

Priorities and Hierarchies in the Location of Multiple Public Facilities*

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August 2006

Abstract

We are interested in rules for locating k public facilities on the real line, as a function of the preferences of the agents. Each agent uses exactly one facility. Moreover, facilities do not suffer from congestion and can thus be used by any arbitrary number of agents. Each agent has a general single-peaked preference over the location of the facility he consumes. A *priority rule* selects locations according to a predetermined priority ordering among “interest groups”. We characterize the family of all priority rules by the conditions of Pareto-efficiency, object-population monotonicity and sovereignty. We characterize each of the subclasses of *priority rules* that respectively satisfy hiding-proofness, consistency, and Maskin-monotonicity. In particular, we show that a *priority rule* is Maskin-monotonic if and only if it is a *hierarchical left-right peaks solution*. All rules in this subclass turn out to be hiding-proof, coalition-strategy-proof, and consistent. Unfortunately, none of them is anonymous.

Keywords: Multiple public facilities, Priority rules, Hierarchical rules, Object-population-monotonicity, Sovereignty, Anonymity, Consistency, Hiding-proofness, Maskin-monotonicity, Group strategy-proofness.

*We thank Haluk Ergin, Bettina Klaus, François Maniquet, William Thomson for helpful conversations and remarks. Part of this paper was written when the first author visited the University of Montreal. Financial support from CIREQ is gratefully acknowledged.

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1 Introduction

We consider a generalization of the model studied in Black (1948) and Moulin (1980). A collective decision problem is described by a set of agents, a profile of single-peaked preferences over the real line and a number k of public facilities to be located. Public facilities do not suffer from congestion and are non-excludable. The first axiomatic treatment of the model –for $k = 2$ – is due to Miyagawa (2001) and Ehlers (2003). Miyagawa (2001) shows that the class of rules satisfying *efficiency* and *replacement-domination* is very narrow when single-peaked preferences satisfy the maximax extension over sets. It consists only of two rules locating either facilities at peaks that are the most to the left or to the right.¹ Ehlers (2003) studies the same set of axioms but with a different extension to preferences over sets.² We follow Miyagawa’s approach. Our first motivation is that only few rules are known and for a small number of public facilities. Furthermore, while Miyagawa’s results are rather disappointing, we wish to investigate other natural axioms that can be imposed on solutions in this model. Our central question is the following. Given a set of basic desirable axioms, what does a social choice expert have to say about the choice of locations?³

In order to motivate our study and the axioms we will use, let us discuss the following example. Consider the case of the European union (henceforth, EU). It is a linguistically diversified “society” that has to select a set of official working languages within the EU among a set comprising all the languages of

¹These rules are called, respectively, the *left-peaks rule* and the *right-peaks rule*.

²In Ehlers (2003), the extension of preferences over single points to preferences over sets of points is the lexicographic extension. Though the axioms used are the same, his findings are in sharp contrast with Miyagawa’s (2001) result.

³Given that public facilities do not suffer from exclusion or congestion, we can abstract from the problem of assigning agents to facilities once their locations have been chosen. We can simply say that agents choose the location they prefer the most—and one out of the set of their preferred facilities in case of indifferences.

existing members.^{4,5,6} In this context, a language is a public facility and its use does not suffer from any form of congestion.⁷ Clearly, not all languages can be selected as official working language, thereby denying some members the possibility to address the political process in their mother tongues. How should have the EU decided which language to select as official working ones? In addition, how should society respond to the language challenges faced by the European Union after its enlargement in 2004? We would like to have a method for selecting official working languages that respects some basic properties.⁸ Formally, we want to characterize solutions satisfying some basic desirable axioms. As a first requirement, we most certainly want to impose *efficiency*. Once a set of languages has been chosen, it is not possible to propose a different set without making some existing members to suffer from this change. In addition, we are interested in two properties –namely, *sovereignty* and *object-population monotonicity*– that are rather natural in such a context. Suppose that there is only one language to choose as communicating language within the institution. The axiom of *sovereignty* loosely says that a language can be given priority over others if the “group of interest” defending it is large enough. Next, we complement our properties with an axiom combining monotonicity in resources and in population. *object-population monotonicity* requires two things. First, adding the possibility to choose an additional language to the set of initially chosen language should not hurt any member. Furthermore, if an additional

⁴Currently, the official working languages of the EU are English, French and German.

⁵According to article 2.11 of the Amsterdam Treaty, every citizen of the union is allowed to use his native language in dealing with the official institution of the EU.

⁶For a more detailed discussion concerning the choice of official languages in the EU, see the paper by Fidrmuc, Ginsburgh and Weber (2006). We borrow a quote of Bretton (1976) from their paper (see below).

⁷Another possible application is the selection of languages in which tax forms are translated in the Netherlands.

⁸Observe that because of its enlargement, the European Union is facing repeatedly such a choice: the entry of new members may call for additional official languages. Hence, we need a method –a rule in the language of collective choice– that can be used in different situations.

language is offered in addition to the arrival of a new member (i.e. a country with, say, a unified language), again no existing member should suffer. The justification should be without controversy since the initial choice of languages is still feasible. While the axioms mentioned above are very natural, the problem underlined is nonetheless not trivial. As Bretton (1976, p.447) points out: “language may be the most explosive issue universally....Fear of being deprived of communicating skills seem to rise political passion to a fever pitch”.

Equipped with these axioms, we provide a complete characterization of the class of solutions that satisfy them. We first prove that the combination of our axioms implies that solutions satisfy *peak-selection*. Furthermore, given the nature of the problem, we feel that rules should obey some criterion of priority. Surprisingly, it turns out that the combination of *efficiency*, *object-population monotonicity* and *sovereignty* characterizes a subfamily of rules that we call *priority rules*. Each of these rules can be derived from a *priority order* that ranks triples of the form (x, S_x, R_{S_x}) and $(y, S'_y, R'_{S'_y})$.^{9,10} The class of rules is fairly large and includes in particular (but not only) a class of rules that we call *majoritarian rules*. Each such rule defines priorities with respect to the highest number of agents having the same preferred point on the real line.¹¹

Next, we study refinements of the family of *priority rules* by considering several axioms of robustness in changes in preferences and in the number of agents, as well as manipulation axioms that have appeared in the literature. We provide two additional characterizations, one for the sub-family of rules satisfying our three axioms and *hiding-proofness*, the other for the sub-family of rules satis-

⁹Where $x \neq y$ are points on the real line. S_x, S_y are the groups of agents having their peaks at x and y , given preference profiles R_{S_x}, R_{S_y} . Finally, notice that $S_x \cap S_y = \emptyset$.

¹⁰We call such a relation a priority order to distinguish it from an order or a partial order. Priority order satisfies only a restricted form of transitivity and of completeness.

¹¹Majoritarian rules differ in the tie-breaking rule embodied into them. Tie-breaking rules are necessary for case in which, for instance, no pair of agents share the same peak. However, not all tie-breaking rules are admissible.

fying our axioms and *Maskin monotonicity*. It turns out that a rule f satisfies *hiding-proofness* if and only if f can be derived from a priority order that is *monotone in population*. Interestingly, any rule in this class satisfies a property of *consistency* (Thomson, 1988) that is not related to our axioms. Furthermore, the combination of *efficiency*, *object-population monotonicity*, *sovereignty* and *Maskin monotonicity* characterize a family of solutions that we call *hierarchical left-right peaks rules*. As the name indicates, these rules are based on rankings of sets of agents, and are thus non-anonymous. Moreover, they are all *consistent* and *group strategy-proof*.

The plan of the paper is as follows. Section 2 introduces the model and necessary definitions. Section 3 presents the class of solutions we characterize. Section 4 presents formal definitions of the axioms used in this paper. Section 5 then presents the first characterization result. Section 6 studies the subclass of rules satisfying our three axioms and *hiding-proofness*. Section 7 presents the characterization of rules satisfying our three axioms and *Maskin monotonicity*. Finally, section 7 offers some concluding remarks.

2 The model

There is a countably infinite set \mathbb{N} of potential agents. A *population* N is a finite and nonempty subset of \mathbb{N} . The population is collectively endowed with a certain number of identical public facilities, each to be located on the real line \mathbb{R} . A typical location on \mathbb{R} is denoted by x . An *assignment* is a menu of locations, i.e. a finite subset $X \subset \mathbb{R}$. A *k-assignment* is an assignment for exactly k facilities, i.e. a subset $X \subset \mathbb{R}$ such that $|X| = k$. Let \mathcal{X}_k be the class of all k -assignments. In particular, a 1-assignment is a single location $x \in \mathbb{R}$, so that $\mathcal{X}_1 = \mathbb{R}$. Let $\mathcal{X} := \cup_{k \geq 1} \mathcal{X}_k$ be the class of all assignments.

A *preference* over \mathcal{X} is a reflexive, transitive and complete binary relation

over \mathcal{X} . Each agent $i \in N$ has a preference R_i over \mathcal{X} . For each preference R_i , let P_i and I_i stand for the strict and indifference relations associated with R_i , respectively. We restrict attention to the class \mathcal{R} of *single-peaked* preferences over \mathcal{X} , defined by the following two conditions. The first condition is the common single-peakedness notion, for preferences over single locations on the real line. The second condition extends preferences from single locations to menus.¹² A preference R_i is single-peaked if the following holds.

- i) There is a location $p(R_i)$, such that for all $x, y \in \mathbb{R}$ satisfying either $x < y \leq p(R_i)$ or $p(R_i) \geq y > x$, $y P_i x$, we have $y P_i x$. The location $p(R_i)$ is called the *peak* of preference R_i .
- ii) For all $X, Y \in \mathcal{X}$, we let $X R_i Y$ if there is $x \in X$ such that for all $y \in Y$, we have $x R_i y$.

For each population N , a *preference profile for N* is a list of preferences $R_N = (R_i)_{i \in N} \in \mathcal{R}^N$ indexed by N . More generally, a *preference profile* is a preference profile for some population N .¹³ For each profile R_N and each subpopulation $K \subseteq N$, let R_K denote the subprofile $(R_i)_{i \in K}$. For each profile $R_N \in \mathcal{R}^N$, let $p(R_N)$ be the set of peak location for R_N , i.e. $p(R_N) \equiv \{p(R_i) : i \in N\}$. For each $k > 0$, let \mathcal{P}_k be the set of preference profiles R_N with a number of peak locations greater than k , i.e. such that $k < |p(R_N)|$. A problem is a pair (k, R_N) such that k is a positive integer, and $R_N \in \mathcal{P}_k$.¹⁴

¹²There are different ways to extend preferences over points to preferences over sets. Consistent with the definition of a public facility used in this paper, we consider the maximax extension of preferences. This specific extension is the one used by Myiagawa (2001).

¹³A profile specifies a population and each of its agents' preferences. Thus the notation R_N .

¹⁴The restriction $k < |p(R_N)|$ allows us to focus on non-trivial cases. When $k \geq |p(R_N)|$, it is possible to locate one facility at each peak location, so that the welfare of each agent is maximized. Locating the remainig facilities (if any) does not affect any agent's welfare.

A *solution* (or rule) is a sequence $f = \{f_1, f_2, \dots\}$ of mappings $f_k : \mathcal{P}_k \rightarrow \mathcal{X}_k$. For each problem (k, R_N) , the solution f prescribes an assignment in \mathcal{X}_k .¹⁵

For each problem (k, R_N) , the *left-peaks rule* f^{LP} is defined by,

$$f_k^{LP}(R_N) \equiv \{x_1 < \dots < x_k : p(R_l) \neq x_l, l = 1, \dots, k \implies p(R_l) > x_l \text{ for each } l = 1, \dots, k\}$$

The *right-peaks rule* f^{RP} is defined in a similar fashion.

Miyagawa (2001) showed for the case $k = 2$ and $n \geq 4$ that f^{LP} and f^{RP} are the only rules satisfying *efficiency* and *replacement-domination*. Though not very appealing, these rules are not particularly undesirable in this model. Indeed, they are *efficient*, anonymous—and hence non-dictatorial—and they moreover satisfy group strategy-proofness, a strong non-manipulation property.

3 A class of solutions

Let us introduce a class \mathcal{F} of solutions which will play an important role in our results. A profile R_M is *unanimous* if all the preferences of this profile have the same peak, i.e. $p(R_M)$ is a singleton. Let \mathcal{T} be the set of unanimous profiles. For any two unanimous profiles R_L and R'_M , we say that R_L and R'_M are *compatible* if they have distinct peaks and disjoint populations, i.e. $p(R_L) \neq p(R'_M)$ and $L \cap M = \emptyset$.

We now introduce priorities over \mathcal{T} . A binary relation is *asymmetric* if for all $R_L, R'_M \in \mathcal{T}$, we have $(R_L \succ R'_M) \implies (\text{not } R'_M \succ R_L)$. It is *restricted transitive* if for all $R_K, R'_L, R''_M \in \mathcal{T}$, such that R_K and R''_M are compatible, we have $(R_K \succ R'_L \text{ and } R'_L \succ R''_M) \implies (R_K \succ R''_M)$. Finally, a binary relation \succ over \mathcal{T} is *restricted complete* if for all $R_L, R'_M \in \mathcal{T}$, we have $(R_L \succ R'_M \text{ or } R'_M \succ R_L) \iff (R'_L \text{ and } R_M \text{ are compatible})$. A *priority* \succ over

¹⁵Our definitions rule out “duplication”, i.e. locating more than one facility at the same point. Under single-peaked preferences, and for the class of problems we consider, Pareto-efficiency would exclude duplication anyway.

\mathcal{T} is a binary relation over \mathcal{T} that is asymmetric, restricted transitive and restricted complete.¹⁶ Let Ω be the set of priorities over \mathcal{T} .

For each profile R_N , let $\{M_x\}_{x \in p(R_N)}$ be the partition of N such that $M_x = \{i \in N : p(R_i) = x\}$ for each $x \in p(R_N)$. The *decomposition of R_N into unanimous subprofiles* is then the collection of subprofiles $\{R_{M_x}\}_{x \in p(R_N)}$. An important property of the decomposition of R_N is that its members are pairwise compatible. As a consequence (see footnote 5), they are strictly ranked by any priority.

We are now ready to define the family of priority rules, parametrized by the set Ω . For each $\succ \in \Omega$, the *priority rule f_\succ* associated with \succ is defined as follows. Let (k, R_N) be an arbitrary problem. Let R_{M_1}, \dots, R_{M_l} be the members of the decomposition of the profile R_N into compatible unanimous subprofiles, ranked in the order of decreasing priority for priority \succ . Let x_1, \dots, x_l be the respective peaks of each of the unanimous subprofiles. Let $f_{\succ k}(R_N)$ be the k -assignment consisting of the peak location of the top k unanimous subprofiles for \succ , i.e. $f_{\succ k}(R_N) = \{x_1, \dots, x_k\}$. Let \mathcal{F} be the set of priority rules.

An important subfamily is the following. Let Ω^* be the subset of Ω such that for all $\succ \in \Omega^*$ and for all $R_H, R'_K \in \mathcal{T}$ satisfying $R_H \succ R'_K$, and for any population L , then there exists a unanimous profile $R''_M \in \mathcal{T}$ such that M is disjoint from H, K and L , and satisfies $p(R''_M) = p(R'_K)$, and $(R'_K, R''_M) \succ R_H$. Let $\mathcal{C} \subseteq \mathcal{F}$ be the subset of rules associated with a priority \succ in Ω^* .

To conclude on this section, observe that the relation \succ compares in effect triples of the form (x, S_x, R_{S_x}) and (y, S_y, R_{S_y}) where R_{S_x} and R_{S_y} are unan-

¹⁶A *restrictive complete* binary relation \succ over \mathcal{T} is in particular *irreflexive*, i.e. for each unanimous profile R_M , we do not have $R_M \succ R_M$. A priority \succ is not a partial order, as it is not fully transitive. However, priorities have the following important property. The restriction of a priority \succ on any set A of pairwise compatible unanimous profiles is a *linear ordering*, i.e. a binary relation that is irreflexive, asymmetric, transitive and complete. If this set is finite, the priority \succ has a greatest (or top) element in A . A top element for \succ typically does not generally exist on a set of unanimous profiles whose elements are not pairwise compatible, even if it is a finite set.

imous profiles with peaks x and y $-x \neq y$. Thus, $R_{S_x} \succ R_{S_y}$ is the compact notation for $(x, S_x, R_{S_x}) \succ (y, S_y, R_{S_y})$. Therefore, \succ uses information on the location of the peak x , the set of agents having their peak at x , and the shape of the indifference curves of each agent having their peak at x . In Section 7, we will show that by adding *Maskin monotonicity* to our set of axioms, non-peak information will not matter. In such a case, \succ will only compare pairs (x, S_x) and (y, S_y) . That is, $R_{S_x} \succ R_{S_y} \implies R'_{S_x} \succ R'_{S_y}$ for any R'_{S_x} and R'_{S_y} such that $p(R_{S_x}) = p(R'_{S_x})$, $p(R_{S_y}) = p(R'_{S_y})$. To simplify notation, we will then write $S_x \succ S_y$.

Example 1: *Majoritarian rule ("counting peaks").*

For any linear ordering \triangleright on \mathbb{R} , let $\succ \in \Omega$ be the majoritarian priority with tie-breaking rule \triangleright be such that for all compatible unanimous profiles R_L and R'_M , with $p(R'_M) = x$ and $p(R_L) = y$ we have $R'_M \succ R_L$ if either $|M| > |L|$, or $(|M| = |L|$ and $x \triangleright y)$. A rule f is a *majoritarian rule* if it is an element of \mathcal{C} that is associated with a majoritarian priority (for some tie-breaking rule \triangleright). Let \mathcal{M} be the set of majoritarian rules.

For example, let \triangleright be the relation "less than", i.e. the relation $<$. Let \succ be the corresponding majoritarian priority. We describe next how f_\succ allocates facilities when facing two particular problems. Let $N = \{1, \dots, 10\}$, $k = 2$ and the two profiles R_N and R'_N with peak locations distributed as follows.

Profile R_N	Agents	1,...,5	6,7,8	9,10
	Peak locations	0.3	0.5	1

Profile R'_N	Agents	1,2,3	4,5,6	7,8,9	10
	Peak locations	0.2	0.3	0.5	1

Let $\{R_{L_x}\}$ and $\{R'_{M_y}\}$ be the decomposition of R_N and R'_N into unanimous subprofiles. The first decomposition is such that $L_{0.3} = \{1, \dots, 5\}$, $L_{0.5} = \{6, 7, 8\}$, and $L_1 = (9, 10)$. The second partition is such that $M_{0.2} = \{1, 2, 3\}$,

$M_{0.3} = \{4, 5, 6\}$, $M_{0.5} = \{7, 8, 9\}$, and $M_1 = \{10\}$. Since $R_{L_{0.3}} \succ R_{L_{0.5}} \succ R_{L_1}$, we have $f_2(R_N) = \{0.3, 0.5\}$. Since $R_{M_{0.2}} \succ R_{M_{0.3}} \succ R_{M_{0.5}} \succ R_{M_1}$, we have $f_2(R'_N) = \{0.2, 0.3\}$. In the second problem, the unanimous subprofiles $R_{M_{0.2}}$, $R_{M_{0.3}}$ and $R_{M_{0.5}}$ are ordered using the tie-breaking rule \triangleright .

Example 2: Not all rules in \mathcal{C} are *majoritarian*. Let \triangleright be a strict complete order over \mathbb{R} such that $x \triangleright y$ if and only if $x < y$. Let \succ be the priority order over unanimous profiles such that for all two compatible unanimous profiles R_L and R'_M , we have $R_L \succ R'_M$ if either ($|L| > |M|$ and $|L| \geq 3$) or ($|L| = 1$ and $|M| = 2$) or ($|L| = |M|$ and $x \triangleright y$). The rule f_\succ is an element of \mathcal{C} that is not majoritarian.

Example 3: *Majoritarian rules* may differ in the criterion they use to break ties. In example 1, ties were broken in favor of the peak that was the most to the left among the peaks without already a facility. Indeed, more complicated tie-breaking rules are admissible. To see this, let $B_1 = (0, 1]$, $B_2 = [-\infty, 0]$, $B_3 = (1, 2]$ and $B_4 = (2, +\infty)$. Let $\succ_{\mathbb{R}}$ be a partial order on \mathbb{R} , such that $w \succ_{\mathbb{R}} x \succ_{\mathbb{R}} y \succ_{\mathbb{R}} z$ for each $w \in B_1$, $x \in B_2$, $y \in B_3$ and $z \in B_4$. Let \succ_{B_1} , \succ_{B_2} , \succ_{B_3} and \succ_{B_4} be partial orders over B_1 , B_2 , B_3 and B_4 respectively, such that for each B_i , each $x, y \in B_i$, $x \succ_{B_i} y$ if $x < y$. Then, let f be a *majoritarian rule* described as follows. Let $\succ \in \Omega$ be the priority order over unanimous profiles such that for all two compatible unanimous profiles R_L and R'_M , we have $R_L \succ R'_M$ if either

- 1) $|L| > |M|$, or
- 2) $|L| = |M|$, and either
 - a) $p(R_L) \in B_i$, $p(R_M) \in B_j$ and $i < j$, or
 - b) $p(R_L), p(R_M) \in B_i$ for some i and $p(R_L) < p(R_M)$.

4 Axioms

For each profile $R_N \in \mathcal{R}^N$ and each $x, y \in \mathbb{R}$, we say that x weakly Pareto-dominates y for profile R_N if $x R_i y$ for each $i \in N$. This is denoted by $x R_N y$. Our first axiom is the usual *efficiency* axiom.

A solution f satisfies **efficiency** if, for each problem (k, R_N) , there is no k -assignment X such that $X R_N f_k(R_N)$, and $X P_j f_k(R_N)$ for some $j \in N$.

Our next axiom states that, starting from an initial problem, if the social endowment is increased by one facility and the profile is enriched with a group of agents who all have the same peak (i.e. a unanimous profile), no agent initially present should suffer a welfare loss, as a result of these changes.

A solution f satisfies **object population-monotonicity** if, for each problem (k, R_N) , we have $f_{k+1}(R_N) R_N f_k(R_N)$, and, in addition, for each unanimous profile $R'_M \in \mathcal{T}$ such that $N \cap M = \emptyset$, and each $i \in N$ such that $p(R_i) \notin p(R'_M)$ we have $f_{k+1}(R_N, R'_M) R_i f_k(R_N)$.

Our third and last axiom requires that for any profile R_N , and any location x , there is at least one unanimous profile R'_M with peak location x , such that the rule would allocate a single facility to location x when facing the profile (R_N, R'_M) .

A solution f satisfies **sovereignty** if, for each profile R_N , each location $x \in \mathbb{R}$, and each population L , either $f_1(R_N) = \{x\}$ or there exists a unanimous profile $R'_M \in \mathcal{T}$ such that M is disjoint from both L and N , that satisfies $f_1(R_N, R'_M) = \{x\} = p(R'_M)$.

5 Main characterization

We will prove that the three axioms introduced in the previous section characterize the class \mathcal{C} . Before this, the following observations are in order. First, any rule that always selects distinct peaks locations is *efficient*. As a consequence, any rule in \mathcal{F} , therefore any rule in \mathcal{C} , therefore any majoritarian rule, is *efficient*. Second, it can be easily verified that any rule in \mathcal{F} , therefore any rule in \mathcal{C} , therefore any majoritarian rule, satisfies *object-population-monotonicity*. Third and last, any rule in \mathcal{C} , therefore any majoritarian rule, satisfies *sovereignty*. This last point follows from the restriction that \succ is an element of Ω^* . It turns out that the three axioms characterize the class \mathcal{C} .

Theorem 1: *A rule f satisfies efficiency, object-population monotonicity and sovereignty if and only if there exists $\succ \in \Omega^*$ such that $f = f_\succ$, i.e. $f \in \mathcal{C}$.*

Before proving the theorem, we first present two useful lemmas. The first one states that *object-population monotonicity* and *sovereignty* imply the following property.

A solution f satisfies **strong sovereignty** if for each problem (k, R_N) , each location $x \in \mathbb{R}$, each population L , either $x \in f_k(R_N)$, or there exists a unanimous profile R'_M such that M is disjoint from both L and N , that satisfies $p(R'_M) = \{x\} \subseteq f_k(R_N, R'_M)$.

Lemma 1: *If f satisfies object-population monotonicity and sovereignty, then it satisfies strong sovereignty*

Proof. Let (k, R_N) be an arbitrary problem, let $x \in \mathbb{R}$ be an arbitrary location, and let L be an arbitrary population such that $N \subseteq L$. By *sovereignty*, either $f_1(R_N) = x$, or there exists a profile $R'_M \in \mathcal{R}^N$ such that $M \cap L = \emptyset$ and $p(R'_M) = \{x\} = f_1(R_N, R'_M)$. In the first case, there is $i \in N$ such that $p(R_i) = x$. Then by *object-population-monotonicity*, we have $f_k(R_N) \subseteq f_i(R_N)$.

Since $f_1(R_N) = \{x\}$ and $p(R_i) = \{x\}$, this implies that $x \in f_k(R_N)$, the desired conclusion. In the second case, by *object-monotonicity*, we have $f_k(R_N, R'_M) R'_M f_1(R_N, R'_M)$. Since $f_1(R_N, R'_M) = \{x\}$ and $p(R'_M) = \{x\}$, this implies that $x \in f_k(R_N, R'_M)$, the desired conclusion. ■

The second lemma shows that the three axioms of the theorem imply that each public facility must be located at some agent's peak location.

A solution f satisfies **peak-selection** if for each problem (k, R_N) , we have $f_k(R_N) \subseteq p(R_N)$.

Lemma 2: *If f satisfies efficiency, object-population-monotonicity, and sovereignty, then it satisfies peak-selection.*

Proof: The proof is by contradiction. Let (k, R_N) be a problem. By contradiction, suppose that there exists a location $x \in f_k(R_N)$ that is not a peak location, i.e. such that $x \notin p(R_N)$. Throughout the proof, let $p_i := p(R_i)$ for each $i \in N$. Let h, l be two agents such that $p_h < x < p_l$, and no peak location for profile R_N is strictly comprised between p_h and p_l . Since $f_k(R_N)$ is *efficient* for R_N , we have $p_h, p_l \notin f_k(R_N)$. Let u, v be two locations such that $p_h < u < x < v < p_l$. By Lemma 1, there are profiles R'_M and R''_L satisfying the following conditions. Let R'_M be a profile, such that $M \cap N = \emptyset$, $p(R'_M) = \{v\}$, and $v \in f_{k+1}(R_N, R'_M)$. Similarly, let R''_L be a profile, such that $L \cap (N \cup M) = \emptyset$, $p(R''_L) = \{u\}$, and $u \in f_{k+1}(R_N, R''_L)$. Let $A := f_{k+2}(R_N, R'_M, R''_L)$. We will prove that this set contains at least $k + 3$ distinct locations, a contradiction, as we know that this set contains exactly $k + 2$ distinct elements.

By *object-population monotonicity*, both u and v are elements of A , since $u \in f_{k+1}(R_N, R''_L)$, $p(R''_L) = \{u\}$, $v \in f_{k+1}(R_N, R'_M)$ and $p(R'_M) = \{v\}$.

Let y_1, \dots, y_p be the elements of $f_k(R_N) \cap (-\infty, x]$, labelled in increasing order. Also let z_1, \dots, z_q be the elements of $f_k(R_N) \cap [x, +\infty)$, labelled in increasing order. We have $y_q = x = z_1$. From the cardinality of $f_k(R_N)$, we have

$p + q = k + 1$. Since $f_k(R_N)$ is *efficient* for R_N , there exist agents i_1, \dots, i_p and j_1, \dots, j_q in N satisfying the following. For each $r = 1, \dots, p$, let $i_r \in N$ satisfying $p(R_{i_r}) \leq y_r$, and, such that for all $z \in f_k(R_N) \setminus \{y_r\}$, we have $y_r P_{i_r} z$. Similarly, for each $s = 1, \dots, q$, let $j_s \in N$ satisfying $p(R_{j_s}) \geq z_s$, and, such that for all $y \in f_k(R_N) \setminus \{z_s\}$, we have $z_s P_{j_s} y$. Since $x \notin p(R_N)$, we have $i_p \neq j_1$. By construction, all agents $i_1, \dots, i_p, j_1, \dots, j_q$ are distinct. For each $r = 1, \dots, p$, let $U_r := \{w \in \mathbb{R} : w R_{i_r} y_r\}$. Similarly, for each $s = 1, \dots, q$, let $V_s := \{w \in \mathbb{R} : w R_{j_s} z_s\}$. By construction, all of the U_r and V_s are closed intervals. In addition, y_r is the upper-bound of U_r for each $r = 1, \dots, p$, and z_s is the lower bound of V_s for each $s = 1, \dots, q$. For each $r = 1, \dots, p-1$, we have $y_{r+1} P_{i_r} y_r$. This implies $y_r \notin U_{r+1}$, i.e. $U_r \cap U_{r+1} = \emptyset$. The U_r are therefore pairwise disjoint. A similar argument shows that the V_s are also pairwise disjoint. We have

$$U_1 < \dots < U_p \leq x \leq V_1 < \dots < V_q.$$

By *object-population monotonicity*, each of the sets $U_r \cap A$ for $r = 1, \dots, p-1$ and each of the sets $V_s \cap A$, for $s = 2, \dots, q$ are nonempty. Let a_1, \dots, a_{p-1} and b_2, \dots, b_q be locations selected from each of these sets. We have $a_1 < \dots < a_{p-1} < b_2 < \dots < b_q$.

By *object-population monotonicity*, the set $U_p \cap f_{k+1}(R_N, R'_M)$ is nonempty. Since $v \in f_{k+1}(R_N, R'_M)$ and v is the only peak of profile (R_N, R'_M) strictly comprised between p_h and p_l , and since $f_{k+1}(R_N, R'_M)$ is *efficient* for profile (R_N, R'_M) , it follows that no location in $f_{k+1}(R_N, R'_M)$ distinct from v is strictly comprised between p_h and p_l . Since $p_h < x < v < p_l$, and x is the upper-bound of U_p , we proved that there exists a location $a \in f_{k+1}(R_N, R'_M)$ such that $a R_{i_p} x$ and $a \leq p_h$ is non-empty. By *object-population-monotonicity*, there is a location $a_p \in A$ such that $a_p R_{i_p} a$. Since $a R_{i_p} x$, we have by transitivity $a_p R_{i_p} x$, i.e. $a_p \in U_p$. Since $p_{i_p} \leq a \leq p_h$, we have $a_p \leq p_h$. We can prove similarly that there is a location $b_1 \in A$ such that $b_1 \in V_1$ and $p_l \leq b_1$. We have

$a_p \leq p_h < u < v < p_l < b_1$. We have shown that A contains the $k + 3$ distinct elements $a_1, \dots, a_p, u, v, b_1, \dots, b_q$, the desired contradiction. ■

We are now ready to prove Theorem 1.

Proof of Theorem 1: As we pointed out at the beginning of this section, it is easy to see that all the elements of the class \mathcal{C} satisfy the three axioms. We still need to prove that all rules that satisfy the three axioms are rules in the set \mathcal{C} . Let f be an arbitrary rule that satisfies *efficiency*, *object-population monotonicity* and *sovereignty*.

Step 1: Construction of a candidate priority $\succ \in \Omega^$ from f .*

By *peak-selection*, for each two compatible unanimous profiles R_L and R'_M , we have $f_1(R_L, R'_M) \in \{p(R_L), p(R_M)\}$. Let \succ be the binary relation over unanimous profiles such that, for each two compatible unanimous profiles R_L and R'_M , we have $R_L \succ R'_M$ if $f_1(R_L, R'_M) = p(R_L)$. By construction, the relation \succ is *asymmetric* and *restricted complete*. It remains to show that \succ is *restricted transitive*. Consider three arbitrary unanimous profiles R_K , R'_L and R''_M such that R_K and R''_M are compatible. Suppose that $R_K \succ R'_L$ and $R'_L \succ R''_M$. Then in particular, R_K and R'_L are compatible, and R'_L and R''_M are compatible. By definition of \succ , we know that $f_1(R_K, R'_L) = p(R_K)$ and $f_1(R'_L, R''_M) = p(R'_L)$. This and *object-population monotonicity* imply that $p(R_K), p(R'_L) \in f_2(R_K, R'_L, R''_M)$. By compatibility, we have $p(R_K) \neq p(R'_L)$. Therefore $f_2(R_K, R'_L, R''_M) = \{p(R_K), p(R'_L)\}$. But this and *object-population monotonicity* imply that $f_1(R_K, R''_M) = p(R_K)$, i.e. $R_K \succ R''_M$, the desired conclusion. Therefore \succ is a priority in Ω .

It only remains to show that \succ belongs to the subset Ω^* . Let R_H and R'_K be two arbitrary unanimous profiles satisfying $R_H \succ R'_K$, and let L be an arbitrary population. By *sovereignty*, since $f_1(R_H, R'_K) \neq p(R'_K)$, then there exists a unanimous profile R''_M satisfying $M \cap (H \cup K \cup L) = \emptyset$, and

$f_1(R_H, R'_K, R''_M) = p(R''_M) = p(R'_K)$. Since (R'_K, R''_M) is a then a unanimous profile, we have $(R'_K, R''_M) \succ R_N$. Therefore $\succ \in \Omega^*$.

Step 2: Let \succ be defined as in step 1 from f . Let R_{M_1}, \dots, R_{M_l} be an arbitrary list of pairwise compatible triples, with respective peak locations x_1, \dots, x_l , such that $R_{M_1} \succ \dots \succ R_{M_l}$. Let $N := M_1 \cup \dots \cup M_l$, so that $R_N := (R_{M_1}, \dots, R_{M_l})$. Then for all $k < l$, we have $f_k(R_N) = \{x_1, \dots, x_k\}$.

The proof is by induction on $d = l - k$. We first prove the claim for $d = 1$. Let R_{M_1}, \dots, R_{M_l} be an arbitrary list. Let h be an arbitrary index in $\{1, \dots, l - 1\}$. By definition of \succ , and since $R_{M_h} \succ R_{M_l}$, we have $f_1(R_{M_h}, R_{M_l}) = x_h$. By *object-population-monotonicity*, we then also have $x_h \in f_{l-1}(R_N)$. Therefore $\{x_1, \dots, x_{l-1}\} \subseteq f_{l-1}(R_N)$. Since these two sets also have the same cardinality, they are therefore equal, which proves the claim for $d = 1$.

Let $d > 1$. Let R_{M_1}, \dots, R_{M_l} be an arbitrary list, and k an arbitrary integer such that $l - k = d$. Suppose that the claim is true for $d - 1$. By contradiction, suppose that there is an index $g \leq k$ such that $x_g \notin f_k(R_N)$. Since $k \leq l - 2$, there exists an index satisfying the following. Let j be an index such that $j \neq g$ and $x_j \notin f_k(R_N)$. *Strong sovereignty* ensures that a profile satisfying the following exists. Let R'_j be a unanimous profile such that $N \cap J = \emptyset$, $p(R'_j) = x_j$, and $x_j \in f_{k+1}(R_N, R'_j)$. By *object-population monotonicity*, we have $f_k(R_N) \subset f_{k+1}(R_N, R'_j)$. Since $x_j \notin f_k(R_N)$, and from the cardinality of the sets, we have $f_{k+1}(R_N \setminus M_j, R'_j) = f_k(R_N) \cup \{x_j\}$. In particular, this set does not contain x_g , in contradiction with the induction hypothesis for $l - (k + 1) = d - 1$. Therefore the claim is true for $l - k = d$, i.e. for all l and k such that $l - k \geq 1$.

It follows from Step 2 that any rule satisfying the axioms is such that $f = f_\succ$ where \succ is defined as in Step 1. ■

We verify that the axioms are independent. First, the left-peaks solution f_k^{LP}

which assigns the k facilities to the first k distinct peak locations when peak locations are counted from left to right satisfies *efficiency* (even *peak selection*), and *object-population monotonicity* but violates *sovereignty*. More generally, any rule in $\mathcal{F} \setminus \mathcal{C}$ satisfies the first two axioms but violates the last. Second, the rule f such that f_1 selects the location x_m of the median voter when $k = 1$, and such that f_k coincides with the left-peaks solution for all $k \geq 2$ satisfies *efficiency* and *sovereignty*, but violates *object-population monotonicity*. Third, the rule f such that f_1 selects the location x_m of the median voter, and in addition (when $k \geq 2$), the $k - 1$ smallest positive integers distinct from x_m satisfies *sovereignty* and *object-population monotonicity*, but violates *efficiency*.

Our result is a partial characterization of the family of priority rules \mathcal{F} . We pointed out already that any rule $f \in \mathcal{F} \setminus \mathcal{C}$ satisfies *efficiency* and *object-population monotonicity* but violates *sovereignty*. Nevertheless, *efficiency* and *object-population monotonicity* are not sufficient to characterize \mathcal{F} . There are rules that satisfy both axioms but cannot be associated with any priority ordering $\succ \in \Omega$. We highlight this claim with the following example.

Example 4: A way to pave the gap between \mathcal{C} and \mathcal{F} is to weaken *sovereignty* to an axiom that we call *asymptotic sovereignty*.

A solution f satisfies **asymptotic sovereignty** if there exists a sequence of solution $\{f^\epsilon\}_{\epsilon=0}^\infty$ converging to f and such that f^ϵ satisfies *sovereignty* for each $\epsilon \geq 0$.

Theorem 2: *A rule f satisfies efficiency, object-population monotonicity and asymptotic sovereignty if and only if there exists $\succ \in \Omega$ such that $f = f_\succ$, i.e. $f \in \mathcal{F}$.*

Proof:

6 Hiding-proofness

In this section, we characterize the subclass of rules in \mathcal{C} that ensure that no agent gains from hiding from the social planner. To motivate this axiom, let us step aside from our example concerning the European union and focus instead on one particular member. In the Netherlands, tax forms are available in several languages (but obviously not all!) other than Dutch. These additional languages are representative of the proportion of “immigrant” communities present in the country. In addition to the three axioms we used previously, we would like agents not to have an incentive not to report their mother tongue to the tax authorities or the city-hall¹⁷ That is, agent i reporting himself to the authorities and declaring that his mother tongue is not Dutch cannot negatively affect him. Let us now define formally this property.

A solution f satisfies **hiding-proofness** if for each problem (k, R_N) , and each $i \in N$, we have $f_k(R_N) R_i f_k(R_{N \setminus \{i\}})$.

The following property plays a role in the characterization. A priority \succ is *monotone in population* if, for all unanimous profiles R_L, R'_M and R''_N such that $L \subseteq N$, R'_M is compatible with both R_L and R''_N , $p(R_L) = p(R''_N)$ and $R''_L = R_L$, we have $R_L \succ R'_M \Rightarrow R''_N \succ R'_M$. Let \mathcal{C}_H be the class of rules in \mathcal{C} such that the priority \succ is *monotone in population*. The following holds.

Proposition 2: *For all rule $f \in \mathcal{C}$, f satisfies hiding-proofness if and only if $f \in \mathcal{C}_H$.*

Proof. The *if* implication is clear. Let us prove the *only if* implication. Let f_\succ be a rule in \mathcal{C} that satisfies *hiding-proofness*. Let R_L, R'_M and R''_N be unanimous profiles such that $L \subseteq N$, R'_M is compatible with both R_L

¹⁷Given that paying taxes is in general painful, one could however understand why agents may not want to report themselves freely to tax authorities!

and R''_N , $p(R_L) = p(R''_N)$ and $R''_L = R_L$, and $R_L \succ R'_M$. We will prove that $R''_N \succ R'_M$ holds. Since $R_L \succ R'_M$, we have $f_1(R_L, R'_M) = p(R_L) = p(R''_N)$. By *hiding-proofness*, we have $f_1(R''_{N \setminus L}, R_L, R'_M) = f_1(R_L, R'_M)$. This and $f_1(R_L, R'_M) = p(R''_N)$ imply $f_1(R''_N, R'_M) = p(R''_N)$. But this implies $R''_N \succ R'_M$, the desired conclusion. ■

Interestingly, the rules in the class \mathcal{C}_H satisfy a property that is not directly related to any of the four axioms (efficiency, *object-population monotonicity*, *sovereignty* and *hiding-proofness*) that characterize it. *Consistency* (Thomson, 1988) requires that if we remove a public good and the group of agents that were assigned to it, the solution applied to the remaining agents and the remaining public facilities is unchanged.¹⁸

A solution f satisfies **consistency** if, for each problem (k, R_N) , each $x \in f_k(R_N)$, and each population L satisfying

$$\{i \in N : x P_i Y\} \subseteq L \subseteq \{i \in N : x R_i Y\},$$

with $Y := f_k(R_N) \setminus \{x\}$, we have $f_{k-1}(R_{N \setminus L}) = Y$.

Proposition 3: *Any rule $f \in \mathcal{C}_H$ satisfies consistency.*

Proof. Let $f \in \mathcal{C}_H$ be associated with the priority \succ . Let (k, R_N) be an arbitrary problem. Let $x \in f_k(R_N)$ and let L be a population satisfying the above inclusions. Let $Y := f_k(R_N) \setminus \{x\}$. We will show that $f_{k-1}(R_{N \setminus L}) = Y$. Let $R_{M_1}, \dots, R_{M_{l-1}}, R_M$ be the decomposition of the profile R_N into unanimous subprofiles, labelled so that $p(R_M) = x$ and the first $l - 1$ subprofiles are ordered in decreasing order for the priority \succ . Let y_1, \dots, y_{l-1} the respective peak locations of these first $l - 1$ subprofiles. We have $f_k(R_N) = \{y_1, \dots, y_{l-1}, x\}$,

¹⁸This definition of *consistency* is the usual one and is stronger than the definition used by Barberà and Beviá (2002) or Ju (2005).

so that $Y = \{y_1, \dots, y_{k-1}\}$. By the second inclusion restriction on L , we have $p(R_L) \subseteq \{y_{k+1}, \dots, y_{l-1}, x\}$. This implies $Y \subseteq p(R_{N \setminus L})$. Let us now examine the decomposition in unanimous profiles of the profile $R_{N \setminus L}$. There is a set $J \subset \{k, \dots, l-1\}$ of indices such that this decomposition consists of the profiles $R_{M_1}, \dots, R_{M_{k-1}}$ and, in addition, the profiles $R_{H_j}, H_j \subseteq M_j$ for $j \in J$, i.e. R_{H_j} is a subprofile of R_{M_j} . For all $h \in \{1, \dots, k-1\}$, and all $j \in J$, we have $R_{M_h} \succ R_{M_j}$. Since \succ is *monotone in population*, then we further have $R_{M_h} \succ R_{H_j}$. Therefore the peak locations $k-1$ highest priority unanimous subprofiles in the decomposition of $R_{N \setminus L}$ are y_1, \dots, y_{k-1} . Therefore $f_{k-1}(R_N) = Y$, the desired conclusion. ■

The converse of Proposition 3 is false. Not all rules in \mathcal{C} that satisfy consistency are elements of \mathcal{C}_H .

7 Maskin-monotonicity

The following is a well-known axiom of stability of a rule with respect to changes in preferences. This axiom was first introduced by Maskin (1977, published in 1999) as a necessary condition for Nash implementation.

A solution f satisfies **Maskin monotonicity** if for each problem (k, R_N) and for each $R'_N \in \mathcal{R}^N$,

$$[LC_i(R_N, f_k(R_N)) \subseteq LC_i(R'_N, f_k(R_N)) \text{ for each } i \in N] \implies [f_k(R'_N) = f_k(R_N)].$$

We first show that an important member of \mathcal{C} violates this condition.

Lemma 4: *No majoritarian rules satisfy Maskin monotonicity*

Proof: Let $N = \{1, 2, 3, 4, 5\}$, $k = 1$, $\mathcal{R}^N = \{R_N, R'_N\}$, and $p(R_1) = w$, $p(R_2) = x$, $x > w$, $p(R_3) = y$, $y > x$ while $p(R_4) = p(R_5) = z$ with $z > y$. Moreover, the profile R_N is chosen so that there exists $u \in \mathbb{R}$ such that $u I_i z$ for $i = 1, 2$. Since f is a majoritarian rule, we obtain that $f_1(R_N) = z$.

The profile R'_N is as follows. We have $R'_N = (R'_1, R'_2, R_3, R_4)$, with $p(R'_1) = p(R'_2) = y$ and $u I'_i z$ for $i = 1, 2$. By construction, $LC_i(R_N, z) \subseteq LC_i(R'_N, z)$ for $i = 1, 2$ while, trivially, the lower contour sets of the remaining agents are unchanged. But notice that $f_1(R'_N) = y$ contradicting *Maskin monotonicity*. As a consequence, no majoritarian rule is Nash implementable. ■

We now introduce a family of rules $\mathcal{H} \subset \mathcal{C}$ that we call *hierarchical left-right peaks rules*. Since $\mathcal{H} \subset \mathcal{C}$, for each $f \in \mathcal{H}$, there exists $\succ \in \Omega^*$ such that $f = f_\succ$. The construction is based on the partition of \mathbb{N} into a family of disjoint subsets representing hierarchies.

A **hierarchy** $S_\ell \subset \mathbb{N}$ is a set of agents such that for each group $s_l, s_m \subset S_\ell$, s_l and s_m are hierarchically equivalent, denoted $s_l \sim_H s_m$. Moreover, for each two distinct hierarchies S_ℓ and S_k , there does not exist $s_l \subset S_\ell, s_k \subset S_k$ such that $s_l \sim_H s_k$.

We define next the notion of hierarchical superiority among different hierarchies and collection of hierarchies. For each two distinct hierarchy S_ℓ and S_k , either S_ℓ is **hierarchically superior** to S_k or the other way around. For each pair of distinct hierarchies S_ℓ, S_k , let $S_\ell >_H S_k$ mean that S_ℓ is hierarchically superior to S_k . Finally, consider two groups L and M . Let $\cup L_l$ and $\cup M_m$ be the decompositions of L and M into the different hierarchies composing them. Then, we say that $L >_H M$ if there exists L_i in $\cup L_l$ such that $L_i >_H M_j$ for each M_j in $\cup M_m$. Likewise, we say that $M >_H L$ if there exists M_j in $\cup M_m$ such that $M_j >_H L_i$ for each L_i in $\cup L_l$. Otherwise, we write $L \sim_H M$.

We next introduce the definition of a *hierarchical system*.

A **hierarchical system** $S = (S_1, S_2, \dots, S_h, S_k, \dots)$ is a collection of subsets S_ℓ of \mathbb{N} , each representing hierarchies, and such that,

- (i) $\mathbb{N} = \cup S_\ell$
- (ii) $S_h \cap S_k = \emptyset$ for each $h \neq k$

- (iii) For each S_h, S_k either $S_h >_H S_k$ or $S_k >_H S_h$
- (iv) For each S_h , there exists S_k such that $S_k >_H S_h$
- (v) For each S_h and $s_l, s_m \subset S_h$, $s_l \sim_H s_m$

A rule f with associated priority order $\succ \in \Omega^*$ such that $f = f_\succ$ is a **hierarchical left-right peak rule** if there exists a *hierarchical system* S such that,

- (i) For each L, M such that $L >_H M$, $f_1(R_L, R_M) = p(R_L)$ for each R_L and R_M
- (ii) For each L, M such that $L \sim_H M$, either
 - a) $f_1(R_L, R_M) = f^{LP}(R_L, R_M)$ for each R_L and R_M , or
 - b) $f_1(R_L, R_M) = f^{RP}(R_L, R_M)$ for each R_L and R_M

In words, given a hierarchy S_ℓ and a non-unanimous profile R_{S_ℓ} , the priority for agents within S_ℓ is given either to the agent $i \in S_\ell$ who is the most to the left or to the right—hence the name *hierarchical left peak-right peak rule*. Therefore, given S_ℓ and $R_{S_\ell} \in \mathcal{R}^{S_\ell}$, partition S_ℓ into subgroups s_l comprising agents with the same peak, i.e. $S_\ell = s_1 \cup s_2 \cup \dots \cup s_n$ such that $|p(R_{s_l})| = 1$ for each $l = 1, \dots, n$ and $p(R_{s_1}) < p(R_{s_2}) < \dots < p(R_{s_n})$. If the *left-peaks rule* applies, $R_{s_l} \succ R_{s_m}$ for each $l < m$. If the *right-peaks rule* applies, $R_{s_l} \succ R_{s_m}$ for each $l > m$. Thus, within group S_ℓ , agents are hierarchically identical. Indeed, within a hierarchy, ties are broken using a unanimous criterion that is function of the location of the peaks.

In order to fix ideas, let us give several examples.

Example 5: Let $f \in \mathcal{H}$, with associated priority order \succ , be such that for each unanimous and compatible profiles R_{S_x} and R_{S_y} , we have that $R_{S_x} \succ R_{S_y}$ if and only if there exists $i \in S_x$ such that $i > j$ for each $j \in S_y$. This implies that for each such i and j , $R_{\{i\}_x} \succ R_{\{j\}_y}$. Thus, for each singletons $\{i\}, \{j\}$

with $i > j$, R_i and R_j with $p(R_i) \neq p(R_j)$, we have $R_i \succ R_j$. The *hierarchical system* associated with f and \succ is $S = (\{1\}, \{2\}, \dots)$ with $\{i\} \succ_H \{j\}$ for each $i > j$.

Indeed, f satisfies *efficiency*, *object-population monotonicity*, *sovereignty* and *Maskin monotonicity*. Observe how strong *Maskin monotonicity* is in this context. Whenever a group S_x is given priority over a group S_y , it implies that there exists an $i \in S_x$ such that i is given priority over each agent $j \in S_y$.

Interestingly, with hierarchies constructed as above, and with N drawn from \mathbb{N} and fixed, we get that f is a *serial-dictatorship*. Indeed, no agent is a global dictator in the usual sense since by considering $N' \supset N$ with N' containing members of hierarchies of higher order, the order of priority is modified.¹⁹

Example 6: Let $f \in \mathcal{H}$, with associated priority order \succ , be such that for each $k > j$, $k > 2$ and each compatible R_k and R_j , we have,

$$R_k \succ R_j.$$

Otherwise,

$$\begin{aligned} R_1 &\succ R_2 \text{ if } p(R_1) < p(R_2), \\ R_2 &\succ R_1 \text{ if } p(R_1) > p(R_2). \end{aligned}$$

The hierarchical system is $S = (\{1, 2\}, \{3\}, \{4\}, \dots)$ with $\{1, 2\}_H < \{3\}_H < \{4\}_H < \dots$. Observe that the *hierarchical system* has indeed no top. Notice that if \mathbb{N} is not partitioned and is simply an infinite indifference class, this

¹⁹In this example, we have a ranking over the set of potential such that the agent with the highest ranking among those actually present—i.e. the set N —chooses his best set of objects, then the agent with the next highest ranking chooses his best set of objects among the remaining objects, and so on. Formally, let π be a permutation on \mathbb{N} . Then the serial dictatorship with respect to π is defined as follows. Given a problem (k, R_N) and $\{1, \dots, n\} = N \subset \mathbb{N}$ with $\pi(1) < \pi(2) < \dots < \pi(n)$, $f_k(R_N) = \{p(R_{\pi(1)}), \dots, p(R_{\pi(k)})\}$.

would imply that either f^{LP} or f^{RP} would apply to each problem (k, R_N) , a contradiction with $f^{LP}, f^{RP} \notin \mathcal{C}$.

Our last example shows that a *hierarchical system* may also have no bottom.

Example 7: Let $f \in \mathcal{H}$ be a rule with associated priority ordering \succ defined as follows. Let $E \subset \mathbb{N}$ be the set of even numbers and $O \subset \mathbb{N}$ be the set of odd numbers.

For each $i, j \in E$ and each $R_i, R_j \in \mathcal{R}$,

$$R_i \succ R_j \implies i > j.$$

For each $l, m \in O$ and each $R_l, R_m \in \mathcal{R}$,

$$R_l \succ R_m \implies l < m.$$

Finally, for each $i \in E$, each $l \in O$ and each $R_i, R_l \in \mathcal{R}$, we have,

$$R_i \succ R_l.$$

Therefore, for each unanimous compatible R_k with associated group S_k , R_h with associated group S_h , we have that $R_k \succ R_j$ if there exists $i \in S_k$ such that $R_i \succ R_j$ for each $j \in S_h$.

We are now ready to state our last theorem. Before this, we introduce one last property that will be useful for its proof.

A solution f satisfies **peak-only** if, for each problem (k, R_N) and each $R'_N \in \mathcal{R}^N$,

$$[p(R_i) = p(R'_i) \text{ for each } i \in N] \implies [f_k(R_N) = f_k(R'_N)].$$

Theorem 3: *A rule f satisfies efficiency, object-population monotonicity, sovereignty and Maskin monotonicity if and only if $f \in \mathcal{H}$.*

Proof: Example 5 to 7 show that the class $\mathcal{H} \subset \mathcal{C}$ is non-empty. It is easy to see that any rule in \mathcal{H} satisfies *efficiency*, *object-population monotonicity*, *sovereignty* and *Maskin monotonicity*. Next, we want to show that any rule satisfying our axioms is necessarily a member of \mathcal{H} . We prove the theorem in several steps.

Step 1: Let f be a rule satisfying efficiency, object-population monotonicity, sovereignty and Maskin monotonicity. Then f satisfies peak-only.

Let f be a rule that satisfies the axioms and with associated priority ordering \succ . Let $(1, R_N)$ be an arbitrary problem.

Case 1: Let $i \in N$ be such that $p(R_i) = x = f_1(R_N)$. This assumption is guaranteed by *peak-selection*. Consider a change in preferences from R_i to R'_i such that $p(R'_i) = x$.

By *Maskin monotonicity*, for each such agent i , $f_1(R_{N \setminus \{i\}}, R'_i) = f_1(R_N)$ for any R'_i such that $p(R'_i) = x$.

Case 2: Let $i \in N$ be such that $p(R_i) = x \neq f_1(R_N)$. Consider a change in preferences from R_i to R'_i such that $p(R'_i) = x$.

Suppose first that $x = f_k(R_{N \setminus \{i\}}, R'_i)$. Transform R'_i back to R_i and observe that this is a monotonic transformation at x implying that $f_1(R_N) = f_k(R_{N \setminus \{i\}}, R'_i)$, a contradiction. Next, suppose that $x \neq f_1(R_{N \setminus \{i\}}, R'_i) \neq f_1(R_N)$. By *peak selection*, there exist groups S_w, S_z with $w = f_1(R_N)$ and $z = f_1(R_{N \setminus \{i\}}, R'_i)$. We obtain that $R_{S_w} \succ R_{S_z}$ and $R_{S_z} \succ R_{S_w}$, a contradiction.

Thus, for each unanimous compatible profiles R_L and R_M , we have,

$$R_L \succ R_M \implies R'_L \succ R'_M \text{ for each } R'_L, R'_M \text{ with } p(R'_L) = p(R_L) \text{ and } p(R'_M) = p(R_M).$$

Since considering problems with one facility is enough to construct the priority order \succ associated with f , we conclude that f satisfies *peak-only*. Given that

only peak-information matters, from now on, whenever $R_{S_x} \succ R_{S_y}$, we will use the notation $S_x \succ S_y$.

Step 2: For all $w, x, y, z \in \mathbb{R}$ such that $w < x$ and $y < z$, for all disjoint populations S and S' we have

$$S_w \succ S'_x \Leftrightarrow S_y \succ S'_z.$$

We consider two different cases that once combined will prove the claim. Let f be a rule satisfying the axioms and with associated priority ordering \succ .

1) $w = y$ and $z \neq x$.

Let $(1, (R_{S_w}, R_{S'_x}))$ be a problem with $f_1(R_{S_w}, R_{S'_x}) = w$, and $u \in \mathbb{R}$, with $u > x$, $u > z$, such that for each $i \in S'_x$, $w I_i u$. By construction, $S_w \succ S'_x$.

Choose $R'_{S'_x}$ such that $p(R'_{S'_x}) = z$ and $w I_i u$ for each $i \in S'_x$. Observe that $R'_{S'_x}$ is a monotonic transformation at w of $R_{S'_x}$. Let $S'_z = S'_x$ and $R'_{S'_z} = R'_{S'_x}$. By *Maskin monotonicity*, $f_1(R_{S_w}, R'_{S'_z}) = w$. Hence, $S_w = S_y \succ S'_z$, as claimed.

2) $z = x$ and $w \neq y$.

Let $(1, (R_{S_w}, R_{S'_x}))$ be a problem with $f_1(R_{S_w}, R_{S'_x}) = w$, $u \in \mathbb{R}$, with $u < y$, $u < w$ such that for each $i \in S_w$, $u I_i x$. By construction, $S_w \succ S'_x$.

Choose R_{S_y} such that $p(R_{S_y}) = y$, $u I_i x$ for each $i \in S_y$ and $|S_y| = |S_w|$. Suppose that $f_1(R_{S_y}, R_{S'_x}) = x$. Observe that R_{S_w} is a monotonic transformation at x of R_{S_y} . By *Maskin monotonicity*, $f_1(R_{S_w}, R_{S'_x}) = x$, a contradiction with our initial assumption. Hence, $S_y \succ S'_z = S'_x$, as claimed.

The only cases left are when both $w \neq y$ and $x \neq z$. A Combination of 1) and 2) proves the claim.

Step 3: For all $x, y \in \mathbb{R}$, all disjoint populations S_x, S_y , all populations S'_x and S'_y such that $S_x \subseteq S'_x$ and $S'_y \subseteq S_y$, we have

$$S_x \succ S_y \Rightarrow S'_x \succ S'_y.$$

Let f be a rule satisfying the axioms and with associated priority ordering \succ . Let $(1, R_N)$ be a problem with $R_N = (R_{S_x}, R_{S_y}, \dots)$, $x < y$ and such that $f_1(R_N) = x$. By construction, we also have that $f_1(R_{S_x}, R_{S_y}) = x$. Given R_N , assume there exists a group $S_z \neq R_{S_x}, R_{S_y}$ with $p(R_{S_z}) = z$. Select $M' \subset S_z$ and $R'_{M'} \in \mathcal{R}^{M'}$ such that $p(R'_{M'}) = x$. Let $R'_{S'_x} = (R_{S_x}, R'_{M'})$. By *Maskin monotonicity*, $f_1(R_{N \setminus M'}, R'_{M'}) = x$. In conclusion, we also have that $f_1(R'_{S'_x}, R_{S_y}) = x$. Thus $S_x \succ S_y \implies S'_x \succ S_y$ for $S'_x \supset S_x$. This establishes that \succ is *monotone in population*.

1) Let $M \subset S_y$, and let $R'_M \in \mathcal{R}^M$ be such that $p(R'_M) = x$. By *Maskin monotonicity*, $f_1(R_{N \setminus M}, R'_M, R'_{M'}) = f_1(R_N) = x$.

2) The last thing that remains to be proven is that $S'_x \succ S_y \implies S'_x \succ S'_y$ for $S'_y \subset S_y$. Let $M \subset S_y$ and $R'_M \in \mathcal{R}^M$ be such that $p(R'_M) = w \neq x, y$. Let $R'_{S'_x} = (R_{S_x}, R'_{M'})$, $S'_x \equiv S_x \cup M'$ and $S'_y \equiv S_y \setminus M$.

Suppose that $f_1(R'_{S'_x}, R_{S'_y}) = y$, i.e. $R_{S'_y} \succ R'_{S'_x}$. By *monotonicity in population*, $R_{S_y} \succ R'_{S'_x}$. This implies that $f_1(R'_{S'_x}, R_{S_y}) = y$, a contradiction with \succ being *monotone in population*.

Step 4: For each (x, S_x) and (y, S_y) with $x \neq y$, we have

$$S_x \succ S_y \iff \exists i \in S_x \text{ such that,}$$

- 1) $\{i\}_x \succ S_y$, and
- 2) $\{i\}_x \succ \{j\}_y$ for each $j \in S_y$.

Let f be a rule satisfying the axioms and with associated priority ordering \succ . Let $(1, R_N)$ be a problem with $N = S_x \cup S_y$, $x < y$, $x = f_1(R_N)$, and such that there exists $u < x$ with $u I_i y$ for each $i \in S_x$.

1) We establish the existence of an agent $i \in S_x$ such that $\{i\}_x \succ S_y$. Suppose that such an agent i does not exist. Consider the following transformation of

R_N and of S_x . Leave one agent i at x and move each $j \in S_x \setminus \{i\}$ to the left of x such that $\bigcap_{j \in S_x \setminus \{i\}} p(R'_j) = \emptyset$ and such that $u I'_j y$ for each $j \in S_x \setminus \{i\}$. Given such a construction, observe that $R'_{S_x \setminus \{i\}}$ can be transformed back to $R_{S_x \setminus \{i\}}$ via a monotonic transformation at y . The choice of any such $i \in S_x$ implies that $S_y \succ \{i\}_x$. For each $j \in S_x \setminus \{i\}$, apply a monotonic transformation at y by changing each R'_j back to R_j . By *Maskin monotonicity*, $y = f_1(R_N)$, a contradiction with our initial assumption. This establishes that there exists at least one agent $i \in S_x$ such that $\{i\}_x \succ S_y$.

2) Once an agent $i \in S_x$ such that $\{i\}_x \succ S_y$ has been found, we need to show that $\{i\}_x \succ \{j\}_y$ for each $j \in S_y$. A direct application of *step 3* finishes the proof.

Step 5: Let f be a rule satisfying the axioms and with associated priority ordering \succ . There exists two relations \succ_R and \succ_L such that for all $x \neq y$, $S_x, S_y, R_{S_x} \in \mathcal{R}^{S_x}$ and $R_{S_y} \in \mathcal{R}^{S_y}$, we have

$$\begin{aligned} R_{S_x} \succ_R R_{S_y} &\iff R_{S_x} \succ R_{S_y}, y < x, \\ R_{S_x} \succ_L R_{S_y} &\iff R_{S_x} \succ R_{S_y}, y > x. \end{aligned}$$

The relations \succ_R and \succ_L are simply a decomposition of \succ into two subpriority orderings. Their existence is thus trivially guaranteed.

Step 6: Let f be a rule satisfying the axioms and with associated priority ordering \succ . For all $x \neq y$, $S_x, S_y, R_{S_x} \in \mathcal{R}^{S_x}$ and $R_{S_y} \in \mathcal{R}^{S_y}$, we have

$$\begin{aligned} R_{S_x} \succ_R R_{S_y} &\iff S_x \succ_R S_y \\ R_{S_x} \succ_L R_{S_y} &\iff S_x \succ_L S_y \end{aligned}$$

The result is immediately implied by *step 1*.

Step 7: Let \succ be a priority order. Then \succ_R and \succ_L have no maximums, i.e. $\nexists \{i\} \subset \mathbb{N}$ such that $\{i\}_x \succ_R \{j\}_y$ or $\{i\}_x \succ_L \{j\}_y$ for each $y \neq x$ and $j \neq i$.

Let f be a rule satisfying the axioms and with associated priority order \succ . Suppose that there exists $\{i\} \subset \mathbb{N}$ such that $\{i\}_x \succ_L \{j\}_y$ for each $y > x$ and $j \neq i$. This implies that for any $N' \subset \mathbb{N}$, $R_{\{i\}_x} \succ_L R_{\{j\}_y}$ for each $y > x$ and each $\{j\} \in N'$.

Let $(1, R_N)$ be a problem with $p(R_i) = x$, $x < p(R_j)$ for each $j \neq i$, $\bigcap_{j \neq i} p(R_j) = \emptyset$ and such that there exists $u \in \mathbb{R}$ with $x I_j u$ for each $j \neq i$. By assumption, $f_1(R_i, R_{N \setminus \{i\}}) = x$.

Select $k \in N$ such that $p(R_k) < p(R_j)$ for each $j \neq i, k$. Apply a monotonic transformation at x , for each $j \neq i, k$, from R_j to R'_j such that $x I'_j u$ and $p(R'_j) = p(R_k)$. By *Maskin monotonicity*, $f_1(R_i, R_k, R'_{N \setminus \{i, k\}}) = x$.

Let $N' \subset \mathbb{N}$ be a set of agents with $N' \cap N = \emptyset$, $R_{N'} \in \mathcal{R}^{N'}$ a profile with $\bigcap_{i \in N', i \in N} p(R_i) = \emptyset$, $p(R_j) > p(R_k)$ and $x I_j u$ for each $j \in N'$. Apply a monotonic transformation at x , for each $j \in N'$, from R_j to R'_j such that $x I'_j u$. By *Maskin monotonicity*, $f_1(R_i, R_k, R_{N \setminus \{i, k\}}, R_{N'}) = x$. We can repeat this process ad infimum and x will always be selected. Thus, there does not exist M with $M \cap N = \emptyset$ and profile R_M with $p(R_M) = p(R_k)$ and such that $f_1(R_N, R_M) = p(R_k)$. Therefore, *sovereignty* is violated, a contradiction. The proof for \succ_R is identical and hence we omit it.

Step 8: For each S_x, S_y , $R_{S_x} \in \mathcal{R}^{S_x}$, $R_{S_y} \in \mathcal{R}^{S_y}$, $i, k \in S_x$ and $j \in S_y$,
 $R_{\{i\}_x} \succ_L R_{\{j\}_y} \succ_R R_{\{k\}_x} \implies \{i\}_x \succ \{k\}_z$ for all $z \neq x$.

1) Suppose there exists z with $x < z < y$ such that $R_{\{k\}_z} \succ_R R_{\{i\}_x}$. By construction, $R_{\{j\}_y} \succ R_{\{k\}_x}$ when $x < y$. By *step 2*, this implies that for any z with $x < z < y$, $R_{\{j\}_y} \succ_R R_{\{k\}_z}$. Select such a z . We have just constructed a cycle $R_{\{k\}_z} \succ_R R_{\{i\}_x} \succ_L R_{\{j\}_y} \succ_R R_{\{k\}_z}$. But this implies that for $x < z < y$, $\{k\}_z \succ \{i\}_x \succ \{j\}_y \succ \{k\}_z$ a contradiction with the fact that \succ is *restricted transitive*.

2) Suppose there exists z with $x < y < z$ such that $R_{\{k\}_z} \succ_R R_{\{i\}_x}$. Then by

step 2, $\{k\}_{z'} \succ \{i\}_x$ for any z' such that $x < z' < y$. Repeating the argument in 1) concludes.

3) Suppose there exists z with $z < x < y$ such that $R_{\{k\}_z} \succ_L R_{\{i\}_x}$. Repeating an argument similar to 1) concludes the proof of step 8.

To complete the proof of the theorem, it remains to show that given $f \in \mathcal{H}$ and an associated priority ordering \succ , the set \mathbb{N} can be partitioned in hierarchies (without top and possibly without bottom) and that within a hierarchy, either the left-peak or the right-peak rule applies.

Let f be a rule satisfying the axioms and with associated priority ordering \succ .

Fix $\{1\} \subset \mathbb{N}$ and the set A_1 with $1 \in A_1$. Define a relation \sim as follows. For each $\{j\} \subset \mathbb{N}$, we say that $1 \sim j$ if,

- 1) $\{1\}_x \succ \{m\}_y \implies \{j\}_x \succ \{m\}_y$
- 2) $\{m\}_y \succ \{1\}_x \implies \{m\}_y \succ \{j\}_x$
- 3) $\{1\}_x \succ_L \{j\}_y \implies \{1\}_x \not\succeq_R \{j\}_y$
- 4) $\{1\}_x \succ_R \{j\}_y \implies \{1\}_x \not\succeq_L \{j\}_y$

By construction, $1 \sim 1$ and $1 \sim j \sim k \implies 1 \sim k$. Whenever items 1 to 4 are satisfied for j , we say that $j \in A_1$. Observe that A_1 is an indifference class. Repeat the same process for each agent $k \neq 1$ and construct the sets A_k . By construction, $\mathbb{N} = A_1 \cup A_2 \cup \dots$. Call this partition, A .

We now show that for each l, m with $l \neq m$, $A_l \cap A_m \neq \emptyset$ implies that $A_l = A_m$. By step 6, \succ_R and \succ_L have no maximums. Hence, \succ has no maximum either.²⁰ Suppose that there exist l and m with $A_l \cap A_m \neq \emptyset$. Thus, there exists an agent $j \in A_l \cap A_m$. By construction, $j \in A_l$ if for each $k \in A_l$, items 1) to 4) are satisfied. Since $j \in A_m$, for each $h \in A_m$, items 1) to 4) are satisfied.

²⁰Otherwise, *sovereignty* would be violated.

But this implies that items 1) to 4) are satisfied for each such k and h . Hence $A_l = A_m$. Derive a new partition of \mathbb{N} from A as follows.

1) Consider first agent 1 and the set A_1 . Pick all the sets A_l such that $A_l \cap A_1 \neq \emptyset$. Since these sets are the same, define a new set $B_1 = A_1$ and delete all such A_l from the initial partition.

2) Consider next agent 2. If $A_1 \cap A_2 \neq \emptyset$, then go directly to 3). Otherwise, repeat 1) for agent 2 and construct the set B_2 . Observe that $B_1 \cap B_2 = \emptyset$.

3) Consider next agent 3. If $B_1 \cap B_2 \cap A_3 \neq \emptyset$, then go directly to 4). Otherwise, repeat 1) for agent 3 and construct the set B_3 .

We continue the process until we obtain a subpartition of $\mathbb{N} = \cup B_\ell$ such that $B_l \cap B_m = \emptyset$ whenever $l \neq m$.

We now want to show that this partition of \mathbb{N} forms a hierarchical system without top.

Take a set B_l . By *step 7*, for each $i \in B_l$ there exists S_i , $S_i \cap B_l = \emptyset$, such that $\{m\}_y \succ \{i\}_x$ for each $m \in S_i$. By construction of the set of B_l , for each $i, j \in B_l$, we have $S_i = S_j = S$. Select an agent m in S . We have that $\{m\}_y \succ \{i\}_x$ for each $i \in B_l$. By *step 4*, $S'_y \succ \{i\}_x$ for each $S'_y \supseteq S$. By *step 3*, $S'_y \succ S_x$ for any $S_x \subseteq B_l$ and any $S'_y \supseteq S$. Therefore, $\mathbb{N} = \cup_\ell B_\ell$, $\cap_\ell B_\ell = \emptyset$ and each B_l is a hierarchy.²¹

We have one final step to add in order to complete the proof of the theorem. It remains to show that within hierarchies, either the *left-peaks rule* f^{LP} or the *right-peaks rule* f^{RP} applies.

²¹If the hierarchical system has no bottom, a similar argument can be applied.

Take a set B_l . For each $i \in B_l$ there exists a set S_i , $S_i \cap B_l = \emptyset$, such that $\{i\}_x \succ \{m\}_y$ for each $m \in S_i$. For each $i, j \in B_l$, we have $S_i = S_j = S$. Select an agent m in S . We have that $\{i\}_x \succ \{m\}_y$ for each $i \in B_l$. By *step 4*, $\{i\}_x \succ S'_y$ for each $S'_y \subseteq S$. By *step 3*, $S_x \succ S'_y$ for any $S_x \subseteq B_l$ and any $S'_y \subseteq S$.

Let B_l be an arbitrary hierarchy taken from the partition B . Let $(1, R_N)$ be a problem with $N = B_l$ and $|p(R_N)| = |B_l|$. Suppose that neither the *left-peaks rule* nor the *right-peaks rule* applies. That is there exist $i, j, k \in B_l$ with $p(R_i) = x < p(R_j) = y < p(R_k) = z$ and such that $f_1(R_N) = y$. Consider the problem $(1, R_{N \setminus \{j\}})$ and assume that $f_1(R_{N \setminus \{j\}}) = a = p(R_m)$. If $a < y$, we obtain that $\{j\}_y \succ_R \{m\}_a \succ_L \{k\}_z$. By *step 8*, this implies that $\{j\}_y \succ \{k\}_z$ for each $z \neq y$. Therefore, $k \notin B_l$, a contradiction with our initial assumption that B_l is a hierarchy. If $a > y$, we obtain that $\{j\}_y \succ_L \{m\}_a \succ_R \{i\}_x$. By *step 8*, this implies that $\{j\}_y \succ \{i\}_x$ for each $x \neq y$. Therefore, $i \notin B_l$, a contradiction with our initial assumption.

Next, let (k, R_N) be a problem with $N = B_l$ and $|p(R_N)| < |B_l|$. Suppose that neither the *left-peaks rule* nor the *right-peaks rule* applies. That is, there exist S_x, S_y, S_z with $x < y < z$ such that $x \notin f_k(R_N)$, $z \notin f_k(R_N)$ but $y \in f_k(R_N)$. By *step 4*, there exists $j \in S_y$ such that $\{j\}_y \succ S_x, S_z$. So consider the problem $(k, R_{N \setminus S_y})$. By peak-selection, there exists $h \in N$ such that $p(R_h) = a \in f_k(R_{N \setminus S_y})$ while $a \notin f_k(R_N)$. If $a < y$, we obtain that $\{j\}_y \succ_R \{m\}_a \succ_L \{k\}_z$ for each $k \in S_z$. By *step 8*, this implies that $\{j\}_y \succ \{k\}_z$ for each $z \neq y$. Therefore, $k \notin B_l$, a contradiction with our initial assumption that B_l is a hierarchy. If $a > y$, we obtain that $\{j\}_y \succ_L \{m\}_a \succ_R \{i\}_x$ for each $i \in S_x$. By *step 8*, this implies that $\{j\}_y \succ \{i\}_x$ for each $x \neq y$. Therefore, $i \notin B_l$, a contradiction with our initial assumption. ■

In addition to invariance axioms, we are also interested in non-manipulation axioms. *Strategy-proofness* requires that each agent i reporting his preferences truthfully is a (weakly) dominant strategy in the associated direct revelation game.

A solution f satisfies **strategy-proofness** if for each $N \in \mathbb{N}$, each $R_N \in \mathcal{R}^N$,

each $k \in \mathbb{N}$, each $i \in N$ and each $R'_i \in \mathcal{R}$,

$$f_k(R_N) R_i f_k(R'_i, R_{-i}).$$

Our last requirement is a strengthening of *strategy-proofness* that applies to coalitional attempts to manipulate a solution.

A solution f satisfies **group strategy-proofness** if for each $N \in \mathbb{N}$, each $R_N \in \mathcal{R}^N$, each $k \in \mathbb{N}$, each $M \subseteq N$ and each $R'_M \in \mathcal{R}^M$,

$$f_k(R_N) R_i f_k(R'_M, R_{-M}) \text{ for some } i \in M.$$

It is easy to see that every $f \in \mathcal{H}$ is *group strategy-proof*. In addition, every rule $f \in \mathcal{H}$ is trivially *hiding-proof* and thus consistent. Finally, observe that no $f \in \mathcal{H}$ is anonymous.

Remark 2: There are also rules in $\mathcal{F} \setminus \mathcal{H}$ that are group strategy-proof, e.g. the left-peaks and the right-peaks rules.

8 Conclusion

We characterized a family \mathcal{C} of solutions satisfying three basic requirements of *efficiency*, *object-population monotonicity* and *sovereignty*. Each solution $f \in \mathcal{C}$ is derived from a priority order over groups of interest. Since the entire family of *priority rules* \mathcal{F} is larger than \mathcal{C} , we pave the gap between the two families by weakening *sovereignty* to *asymptotic sovereignty*. Next, we characterized the subfamily of rules satisfying the additional requirement of *hiding-proofness*. Finally, using *Maskin monotonicity*, we provided the characterization of large subfamily \mathcal{H} of hierarchical solutions that includes the *serial-dictatorship rule*. Each member of \mathcal{H} is *group strategy-proof* and is thus a generalized median rule

(Moulin, 1980) for the case $k = 1$. Rules that belong to \mathcal{H} can thus be seen as an extension of some generalized median rule to the case $k > 1$.

All our results rest on a specific extension of preferences over points to preferences over sets –i.e. the max-extension used in Miyagawa (2001). The results of Ehlers (2003) show that using different extensions lead to different conclusions. It would be interesting to study our axioms when preferences are extended using the lexicographic-extension. We leave this question open for future work.

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