Endogenous link strength in directed communication networks*

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Abstract

Two major drawbacks of many models of communication networks with strategic agents are that (i) there are many Nash networks, and (ii) integrating geometric decay leads to strong difficulties in the determination of stable sets. We address the two points in a directed communication network model with endogenous link strength. Agents are endowed with a fixed amount of resource which they can distribute over links. They obtain indirect benefit by the path maximizing the product of links' strengths. In this environment, the wheel architecture is shown to be not only the unique efficient architecture but also the unique Nash architecture.

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1 Introduction

In most models devoted to the formation of communication networks[1], we observe that the set of stable networks is often large. For instance, in the connections' model, in which links are non directed, there are many pairwise stable architectures as soon as the cost of link formation is neither negligible nor prohibitive. In both one-way and two-way flow model introduced by Bala and Goyal (2000), in which link formation is a unilateral action, there are many Nash architectures (see pp. 1195 for some graphical illustrations of Nash networks in the one-way case and see pp. 1204 in the two-way case). Furthermore, and this is not the least, while geometric decay is a natural and privileged hypothesis in communication models, its integration makes all attempts to determine the exact stable sets unproductive; yet one can conclude that in presence of decay the stable sets are also large (this is true for all seminal works mentioned above).

We address the two points in a directed communication network formation game. For instance, this may fit situations like visiting a web site, but consider also the (directed) relationship in a firm between a manager and a subordinate. With regard to non directed networks, a generic difference govern the formation of connections: in the latter case link formation is a bilateral action, while in the former case it is a unilateral action. Put differently, the coordination problem inherent to the non directed case vanishes in the directed case. Hence, our stability concept is the Nash solution and the closest model it naturally echoes is the one-way flow model. Yet, our model is an extension to directed networks of Bloch and Dutta (2005), who introduce endogenous link strength in a non directed communication network. First, the agents have a fixed and infinitely separable amount of resource which they can distribute over directed connections. Second, individuals obtain indirect benefit by the path maximizing the product of links' strengths. This formulation therefore naturally fits with geometric decay. The difference between the models is that in Bloch and Dutta's work, links are non directed and the value of a link depends on the level of investment exerted by each

[1]See Jackson and Wolinsky (1996), Bala and Goyal (2000) and Bloch and Dutta (2005) for seminal papers.

partner (investments are either substitutable or complementary). In contrast, in our setting the return of some investment on a partner does not depend on the reciprocal investment made by the partner.
In this resource-constraint context, we show that the wheel network is not only the unique efficient architecture, but also the unique Nash one. To our knowledge, this is the first communication model such that integrating geometric decay does not deter a full analysis of the stable set. Moreover, the presence of decay makes remarkable the fact that the set of Nash networks is reduced to a unique architecture. Technically and concerning efficiency issue, we show that the wheel architecture uniquely guarantees a maximal payoff to all agents. The argument is the following: first, consider some agent i in a network g, as well as the subnetwork consisting in the paths used by agent i to extract value from the other agents; this subnetwork is a tree of finite length say k. Then among all possible trees of length k, we show that agent i's payoff cannot exceed that obtained if the tree is a chain of length k; second, that maximal payoff is increasing with the length of the chain. This precipitates the efficiency of the wheel. Concerning stability issue, we show that if the architecture is not a wheel, there always exists some agent who does not play a best-response; indeed, take one agent say i with maximal payoff on any network distinct from the wheel, consider one agent say j at maximal (finite) distance from her. Then, we show that if agent j forms a unique connection with agent i, she obtains a greater payoff than that of agent i, a contradiction.

Our result reminds Bala and Goyal's one-way flow model while sharply distinctive in its conclusion about the stable set. Indeed, in their setting the wheel architecture is Nash and uniquely efficient, but not uniquely Nash. The reason why the set of Nash architectures is reduced to a unique element in our setting is specific to the resource-constraint aspect of the model. In a word, the difference is that in the one-way flow model (with or without decay), the value for say agent i of a path between agent i and say agent j is not related to the adjacent links formed by agent i. This favors multiple individual best-responses and gives rise to a large set of Nash architectures. In contrast, in our context the value that agent i captures from agent j depends on how much she invests in other links, limiting the number of best-responses. And we actually obtain that a unique architecture is Nash[2].

The article is organized as follows. Section 2 introduces the model. The next two sections respectively present the three-player case and the general case. Section 5 discusses our result with regard to the standard one-way flow model. Section 6 concludes. All proofs are presented at the end of the paper. An appendix provides generalizes the set of payoff functions compatible with our result.

2 The model

Let $N = \{1, ..., n\}$ be a set of agents. Every agent is endowed with a finite amount $X \in [0, 1[$ of resources which can be devoted to establishing and maintaining links. Agents derive benefit from accessing other agents, either directly or indirectly. The key distinction with Bloch and Dutta (2005) is that the return of the investment made by agent i on agent j does not depend on how much agent j has invested on agent i. Hence benefits are one-sided, in the sense that agents access others through valued directed paths. A valued directed network represents the structure of allocations.

Individual strategies and directed networks. A strategy for agent i is a vector with $n-1$ coordinates $x_i = (x_{i,1}, ..., x_{i,i-1}, x_{i,i+1}, ..., x_{i,n})$ where $x_{i,j} \in [0, X]$ for each $j \in N \backslash \{i\}$, under the constraint that $\sum_{j \neq i} x_{i,j} \leq X$. When $x_{i,j} > 0$, agent i forms a link with agent j . The set of strategies of agent i is denoted by X_i. The set $X = X_1 \times ... \times X_n$ is the space of strategies of all agents. A strategy profile $x = (x_1, ..., x_n)$ can be represented as a directed and valued network g. We denote by G the set of (non negatively) weighted directed networks. The link $x_{i,j} > 0$ is depicted as an edge between agents i and j with arrowhead pointing at agent j, and we denote by ij this directed and valued link for convenience when there is no confusion; thus $ij \in g$ means that $x_{i,j} > 0$, $ij \notin g$ means that $x_{i,j} = 0$. A path between agent k_0 and agent k_p is a sequence of links $k_0k_1, \cdots, k_{p-1}k_p$ without loops and such that $\prod_{q=0}^{q=p-1} x_{k_q,k_{q+1}} > 0$.

[2] Our result is therefore true under strict Nash refinement. It is interesting to notice that in the one-way flow model the wheel is the unique efficient and strict Nash architecture; this reinforces in our mind the interest of underlying the difference between the respective sets of Nash architectures.

The distance from agent i to agent j, and denoted $d(i,j;g)$, represents the length of the shortest path from agent i to agent j in the network g. By convention if there is no finite path length the distance is infinite. We notice by $d_{max}(i;g) = \max_{j\neq i, d(i,j;g)<+\infty} d(i,j;g)$ the largest among all finite distances from agent i to others in the network g (when it exists).

Individual payoffs. In agreement with the literature on communication networks, we assume that the benefit of an agent in a network is given by the sum of the benefits she receives from accessing every other agent in her component. The benefit $b(i,j;g)$ that agent i captures from accessing agent j is the maximal value of the product of link strength over the set $P(i,j;g)$ of all finite paths from agent i to agent j:

$$b(i,j;g) = \max_{p\in P(i,j;g)} x_{ia_1}\cdots x_{a_{m-1}j}$$

with the convention that $b(i,j;g) = 0$ if there is no finite path from agent i to agent j in the network g. Agent i's payoff in the network g is finally written $\pi_i(g) = \sum_{j\neq i} b(i,j;g)$. This formulation fits naturally with (and actually generalizes to valued links) the standard geometric decay hypothesis in communication models: first only one among all paths between agent i and agent j generates value to agent i, and second the value of that path is decreasing with the distance (in our setting this latter condition obtains when the links' strengths are homogenous).

Efficiency and stability. The welfare $W(g)$ of a network g is the sum of individual payoffs on that network, *i.e.* $W(g) = \sum_{i\in N} \pi_i(g)$. A network is said efficient if there is no other network such that the sum of individual payoffs exceeds that of the former, *i.e.* $W(g) \geq W(g')$ for all $g' \in G$. Further, we apply the usual Nash criterion: a network is Nash if for every agent, her current strategy is a best-response to the current strategies of all other agents. Formally, a profile of individual strategies $x^* = (x_1^*, \cdots, x_n^*)$ is a Nash equilibrium of the game if and only if, for every agent $i \in N$, for every strategy $x_i \in X_i$, $\pi_i(x_i^*, x_{-i}^*) \geq \pi_i(x_i, x_{-i}^*)$.

3 The three-player case

The three-player case exhibits no tension between stability and efficiency.

We begin with efficiency issue. The welfare value of network architecture g is the maximum welfare which can be obtained on that architecture over all possible allocations of resources. Basically, in an efficient network every agent invests her total resource X. The following notation will be useful: for some architectures, the best allocation is such that certain agents reduce the weight of some link to zero; that is, the best alternative consists in modifying the architecture. If z represents the value the architecture asymptotically tends to, we will notice $W(g) \to z$ (and not '='). This means that the value is not reached by this architecture although its value tends to that. This precludes *de facto* this architecture from being efficient. Figure 1 depicts the 16 architectures with three players and gives their values: The only efficient architecture

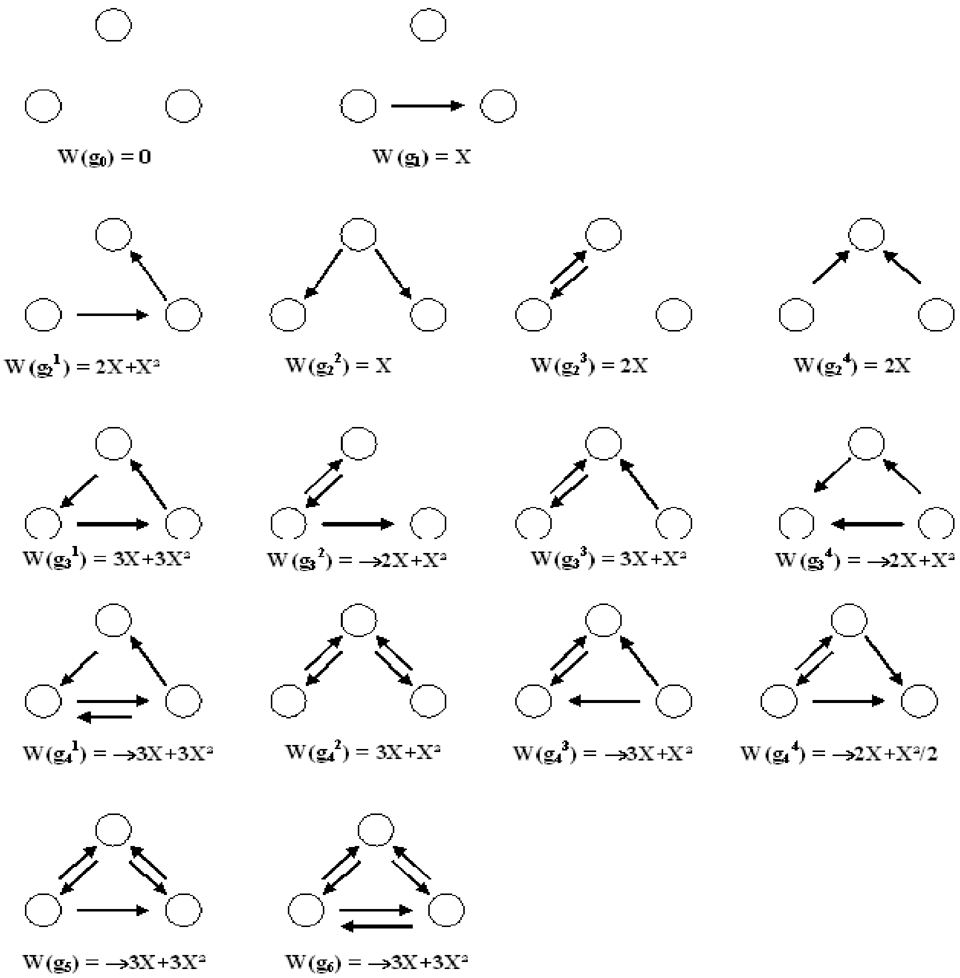

Figure 1: The 16 three-player architectures and their values

is the wheel (labeled g_3^1 in the table), whose value is $W(g_3^1) = 3X + 3X^2$.

We turn to stability analysis. We find that the wheel architecture g_3^1 in which each weight is X is the unique Nash architecture with 3 agents. Its stability is easily grasped as remarking that for each agent the current payoff is $X + X^2$, while any distribution of allocation $(x, X - x), x < X$ (the amount x is invested on the link on which she currently invests X) would induce a payoff equal to either X or $x(1 + X)$. Moreover, no other architecture is Nash. Indeed, we first note that in a Nash network each agent forms at least one link, since there is no link formation cost; we therefore exclude from the stable set the empty network, the whole set of one-link and two-link networks, and also the architectures g_3^2, g_3^4 and g_4^4 depicted in figure 1. Second, we note that for each agent in a Nash network, the sum of direct benefits basically equals X, and consequently a Nash strategy simply maximizes the indirect benefit part of the payoff. Given this point, in some architectures, certain agents obtain no indirect benefit while other strategies would provide them with some indirect benefit: in g_3^3, the agent with one connection pointing at her finds profitable to replace her current link with one entailing the formation of the wheel; in g_4^1, g_4^3, g_5 and g_6 certain agents forming two links are better off by investing all on some appropriate unique connection. Finally, in g_4^2, we remark that the two symmetric agents invest X on their current connection, while the two-link agent shares her allocation into two parts. Therefore, the indirect benefit of a one-link agent is less than X^2. Replacing her current link with a connection of weight X toward the third agent would provide her with an indirect benefit equal to X^2.

4 The general case

Our first result pertains with efficiency. We will use the following lemma:

Lemma 1 *For every agent i in a network g, $\pi_i(g) \leq \sum_{k=1}^{d_{max}(i;g)} X^k$.*

Considering any agent i in some network g, lemma 1 guarantees that her payoff does not exceed that obtained by some agent starting a chain of length $d_{max}(i; g)$ and composed of links with weights equal to X. The reason lies in the resource-constraint nature of our model. When the individual resource constraint is saturated, forming an additional link requires to decrease the investment put over some current links. This entails the following result: whatever a given set of paths of length k issued from some agent i, the sum of their values cannot exceed that of one unique path of length k and composed of maximal weights. The following proposition derives the set of efficient architectures:

Proposition 1 *The complete wheel is the unique efficient network.*

Our first conclusion is that a unique architecture is socially desirable. Indeed, this architecture is such that all agents access others by the means of a chain of length $n-1$ and composed with links of weight equal to X (when all individual resource constraints are saturated). Since no finite distance can be larger than $n-1$, we use lemma 1 to conclude that no individual payoff can exceed that obtained in the wheel architecture. Uniqueness is derived by remarking that the wheel is the unique architecture such that all agents can obtain that maximal payoff (one can even see that whatever a distinct architecture, at most one agent can obtain such a payoff).

Our second proposition presents the set of stable networks:

Proposition 2 *The complete wheel is the unique Nash network.*

The proof relies on two points: first, we consider one agent with maximal payoff on the network, say agent i, and one agent at maximal distance from her, say agent j; noticing that agent j is not critical for agent i, we derive that (i) agent j's payoff is not smaller than that of agent i, and (ii) for obtaining equality, agent i's payoff must be exactly equal to $\sum_{k=1}^{d_{max}(i;g)} X^k$ (we know from lemma 1 that it cannot be greater). We deduce that for each non terminal agent on the path from agent i to agent j, she invests amount X in a unique connection. Second, we notice that if some agent say p is not accessed by agent i, then a possible strategy entails for agent p a greater payoff than that of agent i, a contradiction. We deduce that agent i observes all agents. Combined with the first point, we see that to join agent j, agent i begins a chain of $n-1$ links with weight equal to X. We conclude that agent j's unique best response is to invest amount X on agent i. The complete wheel is therefore uniquely selected.

Hence, a unique architecture is Nash[3] and there is a coincidence between efficient and stable networks. This result is in sharp contrast with the standard communication network games (see the appendix for a generalization of payoff functions compatible with the result).

5 Discussion

We discuss the reason why resource-constraint setting is necessary for our results to emerge. Relaxing the resource-constraint aspect of the model, the closest setting is presumably the one-way flow model with decay (denoted OWD thereafter), so we discuss this latter model.

We begin with efficiency. Actually, lemma 1 is not true in the OWD. Figure 2 illustrates the point in a simple case (we use the network depicted in figure 2 in both models: in the OWD, links' strength are usually represented by parameter $\delta \leq 1$ while for our setting we keep the notations of the present paper - we will therefore assume $\delta = X$ when comparing the two models.). In our context, agent i's payoff is $\pi_i(g) = x_{ik}(1+x_{kj})+x_{il}(1+x_{lm})$, which cannot be greater than $X+X^2$. In the OWD, agent i's payoff would be $\pi_i(g) = 2\delta + 2\delta^2$, which exceeds $\delta + \delta^2$. As the lemma does not apply in the OWD, one may suspect that the wheel is in general not efficient in this context. This can be checked in the three-player case, as the value of the wheel $(3\delta + 3\delta^2)$ is smaller than the value of the four-link star $(4\delta + 2\delta^2)$.

We turn now to stability. Our proof establishes eight successive implications. In fact, points 5 and 6 are inherent to our setting. Implication 5 uses the lemma, and the above discussion indicates that it is not valid in the OWD. Further, implication 6 derives from the expression $b_{ij}(g) = X^{d_{max}(i;g)}$ that agent i initiates a *chain* of length $d_{max}(i;g)$ and composed of weights equal to X. This strong implication is clear in our setting, since if agent i initiates a tree distinct from the chain, at least one weight in

[3] The result on stability is basically robust to stability criteria allowing for collective deviations, like strong Nash equilibrium (Dutta and Mutuswami [1997]) or Strong stability (Jackson and van den Nouweland [2005]). Indeed, we know that (i) the wheel architecture ensures to all agents the maximal individual payoff and (ii) for any distinct architecture at most one agent obtains such a payoff.

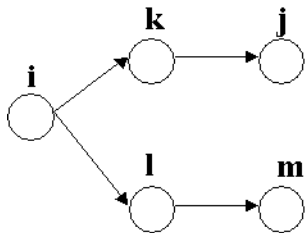

Figure 2:

the path from agent i to agent j would be of weight less than X. For instance, in figure 2 we see that $b_{ij}(g) < X^2$ since $x_{ik} < X$. In contrast, in the OWD $b_{ij}(g) = \delta^{d_{max}(i;g)}$ whether or not agent i initiates a chain (in figure 2 for instance, $b_{ij}(g) = \delta^2$). This is the reason why we obtain a unique Nash architecture, while OWD not.

6 Conclusion

We study a non-cooperative model of communication network formation. In our context, agents invest a fixed and infinitely separable amount of resource into a set of directed connections, a setting which integrates geometric decay. We find that the wheel architecture is both uniquely efficient and uniquely Nash. That the Nash set is reduced to a unique architecture is in contrast with standard communication models. Therefore, taking account resource scarcity has a decisive incidence upon the characterization of stable architectures.

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PROOFS

Proof of lemma 1. The proof is given by induction.
Inductive argument $P(k)$, $k \in \{1, \cdots, n-1\}$: if $d_{max}(i;g) = k$, then $\pi_i(g) \leq \sum_{q=1}^{k} X^q$. We trivially have $P(1)$. Suppose now $P(1), \cdots, P(k)$. We will derive $P(k+1)$. Indeed, if $d_{max}(i;g) = k+1$,

$$\pi_i(g) = \sum_{j \neq i} x_{ij}(g) + \sum_{j \neq i} x_{ij}(g)\pi_j^i(g)$$

where $\pi_j^i(g)$ represent the relevant part of agent j's payoff for agent i. That is, that derived from agents distant until at most order k of agent j; if some agent p is at distance $k+1$ or more than agent j, then either the distance is infinite and it does not count for agent i's payoff, or the distance is finite which means by construction that agent p is at smaller distance from agent i (since $d_{max}(i;g) = k+1$). Since the relevant network used by agent i to access others through agent j is a tree starting from agent j and with path length bounded above by k, we deduce as applying $P(1), \cdots, P(k)$ to agent j (more precisely to the relevant part of agent j's payoff for agent i) that $\pi_j^i(g) \leq \sum_{q=1}^{k} X^q$. Hence,

$$\pi_i(g) \leq \sum_{j \neq i} x_{ij}(g) + \sum_{j \neq i} x_{ij}(g) \sum_{q=1}^{k} X^q$$

And since $\sum_{j \neq i} x_{ij}(g) \leq X$, we obtain $\pi_i(g) \leq \sum_{q=1}^{k+1} X^q$, which proves $P(k+1)$. $\square$

Proof of proposition 1. We notice that a link ij is not necessarily active in the sense that the link is used to extract benefit. For instance, consider the following three-player network such that $x_{12} = X - \epsilon, x_{13} = \epsilon, x_{23} = X$. There are two paths from agent 1 to agent 3, and the two-arc path worths more than the one-arc path whenever $(X - \epsilon)X > \epsilon$. This is true for sufficiently small values of ϵ. But in such a case a more efficient network exists in which agent i reallocates the investment x_{ij}

into the connection x_{ik_1} where the link ik_1 belongs to one path of greatest value from agent i to agent j (note also that by construction no third agent was using the link ij to extract some value, and each agent using the path $ik_1, ..., k_pj$ to extract some value would also be better off). Hence we can restrict attention to networks such that for every agent, the sum of benefits associated with agents at distance 1 from her is exactly equal to X.

From lemma 1 we easily derive that the complete wheel is efficient: in the wheel architecture with link strengths equal to X, every agent accesses others by the means of a chain of length $n-1$. Since no finite distance can be larger than $n-1$, we use lemma 1 to conclude that no individual payoff can exceed that obtained in the wheel architecture.

We now derive uniqueness by observing that whatever network $g \in G$ distinct from the complete wheel, there is at most one agent with payoff equal to $X + \cdots + X^{n-1}$. Consider indeed a network g distinct from the complete wheel and suppose without loss that $\pi_1(g) = X + \cdots + X^{n-1}$. Then, using lemma 1 again, we deduce that there is a chain, starting from agent 1 and containing all agents, and all agents except the terminal node, say agent n, invest X on her link. Thus, these agents have a unique connection. EITHER agent n forms a set of links distinct from investing her total resource on agent 1; and clearly agents 2 to n obtain less than agent 1's payoff; OR agent n invests amount X on agent 1, in which case the complete wheel forms. We deduce that the complete wheel is the unique architecture such that all agents obtain a payoff equal to $X + X^2 + ... + X^{n-1}$. ■

Proof of proposition 2. The wheel architecture with link strengths equal to X is basically Nash, since it guarantees an upper bound on individual payoffs (this can be derived from lemma 1). We then turn to uniqueness. We will first show that some agent with maximal payoff begins a chain composed of links of weight equal to X. Second we will deduce that this agent observes the whole society. Third we will conclude that the wheel is uniquely selected. The proof proceeds in eight successive implications.

Assume without loss that agent $i \in argmax_{k \in N} \pi_k(g)$. Consider agent j such that $d(i, j; g) = d_{max}(i; g)$.

1. Then agent j is not critical to agent i (whatever agent j's strategy, this does not affect agent i's payoff).

2. Then agent j obtains a payoff at least equal to $\pi_j = X + X(\pi_i(g) - b(i,j;g))$; consider indeed the strategy $x_{ji} = X$.

3. Then $X + X(\pi_i(g) - b(i,j;g)) \leq \pi_i(g)$; indeed, agent i's payoff is maximal, so $\pi_j(g) \leq \pi_i(g)$.

4. Then

$$\pi_i(g) \geq \frac{X}{1-X}\left(1 - X^{d_{max}(i;g)}\right)$$

or equivalently $\pi_i(g) \geq X + X^2 + ... + X^{d_{max}(i;g)}$; since $b(i,j;g) \leq X^{d_{max}(i;g)}$.

5. Then it must be the case that $\pi_i(g) = X + X^2 + ... + X^{d_{max}(i;g)}$ and that $b(i,j;g) = X^{d_{max}(i;g)}$; indeed by lemma 1 we know that agent i's payoff does not exceed that value.

6. Then agent i accesses all others through a chain of length $d_{max}(i;g)$ and composed of links with weights equal to X.

7. Then $d_{max}(i;g) = n-1$; otherwise suppose that some agent p is not accessed by agent i, *i.e.* $d_{max}(i;g) < n-1$. Then by investing amount X on agent i, agent p obtains a payoff equal to $X + X\pi_i(g)$. As agent i's payoff is maximal, it must be the case that $X + X\pi_i(g) \leq \pi_i(g)$, which means that $\pi_i(g) \geq \frac{X}{1-X}$. But as $X < 1$, $X + X^2 + ... + X^{n-1} \leq \frac{X}{1-X}$, a contradiction.

8. Then the wheel obtains; we know that there is a unique agent j such that $d(i,j;g) = n-1$ and that $\pi_j(g) = \pi_i(g)$. Given that agent i begins a chain of length $n-1$ and composed of links of weights equal to X, the unique strategy for agent j to obtain agent i's payoff consists in forming a unique connection of weight X with agent i, which completes the wheel. ■

APPENDIX: A MORE GENERAL SETTING ON THE VALUE OF PATHS

We define the set $\Psi_l(i,j;g)$ of all paths of length l from agent i to agent j in the network g: $\Psi_l(i,j;g) = \{(x_{a_0a_1}, \cdots, x_{a_{l-1}a_l}) \in]0,X]^l / a_k \neq a_{k'} \forall k \neq k', a_0 = i, a_l = j\}$. This formulation excludes paths containing some arcs of weight equal to zero. Then we define a collection of functions

$$\begin{cases} \phi_l : \Psi_l(i,j;g) \longrightarrow \mathcal{R}_+ \\ (x_{ia_1}, \cdots, x_{a_{l-1}j}) \longmapsto \phi_l(x_{ia_1}, \cdots, x_{a_{l-1}j}) \end{cases}$$

with l arguments in $]0, X]$, and generating a positive real number: $\phi(z_1, \cdots, z_l) > 0$, *i.e.* a path (composed with positive weights) has positive value. Second, we suppose that the function is increasing and (weakly) convex in each argument. By convexity, we mean that for every indexes l and p:

$$\phi_l(a_1, \cdots, a_p, \cdots, a_l) + \phi_l(a_1, \cdots, b_p, \cdots, a_l) \leq \phi_l(a_1, \cdots, a_p + b_p, \cdots, a_l).$$

The benefit obtained by agent i from agent j is written:

$$b(i,j;g) = \max_{l \in \{1, \cdots, n-1\}} \left[\max_{p \in \Psi_l(i,j;g)} \phi_l(p) \right]$$

with the convention that $b(i,j;g) = 0$ whenever there is no finite path length from agent i to agent j. For instance, the individual payoff in the complete wheel is written

$$\sum_{k=1}^{n-1} \phi_k(\underbrace{X, \cdots, X}_{k \text{ times}})$$

Indeed, there is a unique finite path from agent i to any agent j, and no two paths have equal length. Note that in this formulation we do not impose any ranking on the functions ϕ_k, we only impose positivity.

Among all paths starting from agent i and issuing to agent j, we denote with $P^*_{ij}(g)$ the subset of them with greatest value, *i.e.*

$$P^*_{ij}(g) = \{p_{ij}(g) \in argmax_{\Psi_1(i,j;g), \cdots, \Psi_{n-1}(i,j;g)} \; \phi_l(p)\}.$$

For convenience, we arbitrarily select a unique best path, say $\Theta(i,j;g) \in P^*_{ij}(g)$. We denote for each agent i in the network g the profile $P^*_i(\Theta, g) = \{\Theta(i,j;g)\}_{j \neq i}$. This profile associates with each agent $j \neq i$ one path from i to j of greatest value. We define the set $\Gamma^l_i(\Theta, g)$ composed of the paths in $P^*_i(\Theta, g)$ of length l. Then, convexity implies the following lemma (this lemma is a direct extension of lemma 1):

Lemma 2 *For every index $l \in \{1, \cdots, n-1\}$, and whatever selection process Θ, we have:*

$$\sum_{p \in \Gamma_i^l(\Theta, g)} \phi_l(p) \leq \phi_l(\underbrace{X, \cdots, X}_{l\ times})$$

This lemma bounds above the sum of benefits that agent i captures from accessing agents placed at a given finite distance from her. To fix ideas, consider agent i's payoff in the following network (see figure 2):

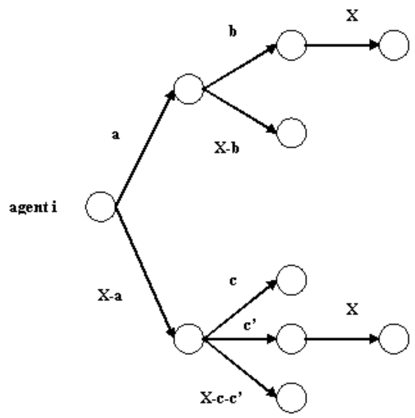

Figure 3:

In this example $P_{ij}^*(g)$ is reduced to a unique element for any $j \neq i$. We obtain:

$$\pi_i = \phi_1(a) + \phi_1(X-a)$$

$$+\phi_2(a,b) + \phi_2(a, X-b) + \phi_2(X-a, c) + \phi_2(X-a, c') + \phi_2(X-a, X-c-c')$$

$$+\phi_3(a,b,X) + \phi_3(X-a, c, X)$$

As the function is increasing,

$$\begin{cases} \phi_3(a,b,X) \leq \phi_3(a,X,X) \\ \phi_3(X-a,c,X) \leq \phi_3(X-a,X,X) \end{cases}$$

Further, the function is convex, so

$$\begin{cases} \phi_1(a) + \phi_1(X-a) \leq \phi_1(X) \\ \phi_2(a,b) + \phi_2(a,X-b) \leq \phi_2(a,X) \\ \phi_2(X-a,c) + \phi_2(X-a,c') + \phi_2(X-a,X-c-c') \leq \phi_2(X-a,X) \end{cases}$$

That is,

$$\pi_i \leq \phi_1(X) + \phi_2(a, X) + \phi_2(X - a, X) + \phi_3(a, X, X) + \phi_3(X - a, X, X)$$

Using again the convexity of ϕ_2 and ϕ_3, we obtain

$$\pi_i \leq \phi_1(X) + \phi_2(X, X) + \phi_3(X, X, X)$$

Hence, confirming lemma 3, we note that for every distance $l = 1, 2, 3$, the sum of benefits that agent i captures from agents placed at distance l from her is bounded above by the value $\phi_l(X, \cdots, X)$.

The following lemma extends lemma 2:

Lemma 3 *Whatever network $g \in G$ distinct from the complete wheel, there is at most one agent with payoff equal to $\sum_{k=1}^{n-1} \phi_k(X, \cdots, X)$. Hence at least $n - 1$ agents have a payoff strictly smaller than that they obtain in the wheel.*

As a direct consequence of lemmata 3 and 4:

Proposition 3 *The complete wheel is both Nash and uniquely efficient.*

Uniqueness (of efficiency set) comes from the fact that the complete wheel is the unique architecture providing $\phi_k(X, \cdots, X)$ from $k = 1$ up to $k = n - 1$ as benefit to all agents. Hence, we observe that a crucial determinant of uniqueness is that all functions $\phi_k(X, \cdots, X)$ are positive.

We obtain uniqueness of the stable set under the following more specific environment:

$$\phi_l(x_{ia_1}, \cdots, x_{a_{l-1}j}) = \theta(x_{ia_1}) \times \cdots \times \theta(x_{a_{l-1}j})$$

with function $\theta(.) : \mathcal{R}_+ \longrightarrow [0, 1[$, increasing and convex, and such that $\theta(x) > 0$ iff $x > 0$. The additional result obtains:

Proposition 4 *In such a specification, the complete wheel is the unique Nash network.*

Indeed, when ϕ is a product of identical functions, the proof of proposition 2 is straightforwardly replicated.

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