach is the manner in which crowding characteristic arise endogenously in the model. To be precise, in Conley and Wooders (2001) players essentially 'buy' their crowding types from a set of crowding characteristics. In our model the crowding characteristics of a player are a by-product of his private consumption. Therefore, we accommodate not only external effects that are strategic choices make in response to market signals but we also accommodate external effects that arise indirectly. In a sense, our framework simultaneously extends the models of Conley and Wooders (1996,1997,2001). The relation between private consumption and crowding type is given by a function that maps from private consumption into crowding types and depends on the unobservable genetic type of the player. In other words, when a player decide his private consumption, he makes choices that could affect his crowding characteristics. For example, a player may like to dance. Dances are a club good and clubs with money to spend (private good) can afford to hire great dance bands while poorer clubs can only afford disc jockeys or perhaps only radios. Diversity of tastes for public goods and also negative aspects of 'huge' dance clubs (think of a huge meeting of economists) may lead to a preference for dance clubs of moderate size. A player can spend his money on private consumption (dance classes, for example) that he may enjoy and may influence his desirability as a salsa club member. Or he could spend less on dance classes and have more money to contribute to hiring a dance band. Alternatively, a player may spend income on acquiring business skills or retain income to go into a business partnership.

To study our context we explore a strategic approach of the model, that is, we assume that players behave strategically. We design a Nash game with two stages. In the first stage of the game, players choose private consumption levels and consequently their crowding types. In the second stage, with the crowding profile of the economy already fixed, the players choose their club memberships. Since in the second stage of the game the strategy set is finite we consider mixed strategies in order to obtain a solution for the game, that is, we suppose that players choose lotteries - a probability distribution - over jurisdictions. First we illustrate this conception with an example and then we show that the game has a subgame perfect equilibrium. We note that our use of lotteries was motivated by Garratt and Qin (1996).

In the context of choice of crowding type, a two-stage game is more natural than the static frameworks of Conley and Wooders, especially when crowding types are endogenous. For many crowding types, it is easy to imagine that choice or formation of crowding characteristics occurs before individuals join jurisdictions, clubs or firms. Education is a prime example.

Another important issue in this literature is how players sort by tastes across clubs. Indeed, if taste-homogenous clubs are optimal that means that clubs where players have the same tastes are more efficient for the provision of public good. Taste-homogenous clubs are optimal when crowding is anonymous but not necessarily optimal when crowding is differentiated, namely if crowding types are endogenously given and the players have the same genetic abilities. However, if differentiated crowding types are exogenous given or are endogenously given but players have differential genetic endowments or preferences over the crowding type, in general taste-homogenous jurisdiction are not optimal, [see Conley and Wooders 2001, table 1]. In Conley and Wooders (2001) it is proved that a strong form of small group effectiveness implies weak essentially taste homogeneity, that is, it is possible to homogenize players with the same crowding type by taste in a core club. In our context we show that this is true only if we add a restriction. This restriction could be that some critical levels of consumption are required to reach a specific crowding type or that more expensive crowding types are more preferred.

Finally, we note that this version of our paper is preliminary. In addition, we have not included a complete set of references. We note, however, that our model is related to several papers in Demange and Wooders (2005), in particular, the contributions of Demange (2005), Conley and Smith (2005), Le Breton and Weber (2005) and Jaramillo, Kempf and Moizeau (2005) and a number of references in these works.

2 The Model

2.1 Formal Elements

We consider an economy with one private good and I players indexed by $i \in \{1, ..., I\} \equiv \mathcal{I}$.

There are T different types of preferences, indexed by $t \in \{1, ..., T\} \equiv \mathcal{T}$. A mapping $\tau : \mathcal{I} \rightarrow \mathcal{T}$ ascribes a taste type to each player in the economy.

Each player in the economy acquires a crowding characteristic, $c \in \{1, ..., C\} = \mathcal{C}$ as a result of his private consumption. However the effect of private consumption on crowding profile depends on characteristics inherent to the player, which we will call the player's genetic type. Accordingly, we assume that each player is also endowed with a genetic type, $g \in \{1, ..., G\} = \mathcal{G}$. Let $\gamma : \mathcal{I} \rightarrow \mathcal{G}$ be a function assigning each player $i \in \mathcal{I}$ a genetic type, that is, $\gamma(i) = g$ for some $g \in \mathcal{G}$. Thus, a player with genetic type g is characterized by a function that associates to each level of his private consumption a crowding characteristic.

$$\mathbf{c}_g : X \rightarrow \mathcal{C},$$

A main feature of a Tiebout economy is that it is optimal or near-optimal to players to be separated into relatively small clubs/jurisdictions in order to consume public goods within their jurisdictions of residence. Let $\mathcal{S} = \{s_1, ..., s_J\}$ be the set of potential jurisdictions and let $\mathcal{Y}$ be the set of public projects. Each club s_i produce a public project $y \in \mathcal{Y}$.

As usual in differentiated crowding models we assume that the utility of a player is affected (positively or negatively) by the profile of crowding characteristics of the players living in his jurisdiction. Denote a profile of crowding characteristics by

$$n = (n_1, ..., n_C) \in \mathbb{Z}^C,$$

where $\mathbb{Z}$ is the set of non-negative integers and n_c is the number of players who has crowding type c. For any given vector of private consumption $x = (x_1, ..., x_I)$, we define an assignment

$$A(x) : \mathcal{I} \rightarrow C$$

which assigns a crowding type to each player, $A(x)(i) = \mathbf{c}_{\gamma(i)}(x_i)$.

A taste type t is described by an endowment of private good $w_t \in R^+$ and a preference relation $\succeq_t$ defined over $R \times \mathcal{Y} \times \mathcal{C} \times Z_+^C$. Let us denote a consumption bundle as $(x, y, c(x), n)$, where x is the level of private good, y is a public project, $c(x)$ is the corresponding crowding type and n is the crowding profile of the jurisdiction in which the player resides.

2.2 Monotonicity:

Crowding affects production. The production technology, commonly available to all, is given by the cost function

$$f : \mathcal{Y} \times Z_+^C \rightarrow R,$$

where $f(y, CP(A, s))$ is the cost in terms of private good of carrying out a public project for jurisdiction s under assignment A.

A *feasible state of the economy*, $(X, Y, A(X), S)$, is an allocation of private good for each player, $X = (x_1, ..., x_I)$, a public project for each jurisdiction, $Y = (y^1, ..., y^K)$, an assignment $A(X)$ of players to crowding types and a partition S of the population such that for all $i \in \mathcal{I}$,

$$\sum_{i \in \mathcal{I}} (\omega_i - x_i) - \sum_k f\left(y^k, CP\left(A(X), s^k\right)\right) \geq 0.$$

Denote the set of feasible states by F. The pair $(\overline{x}, \overline{y})$ is a feasible allocation for a club $\overline{s}$ under assignment $A(\overline{x})$ if, for all $i \in \overline{s}$,

$$\sum_{i \in \overline{s}} (\omega_i - \overline{x}_i) - f\left(\overline{y}, CP\left(A(\overline{x}), \overline{s}\right)\right) \geq 0.$$

A club $\overline{s} \in \mathcal{S}$ producing a feasible allocation $(\overline{x}, \overline{y})$ under assignment $A(\overline{x})$ can *improve upon* a feasible state $(X, Y, A(X), S) \in F$ if, for all $i \in \overline{s}$,

$$\left(\overline{x}_i, \overline{y}, \mathbf{c}_{\gamma(i)}(\overline{x}_i), CP(A(\overline{x}), \overline{s})\right) \succeq_{\tau(i)} \left(x_i^k, y^k, \mathbf{c}_{\gamma(i)}\left(x_i^k\right), CP\left(A(x), s^k\right)\right),$$

where $i \in s^k \in S$ in the original feasible state, and for some $j \in \overline{s}$ it holds that

$$\left(\overline{x}_j, \overline{y}, \mathbf{c}_{\gamma(j)}(x_j), CP(A(\overline{x}), \overline{s})\right) \succ_{\tau(j)} \left(x_j^k, y^k, \mathbf{c}_{\gamma(j)}\left(x_j^k\right), CP\left(A(x), s^k\right)\right),$$

where $j \in s^k \in S$ in the original feasible state. A feasible state $(X, Y, A(X), S) \in F$ is in the ***core*** of the economy if it cannot be improved upon by any coalition.

2.3 Equal Treatment and Strict Small Group Effectiveness

We consider now economies in which gains to coalition size are limited. Formally, an economy is said to satisfy ***strict small group effectiveness, (SSGE)***, if there exists a positive integer $B \in Z^+$ such that:

1. For all core states $(X, Y, A(X), S)$ and for all $s^k \in S$ it holds that $\left|s^k\right| \leq B$.

2. If a feasible state $(X, Y, A(X), S)$ can be improved upon, there exists a coalition $\overline{s} \in \mathcal{S}$ such that $|\overline{s}| \leq B$ which can also improve upon $(X, Y, A(X), S)$.

3. For all $t \in \mathcal{T}$, and $g \in \mathcal{G}$, it holds that either $|\{i \in \mathcal{I} : \tau(i) = t,\ \gamma(i) = g\}| > B$ or $|\{i \in \mathcal{I} : \tau(i) = t,\ \gamma(i) = g\}| = 0$.

SSGE implies that players of a given type are equally treated in the core. We note that the above form of SSGE is made for convenience.

The following result is demonstrated in Conley and Wooders (1997) for their model and is crucial to decentralizability of the core by prices that depend only on the crowding characteristics of players. In brief, the result states that players in clubs with the same crowding profile and the same public projects must make the same implicit contribution to public good production. We state the result and then address whether it holds in the context of our model.

Theorem. (Conley and Wooders 1997) Let $(X, Y, A(X), S)$ be a core state of an economy satisfying SSGE. For any two individuals $i, \hat{\imath} \in \mathcal{I}$ such that $\tau(i) = \tau(\hat{\imath}) = t$ and $\gamma(i) = \gamma(\hat{\imath}) = g$, if $i \in s^k$, and $\hat{\imath} \in s^{\widehat{k}}$ then

$$\left(x_i, y^k, \mathbf{c}_g(x_i), CP\left(A(X), s^k\right)\right) \sim_t \left(x_{\hat{\imath}}, y^k, \mathbf{c}_g(x_{\hat{\imath}}), CP\left(A(X), s^{\widehat{k}}\right)\right)$$

To prove this result we have to make some additional assumptions. We exhibit two different proofs that rely on different assumptions.

Assumption 1: *For all genetic types g the function $\mathbf{c_g} : X \to C$ is a step function and for every $a \in X$, $\lim_{x \to a^+} c_{\mathbf{g}}(x) = c_{\mathbf{g}}(a)$, that is the function $c_{\mathbf{g}}(\cdot)$ is right continuous.*

Assumption 1 can be motivated by the idea that sometimes certain critical levels of consumption are required to reach a specific crowding type. For example, for some universities one must be in residence for a period of a minimal number of years to obtain a degree.

Assumption 2:

(*a*) *If $x_1 > x_2$ then every player type of type t prefers crowding type $c_g(x_1)$ to $c_g(x_2)$ for every genetic type g.*

(*b*) *If in a jurisdiction we replace a player with crowding type $c_g(x_2)$ by a player with crowding type $c_g(x_1)$ (with $x_1 > x_2$), this new crowding profile of the jurisdiction still able to produce the same public good and the utility of the players that stay in the jurisdiction does not decrease with this new jurisdiction crowding profile.*

In other words, Assumption 2 states that more expensive crowding types are more preferred (a sort of Veblen effect for crowding types) and a sort of free disposal of crowding type level– players who consume more of the private input that determines crowding type are at least as desirable and capable as those who consume less of the private input.

Theorem. Let $(X, Y, A(X), S)$ be a core state of an economy satisfying SSGE, and satisfying Assumption 1 or Assumption 2 and let s^k, $s^{\widehat{k}} \in \mathcal{S}$ be a pair of jurisdictions in the core partition such that $y^k = y^{\widehat{k}}$, and $CP\left(A(X), s^k\right) = CP\left(A\left(\widehat{X}\right), s^{\widehat{k}}\right)$. Then for any crowding type $c \in \mathcal{C}$, and any pair of players $i \in s^k$ and $\hat{\imath} \in s^{\widehat{k}}$ such that $\mathbf{c}_{\gamma(i)}(x_i) = \mathbf{c}_{\gamma(\hat{\imath})}(x_{\hat{\imath}}) = c$, it holds that

$$w_{\tau(i)} - x_i = w_{\tau(\hat{\imath})} - x_{\hat{\imath}}.$$

The proofs are given in the Appendix, the first proof uses Assumption 1 and the second proof uses Assumption 2.

Remark: Assumption 1 implies that for every $x \in X$, if $\mathbf{c}_{\mathbf{g}}(x) = c$, there is $\varepsilon > 0$ such that, $\mathbf{c}_{\mathbf{g}}(x + \varepsilon) = c$. Assumption 2 is not appropriate if we consider a model with only two crowding types, where a player could be smoker or non-smoker, for example. In this context it is acceptable that if a player increases own private consumption, initially he could be a non-smoker and then become a smoker. Providing that being a smoker is not a valuable characteristic for other players, increasing consumption of cigarettes does not make the player more attractive to other players. Furthermore, a jurisdiction crowding profile resulting from replacing a non-smoker by a smoker does not leave the other players in the jurisdiction indifferent or better off, the utility of the other players may decrease.

Let $\theta_c(A, s) \equiv \{t \in T : \exists i \in s \text{ such that } \tau(i) = t \text{ and } A(i) = c\}$.
Weak Essentially Taste Homogeneity (WET): Consider any $c \in \mathcal{C}$ and any jurisdiction in the core partition $s^k \in \mathcal{S}$ such that $\left|\theta_c\left(A, s^k\right)\right| > 1$. Consider any player $i \in s^k$ such that $A(i) = c$, and suppose that $\tau(i) = t$ and $\gamma(i) = g$. Then there exist a jurisdiction $\overline{s} \in \mathcal{S}$ and an allocation $(\overline{x}, \overline{y})$ which is feasible for $\overline{s}$ under with the crowding profile $A(\overline{x})$ such that $\theta_c(A(\overline{x}), \overline{s}) = \{(t, g)\}$ and for all $j \in \overline{s}$ it holds that

$$\left(\overline{x}_j, \overline{y}, \mathbf{c}_{\gamma(i)}(\overline{x}_j), CP(A(\overline{x}), \overline{s})\right) \succeq_{\tau(j)} \left(x_j^{\widehat{k}}, y^{\widehat{k}}, \mathbf{c}_{\gamma(j)}\left(x_j^{\widehat{k}}\right), CP\left(A\left(x^{\widehat{k}}\right), s^{\widehat{k}}\right)\right)$$

where $j \in s^{\widehat{k}} \in \mathcal{S}$ in the core state.

In other words, a core state is WET if for any $c \in \mathcal{C}$ and any jurisdiction s^k in the core partition which contains at least one player of taste type t and crowding type c it is possible to form a new jurisdiction in which all players of that crowding type c are of type t and which leaves all of its members at least as well off as they were in the core state.

Theorem. If the economy satisfies SSGE, the core of the economy satisfies WET.

Example: This example illustrates an economy where Assumption 2 is not valid and suggests that weak essentially taste homogeneity could not be valid in an economy where the crowding profile is determined by private consumption.

Crowding types: {smoker, no smoker }=$\{S, NS\}$.

Genetic types: $\{g_1, g_2\}$ with

$$\mathbf{c}_{g_1}(x) = \begin{cases} NS & \text{if } \ x \le \underline{x} \\ S & \text{if } \ x > \underline{x} \end{cases} \quad , \quad \mathbf{c}_{g_2} = \begin{cases} NS & \text{if } \ x \le \overline{x} \\ S & \text{if } \ x > \overline{x} \end{cases} \quad \text{and } \underline{x} < \overline{x}.$$

Preference types: {Indifferent, Almost Hater}=$\{I, AH\}$

The utility received by a player who lives in a jurisdiction with two players:

$$\begin{array}{lcl} U_I(NS; NS) & = & 5 \\ U_I(NS; S) & = & 5 \\ U_I(S; NS) & = & 5 \\ U_I(S; S) & = & 5 \end{array} \qquad \begin{array}{lcl} U_{AH}(NS; NS) & = & 10 \\ U_{AH}(NS; S) & = & 5 \\ U_{AH}(S; NS) & = & 0 \\ U_{AH}(S; S) & = & 0 \end{array}$$

The first coordinate in the utility denotes the crowding type of the player of who we are evaluating the utility and the second coordinate denotes the crowding type of the other player in the jurisdiction.

Let there be 100 players of each of all four possible taste and genetic types: Ig_1, Ig_2, AHg_1 and AHg_2.

In a core jurisdiction with two types, both with the same crowding type but different taste and genetic types, for example: (NS_{Ig_1}, NS_{AHg_2}) it is not necessarily true that the players make the same contribution to the public good. Let us denote by i' and i'', respectively the players in this jurisdiction.

It is possible that the player i' is making a smaller contribution to public good provision than i'', that is, $w_{i'} - x_{i'} < w_{i''} - x_{i''}$. In fact, we can not improve the core allocation if $x_{i''} = \overline{x}$. If we replace player i' by a player $\hat{\imath}$ of type AHg_2,giving him $x_{\hat{\imath}}$ such that $w_{i'} - x_{i'} \le w_{\hat{\imath}} - x_{\hat{\imath}} < w_{i''} - x_{i''}$, then $x_{\hat{\imath}} > x_{i''}$, the private consumption is above $\overline{x}$ and the player became a smoker which worsen the utility of the player i''.

2.4 Tiebout Equilibrium

A Tiebout price system for crowding type c is a mapping

$$\rho_c : \mathcal{Y} \times N_c \to R,$$

where $\rho_c(y, n)$ is the price that a player who is crowding type c would have to pay to join a jurisdiction producing public good levels y and having a crowding profile n. The price system is anonymous in the sense that depends only on the observable characteristics of players (crowding types) and not on unobservable characteristics (tastes and endowments). The set of price systems is denoted by P.

A **Tiebout admission price system** denoted by ρ is simply a collection of such price systems (one for each crowding type).

Definition. A **Tiebout equilibrium** is a feasible state $(X, Y, A(X), S) \in F$ and a price system $\rho \in P$ such that:

1. For all $s^k \in S$, all individuals $i \in s^k$, all alternative crowding profiles $\overline{n} \in Z_+^C$, all alternative consume choice $\overline{x}_i$ with $c = \mathbf{c}_{\gamma(i)}(\overline{x}_i)$ such that $\overline{n}_c > 0$, and for all alternative public projects $\overline{y} \in \mathcal{Y}$,

$$\left(w_{\tau(i)} - \rho_{A(x)(i)}\left(y^k, CP\left(A(X), s^k\right)\right), y^k, \mathbf{c}_{\gamma(i)}(x_i), CP\left(A(X), s^k\right)\right) \succeq_{\tau(i)}$$
$$\left(w_{\tau(i)} - \rho_{A(\overline{x})(i)}\left(\overline{y}, CP\left(A\left(\overline{X}\right), \overline{s}\right)\right) \overline{x}_i, \overline{y}, \mathbf{c}_{\gamma(i)}(\overline{x}_i), \overline{n}\right),$$

2. For all potential jurisdictional crowding profiles $\overline{n} \in Z_+^C$ and public projects $\overline{y} \in \mathcal{Y}$,

$$\sum_{\{c \in \mathcal{C} : \overline{n}_c > 0\}} \overline{n}_c \rho_c(\overline{y}, \overline{n}) - f(\overline{y}, \overline{n}) \leq 0.$$

3. For all $s^k \in S$,

$$\sum_{i \in s^k} \rho_{A(x)(i)}\left(y^k, CP\left(A(X), s^k\right)\right) - f\left(y^k, CP\left(A(X), s^k\right)\right) = 0$$

Theorem. If a feasible state $(X, Y, A(X), S)$ and a price system ρ is a Tiebout equilibrium then $(X, Y, A(X), S)$ is in the core.

3 A Strategic Approach

In this section we assume that the preference relation of each type in the economy, $\geq_t$, is represented by a quasi-linear utility function,

$$u_t(x, y, \mathbf{c}(x), n) = x + h_t(y, \mathbf{c}(x), n).$$

We establish a new approach of the previous model, by assuming that the players behave strategically. We consider a Nash game with two stages with observed actions, that is, all players knew in the second stage, the actions chosen at the first stage and all players move simultaneously in each stage. In the first stage the players choose private consumptions, which, given the relationship between private consumption and crowding type, entails the simultaneously choice of crowding types. In the second stage, with the crowding profile and private consumption already fixed, players choose public good consumption clubs.

In the first stage the strategy set for each player-player i is the consumption set:

$$W_i = \left[0, w_{\tau(i)}\right].$$

Given that the choice of private consumption means also the choice of the crowding profile we describe the strategy profile for the first stage of the game as a vector of private consumptions and related crowding types,

$$(X, A(X)) = (x_1, ..., x_I, \mathbf{c}_{\gamma(1)}(x_1), ..., \mathbf{c}_{\gamma(I)}(x_I)) \in W_1 \times \cdots \times W_I \times \mathcal{C} \times \cdots \times \mathcal{C}.$$

In the second stage of the game, the players know the history of stage 1, that is, they know the crowding profile of the population and they choose which club to enter. We assume that the player play mixed strategies, that is, they choose a lottery - a probability distribution - over clubs.

The strategy set for each player is given by

$$\Delta^J = \left\{ p_i \in \mathbb{R}_+^J : \sum_{j=1}^{J} p_{ij} = 1 \right\},$$

where p_{ij} is the probability that players i is in jurisdiction s^j. Let

$$\mathbb{S} = \left\{ S = \left(s^1, ..., s^J\right) : S \text{ is a partition of the population } \right\},$$

that is, $\mathbb{S}$ is the set of all possible partitions of players through the J jurisdictions. An allocation of individual lotteries for the economy is a I-tuple $(p_i)_{i=1,\ldots,I}$ of elements in Δ^J and is denoted by P. To each allocation of individual lotteries, P, corresponds a joint lottery over the set $\mathbb{S}$. Thus, individual lotteries are marginal distributions of the joint lotteries. We denote the joint lottery over $\mathbb{S}$, corresponding to P, by L. These concepts are illustrated in the following example:

Example 1. Let $I = 3$ and $J = 2$. Suppose that players 1 and 2 have crowding type c and player 3 has crowding type c'. Consider the following individual lotteries over clubs:

Individual Lotteries		
Clubs	s_1	s_2
Probabilities	p_1	p_2
player 1	$\frac{1}{2}$	$\frac{1}{2}$
player 2	$\frac{1}{3}$	$\frac{2}{3}$
player 3	$\frac{1}{4}$	$\frac{3}{4}$

Then the corresponding joint lottery is given by,

Joint Lottery - L		
Potential clubs-$\mathbb{S}$	Crowding Profile	Probability
$(\emptyset, \{1,2,3\})$	$(\emptyset, \{c,c,c'\})$	$\frac{6}{24}$
$(\{1\},\{2,3\})$	$(\{c\},\{c,c'\})$	$\frac{6}{24}$
$(\{2\},\{1,3\})$	$(\{c\},\{c,c'\})$	$\frac{3}{24}$
$(\{3\},\{1,2\})$	$(\{c'\},\{c,c\})$	$\frac{2}{24}$
$(\{1,2\},\{3\})$	$(\{c,c\},\{c'\})$	$\frac{3}{24}$
$(\{1,3\},\{2\})$	$(\{c,c'\},\{c\})$	$\frac{2}{24}$
$(\{2,3\},\{1\})$	$(\{c,c'\},\{c\})$	$\frac{1}{24}$
$(\{1,2,3\},\emptyset)$	$(\{c,c,c'\},\emptyset)$	$\frac{1}{24}$

Let

$$\mathbb{S}_i^j = \left\{ s^j : i \in s^j \right\},$$

that is, the set $\mathbb{S}_i^j$ is the set of potential allocations of players to club s^j contingent on player i belonging to s^j. In Example 1, $\mathbb{S}_1^1 = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\}\}$. In order to evaluate the utility of each player it is necessary to consider the crowding profile of the clubs to which the player might belong. Let

$\mathcal{CP}\left(\mathbb{S}_i^j\right)$ denotes the corresponding set of potential crowding profiles of the club s^j conditional on player i belonging to club s^j. In Example 1, $\mathcal{CP}\left(\mathbb{S}_1^1\right) = \{\{c\}, \{c, c\}, \{c, c'\}, \{c, c, c'\}\}$.We denote the elements of the set $\mathbb{S}_i^j$ by s_i^{jh} for $h = 1, .., \#\mathbb{S}_i^j$ and the corresponding crowding profile by $cp(s_i^{jh})$ for $h = 1, .., \#\mathbb{S}_i^j$.

The joint lottery, L, induce a conditional distribution over $\mathbb{S}_i^j$, that we denote by $l_i^{s^j}$, that is, $l_i^{s^{jh}}$ denotes the probability of the allocation s_i^{jh} for the jurisdiction s^j given that $i \in s^j$, given by the distribution $l_i^{s^j}$. In summary, a given allocation of individual lotteries $P = (p_1, ..., p_I)$ induces a joint lottery L over the set of pure allocation of player through jurisdictions and a collection of conditional distributions over jurisdictions that contains the player i, $\left\{l_i = \left(l_i^{s^1}, ..., l_i^{s^J}\right), i = 1, ..., I\right\}$ with $\mathbb{S}_i^j$ being the support of the distribution $l_i^{s^j}$ for all $j = 1, .., J$ and $i = 1, ..., I$. In the following three tables we describe the distributions $l_1^{s^1}$, $l_2^{s^1}$, $l_3^{s^1}$, $l_1^{s^2}$, $l_2^{s^2}$ and $l_3^{s^2}$ for the example 1.

$\mathbb{S}_1^1$	$\mathcal{CP}\left(\mathbb{S}_1^1\right)$	$l_1^{s^1}$	$\mathbb{S}_2^1$	$\mathcal{CP}\left(\mathbb{S}_2^1\right)$	$l_2^{s^1}$	$\mathbb{S}_3^1$	$\mathcal{CP}\left(\mathbb{S}_3^1\right)$	$l_3^{s^1}$
$\{1\}$	$\{c\}$	$\frac{6}{12}$	$\{2\}$	$\{c\}$	$\frac{3}{8}$	$\{3\}$	$\{c'\}$	$\frac{2}{6}$
$\{1,2\}$	$\{c,c\}$	$\frac{3}{12}$	$\{1,2\}$	$\{c,c\}$	$\frac{3}{8}$	$\{1,3\}$	$\{c,c'\}$	$\frac{2}{6}$
$\{1,3\}$	$\{c,c'\}$	$\frac{2}{12}$	$\{2,3\}$	$\{c,c'\}$	$\frac{1}{8}$	$\{2,3\}$	$\{c,c'\}$	$\frac{1}{6}$
$\{1,2,3\}$	$\{c,c,c'\}$	$\frac{1}{12}$	$\{1,2,3\}$	$\{c,c,c'\}$	$\frac{1}{8}$	$\{1,2,3\}$	$\{c,c,c'\}$	$\frac{1}{6}$

$\mathbb{S}_1^2$	$\mathcal{CP}\left(\mathbb{S}_1^2\right)$	$l_1^{s^2}$	$\mathbb{S}_2^2$	$\mathcal{CP}\left(\mathbb{S}_2^2\right)$	$l_2^{s^2}$	$\mathbb{S}_3^2$	$\mathcal{CP}\left(\mathbb{S}_3^2\right)$	$l_3^{s^2}$
$\{1,2,3\}$	$\{c,c,c'\}$	$\frac{6}{12}$	$\{1,2,3\}$	$\{c,c,c'\}$	$\frac{3}{8}$	$\{1,2,3\}$	$\{c,c,c'\}$	$\frac{2}{6}$
$\{1,3\}$	$\{c,c'\}$	$\frac{3}{12}$	$\{2,3\}$	$\{c,c'\}$	$\frac{3}{8}$	$\{2,3\}$	$\{c,c'\}$	$\frac{2}{6}$
$\{1,2\}$	$\{c,c\}$	$\frac{2}{12}$	$\{1,2\}$	$\{c,c\}$	$\frac{1}{8}$	$\{1,3\}$	$\{c,c'\}$	$\frac{1}{6}$
$\{1\}$	$\{c\}$	$\frac{1}{12}$	$\{2\}$	$\{c\}$	$\frac{1}{8}$	$\{3\}$	$\{c'\}$	$\frac{1}{6}$

Since the private consumption of each player is already fixed, the contribution of player i to the public good is given by the surplus of his endowment over his private consumption, $w_{\tau(i)} - x_i$. Therefore, we consider that a club s^j with crowding profile $cp\left(s^j\right)$ make use of all available private good $\sum_{i \in s^j}\left(w_{\tau(i)} - x_i\right)$ to produce the public good, that is, in each club the public good y^j produced solves the equation:

$$f(y^j, cp\left(s^j\right)) = \sum_{i \in s^j}\left(w_{\tau(i)} - x_i\right).$$

That is, a jurisdiction s^j with crowding profile $cp\left(s^j\right)$ make use of all available private good $\sum_{i\in s^j}\left(w_{\tau(i)}-x_i\right)$ to produce the public good for its given crowding profile.

For a given sequence of strategies for the two stages of the economy, the payoff of player i is the expected utility described by,

$$U_i : W_1 \times \cdots \times W_I \times \Delta^J \times \cdots \times \Delta^J \longrightarrow R$$

$$U_i\left(x_1,...,x_I,p_1,...,p_I\right) = \sum_{j=1}^{J} p_j \sum_{h=1}^{\#\mathbb{S}_i^j} l_i^{s^{jh}} u_{\tau(i)}\left(x_i, y^{jh}, \mathbf{c}_{\gamma(i)}\left(x_i\right), cp\left(s_i^{jh}\right)\right).$$

or yet

$$U_i\left(x_1,...,x_I,p_1,...,p_I\right) = x_i + \sum_{j=1}^{J} p_j \sum_{h=1}^{\#\mathbb{S}_i^j} l_i^{s^{jh}} h_{\tau(i)}\left(y^{jh}, \mathbf{c}_{\gamma(i)}\left(x_i\right), cp\left(s_i^{jh}\right)\right).$$

Definition. A strategy profile $(X^*, P^*) \in W_1 \times \cdots \times W_I \times \Delta^J \times \cdots \times \Delta^J$ is a Tiebout equilibrium for the economy if it is a subgame-perfect equilibrium for the two-stage game $\mathcal{G} \equiv \left\{\left(W_i \times \Delta^J, U_i\right); i = 1,...,I\right\}$.

Theorem. An equilibrium exists.

Proof. To be inserted. We note that the proof depends on compactness of the choice sets in both stages of the game, and our assumptions enduring upper-semi-continuity. Our use of lotteries then enables us to apply a fixed point theorem to obtain existence.

4 Appendix

Theorem. Let $(X, Y, A(X), S)$ be a core state of an economy that satisfies SSGE. For any two players $i,\ \hat{\imath} \in \mathcal{I}$ such that $\tau(i) = \tau(\hat{\imath}) = t$ and $\gamma(i) = \gamma(\hat{\imath}) = g$, if $i \in s^k$ and $\hat{\imath} \in s^{\hat{k}}$ then

$$\left(x_i, y^k, \mathbf{c}_g(x_i), CP\left(A(X), s^k\right)\right) \sim_t (x_{\hat{\imath}}, y^{\hat{k}}, \mathbf{c}_g(x_{\hat{\imath}}), CP(A(X), s^{\hat{k}})).$$

Proof. Suppose not. By SSGE, for all $s^k \in S$, $\left|s^k\right| \leq B$, and for this particular t, $|\{i \in \mathcal{I} : \tau(i) = t, \text{ and } \gamma(i) = g\}| > B$. Then, there exist at least two players with the same type (t, g) who are in different clubs in the core partition S. Let us assume without loss of generality the two players with the same type but members of different clubs are not equally treated. Suppose that

$$\left(x_i, y^k, \mathbf{c}_g(x_i), CP\left(A(X), s^k\right)\right) \succ_t (x_{\hat{\imath}}, y^{\hat{k}}, \mathbf{c}_g(x_{\hat{\imath}}), CP(A(X), s^{\hat{k}})).$$

We claim that this is not possible in a core state. Consider the club $\overline{s} \equiv \left\{s^k \backslash i\right\} \cup \hat{\imath}$. Let the allocation for $\overline{s}$ be $(\overline{x}, \overline{y})$ with $\overline{y} = y^k$, $\overline{x}_i = x_i^k$ if $i \neq \hat{\imath}$ and for player $\hat{\imath}$, $\overline{x}_{\hat{\imath}} = x_i^k$. Then, the club $\overline{s}$ is formed by replacing player i with player $\hat{\imath}$. The allocation $(\overline{x}, \overline{y})$ for $\overline{s}$ is identical to $\left(x^k, y^k\right)$ except that player $\hat{\imath}$ has the same consumption of player i and because he is of the same genetic type he has the same crowding type as player i. Then, by construction $CP(\overline{s}) = CP\left(s^k\right)$ and $\sum_{j \in s^k} w_{\tau(j)} = \sum_{j \in \overline{s}} w_{\tau(j)}$ it follows that $(\overline{x}, \overline{y})$ is feasible under assignment $A\left(\overline{X}\right)$ for jurisdiction $\overline{s}$. It is true that for all $j \in \overline{s}$, $j \neq \hat{\imath}$,

$$\left(\overline{x}_j, \overline{y}, \mathbf{c}_g(\overline{x}_j), CP\left(A\left(\overline{X}\right), \overline{s}\right)\right) \sim_{\tau(j)} (x_j, y^k, \mathbf{c}_g(x_j), CP(A(X), s^k)),$$

and for player $\hat{\imath}$

$$\begin{aligned}\left(\overline{x}_{\hat{\imath}}, \overline{y}, \mathbf{c}_g(\overline{x}_{\hat{\imath}}), CP\left(A\left(\overline{X}\right), \overline{s}\right)\right) &\sim_t \left(x_i, y^k, \mathbf{c}_g(x_i), CP\left(A(X), s^k\right)\right) \\ &\succ_t (x_{\hat{\imath}}, y^{\hat{k}}, \mathbf{c}(x_{\hat{\imath}}), CP(A(X), s^{\hat{k}})).\end{aligned}$$

That is, all players in $\overline{s}$ are at least as well off, and player $\hat{\imath}$ is strictly better off. This allocation improves upon $(X, Y, A(X), S)$ which contradicts the hypothesis that it is a core state.

Theorem. Let $(X, Y, A(X), S)$ be a core state of an economy satisfying SSGE, and let s^k, $s^{\hat{k}} \in S$ be a pair of clubs in the core partition such that

$y^k = y^{\widehat{k}}$, and $CP(A(X), s^k) = CP(A(\widehat{X}), s^{\widehat{k}})$. Then for any crowding type $c \in \mathcal{C}$, and any pair of players $i \in s^k$ and $\hat{\imath} \in s^{\widehat{k}}$ such that $\mathbf{c}_{\gamma(i)}(x_i) = \mathbf{c}_{\gamma(\hat{\imath})}(x_{\hat{\imath}})$, it holds that

$$w_{\tau(i)} - x_i = w_{\tau(\hat{\imath})} - x_{\hat{\imath}}.$$

<u>First proof:</u>

Suppose not. Without loss of generality, assume

$$w_{\tau(i)} - x_i > w_{\tau(\hat{\imath})} - x_{\hat{\imath}},$$

$\gamma(i) = g$ and $\tau(i) = t$.

By SSGE, for all $s^k \in S$, it holds that $\left|s^k\right| \leq B$, and for this particular g and t it holds that $|\{i \in \mathcal{I} : \tau(i) = t, \text{ and } \gamma(i) = g\}| > B$. It follows that there are at least one player of type g and t who is not in club s^k. Denote this player by $\bar{\imath} \in s^{\bar{k}} \neq s^k$.

We can consider the club $\bar{s} \equiv \left\{s^{\widehat{k}} \backslash \hat{\imath}\right\} \cup \bar{\imath}$, which follows by replacing player $\hat{\imath}$ with $\bar{\imath}$ in club $s^{\widehat{k}}$. Let the allocation for $\bar{s}$ be $(\overline{x}, \overline{y})$ with $\overline{y} = y^k$, $\overline{x}_j = \widehat{x}_j$ if $j \neq \bar{\imath}$ and for player $\bar{\imath}$, $\overline{x}_{\bar{\imath}} = x_i + \varepsilon$ with $\varepsilon > 0$ such that such that $w_{\tau(i)} - x_i > w_{\tau(i)} - (x_i + \varepsilon) > w_{\tau(\hat{\imath})} - x_{\hat{\imath}}$, and $\mathbf{c}_g(x_i) = \mathbf{c}_g(x_i + \varepsilon)$. Then, the club $\bar{s}$ is formed by replacing in the club $s^{\widehat{k}}$ player $\hat{\imath}$ with player $\bar{\imath}$. The allocation of player $\bar{\imath}$ is feasible because $w_t - \overline{x}_{\bar{\imath}} = w_t - (x_i + \varepsilon) > w_{\tau(\hat{\imath})} - x_{\hat{\imath}} \geq 0$.

This reallocation stills a feasible plan for $\bar{s}$ because player $\hat{\imath}$ is replaced by player $\bar{\imath}$, the net collection of private goods for the public projects production is superior for club $\bar{s}$ than it was for $s^{\widehat{k}}$:

$\sum_{j \in s^{\widehat{k}}} \left(w_{\tau(j)} - \widehat{x}_j\right) > \sum_{j \in \bar{s}} \left(w_{\tau(j)} - \overline{x}_j\right)$.

For all $j \in \bar{s}$, such that $j \neq \bar{\imath}$,

$$\left(\overline{x}_j, \overline{y}, \mathbf{c}_{\gamma(j)}(\overline{x}_j), CP\left(A\left(\overline{X}\right), \bar{s}\right)\right) \sim_{\tau(j)} (\widehat{x}_j, y^{\widehat{k}}, \mathbf{c}_{\gamma(j)}(\widehat{x}_j), CP(A(\widehat{X}), s^{\widehat{k}})).$$

By construction $\overline{x}_{\bar{\imath}} > x_i$, then monotonicity implies:

$$\left(\overline{x}_{\bar{\imath}}, \overline{y}, \mathbf{c}_g(\overline{x}_{\bar{\imath}}), CP\left(A\left(\overline{X}\right), \bar{s}\right)\right) \succ_t \left(x_i, y^k, \mathbf{c}_g(x_i), CP\left(A(X), s^k\right)\right).$$

Since player i and $\bar{\imath}$ are both type g and t by Theorem 1 they are both equally treated in the core, $\left(x_i, y^k, \mathbf{c}(x_i), CP\left(A(X), s^k\right)\right) \sim_t (x_{\bar{\imath}}, y^{\bar{k}}, \mathbf{c}(x_{\bar{\imath}}), CP(A(\overline{X}), s^{\bar{k}}))$ and finally:

$$\left(\overline{x}_{\bar{\imath}}, \overline{y}, \mathbf{c}(\overline{x}_{\bar{\imath}}), CP\left(A\left(\overline{X}\right), \bar{s}\right)\right) \succ_t (x_{\bar{\imath}}, y^{\bar{k}}, \mathbf{c}(x_{\bar{\imath}}), CP(A(\overline{X}), s^{\bar{k}})).$$

In words, all players in $\overline{s}$ are at least as well off and player $\overline{i}$ is strictly better off. This allocation improves upon $(X, Y, A(X), S)$, which contradicts the hypothesis that it is a core state.

Second proof:

Suppose not. Without loss of generality, assume

$$w_{\tau(i)} - x_i > w_{\tau(\hat{\imath})} - x_{\hat{\imath}},$$

$\gamma(i) = g$ and $\tau(i) = t$.

By SSGE, for all $s^k \in S$, it holds that $|s^k| \le B$, and for this particular g and t it holds that $|\{i \in \mathcal{I} : \tau(i) = t, \text{ and } \gamma(i) = g\}| > B$. It follows that there are at least one player of type g and t who is not in club s^k. Denote this player by $\overline{i} \in s^{\overline{k}} \neq s^k$.

We can consider the club $\overline{s} \equiv \left\{ s^{\widehat{k}} \backslash \hat{\imath} \right\} \cup \overline{i}$, which follows by replacing player $\hat{\imath}$ with $\overline{\imath}$ in club s^k. Let the allocation for $\overline{s}$ be $(\overline{x}, \overline{y})$ with $\overline{y} = y^k$, $\overline{x}_j = x_{\widehat{j}}$ if $j \neq \overline{i}$ and for player $\overline{i}$, $\overline{x}_{\overline{i}}$ is such that such that $w_t - \overline{x}_{\overline{i}} = w_{\tau(\hat{\imath})} - x_{\hat{\imath}}$. For player $\overline{i}$ note that $w_t - \overline{x}_{\overline{i}} = w_{\tau(\hat{\imath})} - x_{\hat{\imath}} < w_t - x_i$, then $\overline{x}_{\overline{i}} > x_i$. Then the crowding type of player $\overline{i}$ which is $\mathbf{c}_g(\overline{x}_{\overline{i}})$ is preferred to $\mathbf{c}_g(x_i) = \mathbf{c}_{\gamma(\hat{\imath})}(x_{\hat{\imath}})$.

This reallocation stills a feasible plan for $\overline{s}$ because player $\hat{\imath}$ is replaced by player $\overline{i}$, the net collection of private goods for the public projects production is the same for club $\overline{s}$ as it was for $s^{\widehat{k}}$:
$\sum_{j \in s^{\widehat{k}}} \left(w_{\tau(j)} - \widehat{x}_j \right) = \sum_{j \in \overline{s}} \left(w_{\tau(j)} - \overline{x}_j \right)$, because $w_{\tau\left(\overline{i}\right)} - \overline{x}_{\overline{i}} = w_{\tau(\hat{\imath})} - x_{\hat{\imath}}$. By other side, the crowding profile of the club $s^{\overline{k}}$ is the same except for player $\overline{i}$ whose crowding type was enhanced relatively to the crowding type of player $\hat{\imath}$.

For all $j \in \overline{s}$, such that $j \neq \overline{i}$,

$$\left(\overline{x}_j, \overline{y}, \mathbf{c}_{\gamma(j)}\left(\overline{x}_j\right), CP\left(A\left(\overline{X}\right), \overline{s}\right)\right) \succeq_{\tau(j)} (\widehat{x}_j, y^{\widehat{k}}, \mathbf{c}_{\gamma(j)}(\widehat{x}_j), CP(A(\widehat{X}), s^{\widehat{k}})).$$

By construction $\overline{x}_{\overline{i}} > x_i$ and $\mathbf{c}_g(\overline{x}_{\overline{i}})$ is preferred to $\mathbf{c}_g(x_i)$, then monotonicity implies: $\left(\overline{x}_{\overline{i}}, \overline{y}, \mathbf{c}_g\left(\overline{x}_{\overline{i}}\right), CP\left(A\left(\overline{X}\right), \overline{s}\right)\right) \succ_t \left(x_i, y^k, \mathbf{c}_g\left(x_i\right), CP\left(A\left(X\right), s^k\right)\right)$. Since player i and $\overline{i}$ are both type (g, t) by Theorem 1 they are both equally treated in the core, $\left(x_i, y^k, \mathbf{c}\left(x_i\right), CP\left(A\left(X\right), s^k\right)\right) \sim_t (x_{\overline{i}}, y^{\overline{k}}, \mathbf{c}(x_{\overline{i}}), CP(A(\overline{X}), s^{\overline{k}}))$ and finally:

$$\left(\overline{x}_{\overline{i}}, \overline{y}, \mathbf{c}_{\gamma(i)}\left(\overline{x}_{\overline{i}}\right), CP\left(A\left(\overline{X}\right), \overline{s}\right)\right) \succ_t (x_{\overline{i}}, y^{\overline{k}}, \mathbf{c}_{\gamma(i)}(x_{\overline{i}}), CP(A(\overline{X}), s^{\overline{k}})).$$

In words, all players in $\overline{s}$ are at least as well off and player $\overline{i}$ is strictly better off. This allocation improves upon $(X, Y, A(X), S)$, which contradicts the hypothesis that it is a core state.

Theorem. If the economy satisfies SSGE, the core of the economy satisfies WET.

Proof. Let us take a crowding type $c \in \mathcal{C}$, and a club s^k in the core such that $\left|\theta_c\left(A, s^k\right)\right| > 1$. Let us consider a player $\overline{i} \in s^k$ such that $\mathbf{c}_{\gamma\left(\overline{i}\right)}\left(x_{\overline{i}}\right) = c$, $\gamma\left(\overline{i}\right) = g$ and $\tau\left(\overline{i}\right) = t$. For the club s^k we have $CP\left(A\left(X\right), s^k\right) = \left(n_1^k, ..., n_C^k\right)$. By SSGE, there exist enough players in the population to create a club s_c of n_c^k players such that for all $j \in s_c$, $\gamma\left(j\right) = g$ and $\tau\left(j\right) = t$, and if $j \in s^k$, then $\mathbf{c}_g(x_j) = c$. Consider also the club: $s_{\overline{c}} = \left\{i \in s^k : A\left(x_i\right) \neq c\right\}$. This two club s_c and $s_{\overline{c}}$ are disjoint by construction. Let $\overline{s}$ be the union of these two club,

$$\overline{s} = s_c \cup s_{\overline{c}}.$$

In words, $s_{\overline{c}}$ consists of all players in s^k who were not of crowding type c. The other group of players s_c consists of players of type (g, t) from either outside coalition s^k or from the set of players in s^k who already was of crowding type c.

The allocation for this club is the following:

(1) let $\overline{y} = y^k$;

(2) for all $j \in s_{\overline{c}}$ let $\overline{x}_j = x_j$;

(3) for all $j \in s_c$ let $\overline{x}_j = x_{\overline{i}}$.

By (3) for all $j \in s_c$, $\mathbf{c}_{\gamma(j)}\left(\overline{x}_j\right) = \mathbf{c}_g\left(x_{\overline{i}}\right) = c$, therefore $CP\left(A\left(\overline{x}\right), \overline{s}\right) = CP\left(A\left(x\right), s^k\right)$.

Let us show that this is feasible for $\overline{s}$.

By construction $\theta_s\left(\overline{A}, \overline{s}\right) = t \in \theta_c\left(A, s^k\right)$,

$$\sum_{i \in s_{\overline{c}}}\left(w_{\tau(i)} - \overline{x}_i\right) = \sum_{\left\{i \in s^k : \mathbf{c}(x_i) \neq c\right\}}\left(w_{\tau(i)} - x_i\right).$$

By Theorem 2, for all $i, j \in s^k$, such that $\mathbf{c}_{\gamma(i)}\left(x_i\right) = \mathbf{c}_{\gamma(j)}\left(x_j\right)$,

$$w_{\tau(i)} - x_i = w_{\tau(j)} - x_{j.}$$

Thus

$$\sum_{i \in s_c}\left(w_t - \overline{x}_t\right) = \sum_{\left\{i \in s^k : \mathbf{c}(x_i) = c\right\}}\left(w_{\tau(i)} - x_i\right).$$

In follows that,

$$\sum_{i\in\overline{s}}\left(w_{\tau(i)}-\overline{x}_i\right)=\sum_{i\in s^k}\left(w_{\tau(i)}-x_i\right).$$

Let us now check that all players in $\overline{s}$ are at least as well off as they were in the club from which they came. If $j\in s_{\overline{c}}$ it is immediate since they receive exactly the same public projects and private good level as they did in club s^k. If $i\in s_c$, since $\gamma(i)=\gamma\left(\overline{i}\right)=g$ and $\tau(i)=\tau\left(\overline{i}\right)=t$, by Theorem 1,

$$\left(x_i,y^{k_i},\mathbf{c}_g(x_i),CP\left(A(X),s^{k_i}\right)\right)\sim_t(x_{\overline{i}},y^k,\mathbf{c}_g(x_{\overline{i}}),CP(A(x),s^k)).$$

where $i\in s^{k_i}\in S$ in the core partition. Then since $\overline{x}_i=x_{\overline{i}}$, $\overline{y}=y^k$ and $CP\left(\overline{A}(\overline{x}),\overline{s}\right)=CP\left(A\left(x^k\right),s^k\right)$, we conclude that for all $j\in\overline{s}$,

$$\left(\overline{x}_j,\overline{y},\mathbf{c}_{\gamma(j)}(\overline{x}_j),CP\left(A\left(\overline{X}\right),\overline{s}\right)\right)\sim_{\tau(j)}\left(x_j,y^{k_j},\mathbf{c}_{\gamma(j)}(x_j),CP\left(A(X),s^{k_j}\right)\right).$$

where $j\in s^{k_j}\in S$ in the core state.

Theorem. If a feasible state $(X,Y,A(X),S)$ and a price system ρ is a Tiebout equilibrium then $(X,Y,A(X),S)$ is in the core.

Proof. Suppose not. Then there exists some club $\overline{s}\in\mathcal{S}$ producing a feasible allocation $(\overline{x},\overline{y})$, under assignment $A\left(\overline{X}\right)\in\mathcal{A}$ that improves upon the Tiebout equilibrium. Consider an arbitrary player $i\in\overline{s}$ and suppose that he contributes more to public projects production in the improving club than he would have been required to contribute under the price system:

$$w_{\tau(i)}-\overline{x}_i>\rho_{A(\overline{x})(i)}(\overline{y},CP(A\left(\overline{X}\right),\overline{s})).$$

But by definition of a Tiebout equilibrium,

$$\left(w_{\tau(i)}-\rho_{A(x)(i)}\left(y^{k_i},CP(A(X),s^{k_i})\right),y^{k_i},\mathbf{c}_{\gamma(i)}(x_i),CP(A(X),s^{k_i}))\right)\succeq_{\tau(i)}$$

$$\left(w_{\tau(i)}-\rho_{A(\overline{x})(i)}\left(\overline{y},CP(A(\overline{X}),\overline{s})\right),\overline{y},\mathbf{c}_{\gamma(i)}(\overline{x}_i),CP(A(\overline{X}),\overline{s}))\right)$$

where $i\in s^{k_i}\in S$ in the equilibrium state. Then by monotonicity,

$$\left(w_{\tau(i)}-\rho_{A(x_i)}\left(y^{k_i},CP(A(X),s^{k_i})\right),y^{k_i},\mathbf{c}_{\gamma(i)}(x_i),CP(A(X),s^{k_i}))\right)$$

$$\succ_{\tau(i)} \left(\overline{x}_i, \overline{y}, \mathbf{c}_{\gamma(i)}\left(\overline{x}_i\right), CP(A(\overline{X}), \overline{s})\right),$$

which contradicts the hypothesis that $(\overline{x}, \overline{y})$ is an improving allocation.

Therefore, for all $i \in \overline{s}$,

$$w_{\tau(i)} - \overline{x}_i \leq \rho_{A(\overline{x})(i)}(\overline{y}, CP(A\left(\overline{X}\right), \overline{s})).$$

Remark that if $\sum_{i \in \overline{s}} \left(w_{\tau(i)} - \overline{x}_i\right) > f(\overline{y}, CP(\overline{A}\left(\overline{X}\right), \overline{s}))$ then this club can also improve on $(X, Y, A(X), S)$ with another feasible allocation in which the surplus private good is redistributed back to the players in $\overline{s}$. Thus, without loss of generality we may assume that

$$\sum_{i \in \overline{s}} \left(w_{\tau(i)} - \overline{x}_i\right) = f(\overline{y}, CP(A\left(\overline{X}\right), \overline{s})).$$

Suppose now that for some $i \in \overline{s}$,

$$w_{\tau(i)} - \overline{x}_i < \rho_{A(\overline{x})(i)}(\overline{y}, CP(A\left(\overline{X}\right), \overline{s})).$$

This imply that for some player $j \in \overline{s}$,

$$w_{\tau(j)} - \overline{x}_j > \rho_{\overline{A}(\overline{x}_j)}(\overline{y}, CP(\overline{A}\left(\overline{X}\right), \overline{s})).$$

This however, contradicts what we previously have shown. We conclude that for all $i \in \overline{s}$,

$$w_{\tau(i)} - \overline{x}_i = \rho_{A(\overline{x})(i)}(\overline{y}, CP(A\left(\overline{X}\right), \overline{s})).$$

By hypothesis, the allocation $(\overline{x}, \overline{y})$ and the assignment $A(\overline{X})$ is improving for club $\overline{s}$. This implies that for some $j \in \overline{s}$,

$$(w_{\tau(j)} - \rho_{\overline{A}(\overline{x}_j)}\ (\overline{y}, CP(A(\overline{X}), \overline{s})), \overline{y}, \mathbf{c}_{\gamma(j)}\left(\overline{x}_j\right), CP(A(\overline{X}), \overline{s})) \sim_{\tau(j)}$$

$$\left(\overline{x}_j, \overline{y}, \mathbf{c}_{\gamma(j)}\left(\overline{x}_j\right), CP(\overline{A}(\overline{X}), \overline{s})\right) \succ_{\tau(j)}$$

$$\left(w_{\tau(j)} - \rho_{A(x)(j)}\ (y^{k_j}, CP(A(X), s^{k_j})), y^{k_j}, \mathbf{c}_{\gamma(j)}\left(x_j\right), CP(A(X), s^{k_j}))\right),$$

where $j \in k_j \in S$ in the Tiebout equilibrium. However, this violates the definition of a Tiebout equilibrium.

5 References (incomplete)

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