The core-partition of a hedonic game

(revised version)

Vincent Iehlé

Universitat Autònoma de Barcelona - CODE
Edifici B, 08193 Bellaterra (Barcelona), Spain.
Phone: +34-935812933, Fax +34-935812461.

Abstract

A pure hedonic game describes the situation where player's utility depends only on the identity of the members of the group he belongs to. The paper provides a necessary and sufficient condition for core-partition existence in a hedonic game. The condition is based on a new concept of balancedness involving the notion of pivotal distribution that associates to each coalition a particular sub-group of players in the coalition. We also review several sufficient conditions for core-partition existence to show how the extant results can be unified through suitably chosen pivotal distributions.

JEL classification: C71.

Key words: balancedness, core-partition, group formation, hedonic game.

Email address: viehle@pareto.uab.es (Vincent Iehlé).

Submitted to XXXXX *9 November 2006*

1 Introduction

Drèze and Greenberg (1980) called hedonic aspect the dependence of a player's utility on the identity of the members of his group. In many different areas of economics the hedonic aspect plays a central role since it explains the formation and the existence of groups, clubs and communities. [1] Such group-dependent preferences also encompass most of matching models that can be found in marriage, roommate or indivisible goods problems (e.g. Gale and Shapley, 1962; Scarf and Shapley, 1974). By focusing exclusively on this feature, a hedonic game describes the situation where player's utility depends *solely* on the hedonic aspect, as defined firstly by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002). [2]

The paper deals with the theoretical analysis of a hedonic game, specifically with the existence of the core-partition which is the natural cooperative game solution in the hedonic game setting. A core-partition is a partition of the players such that there is no coalition of players where each player in the coalition is better off (with respect to his utility in the coalition) than in the partition. Hence in essence, the core-partition has the same requirement as the core with coalition structure studied in standard cooperative games (see Aumann and Drèze, 1974). The paper provides a necessary and sufficient condition for core-partition existence in a hedonic game.

In the literature, authors have alternatively offered conditions for core-partition existence by specifying restrictions on feasible coalitions, on individual preferences or on the preference profile. In the case of restrictions on feasible coalitions, the existence and the uniqueness of the core-partition is characterized by Pápai (2004). In the case of restrictions on individual preferences, the existence is studied by Burani and Zwicker (2003), Dimitrov et al. (2004) and Pápai (2005). In our paper the restrictions hold on the whole preference profile. [3] In that general line, the main contributions are due to Banerjee et al. (2001) and Bogomolnaia and Jackson (2002), where they both show mainly two disconnected approaches to obtain sufficient but not necessary conditions for core-partition existence. In the first approach, it can be shown that a hedonic game admits a representation in terms of an NTU game (Non Trans-

[1] As example, the utility of an individual may depend both on the consumption level of a public good and on the identity/number of the agents with who the good is shared. The dependency suggests how agents eventually match to form clubs.
[2] Earlier cooperative game analysis in that line also includes Greenberg and Weber (1986) and Kaneko and Wooders (1982).
[3] Observe that a restriction on individual preferences is a restriction on preference profile. Second, restricting feasible coalitions can be restated on the preference profile by evaluating the disregarded coalitions as the worst coalitions (with respect to preferences).

ferable Utility). Then it follows that the balanced game condition of Scarf (1967) provides a sufficient condition for core-partition existence in the original hedonic game, the NTU balanced game condition being restated into the hedonic game setting. As usual in many economic models, balanced game conditions provide powerful results on core-like solutions, while the difficulty with these conditions is the limited interpretative range of the balancedness. The second approach is based on more interpretative mechanisms. Among other stronger results Banerjee et al. (2001) show that the existence of at least one top coalition in each coalition guarantees the core-partition existence, where top coalition refers to a subset of the coalition such that each agent in the subset is better off than in other subsets of the coalition.

We define a new notion of balancedness, called pivotal balancedness, that gathers the above two leading intuitions. To define the notion, consider the notion of balanced family (Bondareva, 1963; Shapley, 1967). A family of coalition is said to be balanced if for each coalition in the family there is a weight such that, for each player, the weights of the coalitions to which he belongs sum to 1. To define a pivotally balanced family, one first associates to each element of a family of coalitions a non-empty subset of the coalition, the resulting subsets form a so called pivotal distribution of the family. A family of coalitions is said to be pivotally balanced if its pivotal distribution is balanced. In the setting of a hedonic game, one deduces naturally a definition of pivotally balanced game. The game is said to be pivotally balanced if there exists a pivotal distribution such that for each pivotally balanced family there exists a partition of set of players such that each player prefers his coalition in the partition than his worst coalition in the family.

Our main result, Theorem 2, states that a hedonic game admits a core-partition if and only if the game is pivotally balanced. Importantly observe that the result is actually in line with the prior contributions in standard cooperative games since the extant core non-emptiness characterizations always rely on balancedness conditions. Beyond our new balancedness condition, the result also points out the necessary role of pivotal agents to stabilizing a social system whenever agents are endowed with group-dependent preferences.

The model is given in Section 2. We first recall the existence result of Bogomolnaia and Jackson (2002) based on standard balancedness. Then, we define the notion of pivotal balanced game in order to establish the main result of the paper, Theorem 2. A first intuition about the result is given through a very simple game which is pivotally balanced but not balanced. In Section 3, we review several sufficient conditions for the existence of core-partition, already established in the literature. We show how the properties of consecutiveness (Bogomolnaia and Jackson, 2002) and top coalition (Banerjee et al., 2001) imply pivotal balancedness. To demonstrate these results, we construct explicitly the appropriate pivotal distributions in each case. Section 4 contains the proof of Theorem 2. To prepare the proof, we describe the representation of a hedonic game in terms of an NTU game. The NTU game modeling of a hedonic game has been already used in prior literature to deduce the existence of core-partition through Scarf's theorem. The difference here is that the result of Scarf (1967) is not sufficiently strong to prove our result. Instead, we use a result of Billera (1970) for b-balanced games. Section 5 is devoted to concluding remarks.

2 Model and result

Let N be the finite set of players and $\mathcal{N}$ be the set of all non-empty subsets of N. A group of players $S \in \mathcal{N}$ is called a coalition. Given $\mathcal{B} \subset \mathcal{N}$ and $i \in N$, let $\mathcal{B}(i) = \{S \in \mathcal{B} \mid i \in S\}$ be the set of coalitions in $\mathcal{B}$ that contain i. A partition of N is a family $\pi = \{S_1, ..., S_K\}$ where $\cup_{k=1}^{K} S_k = N$, and $S_k \cap S_\ell = \emptyset$ for any $k, \ell \in \{1, ..., K\}$, $k \neq \ell$. The set of all partitions in N is denoted $\Pi(N)$. For any partition $\pi \in \Pi(N)$ and any player $i \in N$, let $\pi(i)$ be the unique coalition such that $i \in \pi(i)$. For each $S \in \mathcal{N}$, $\mathbf{1}^S \in \mathbb{R}^N$ is the vector with coordinates equal to 1 in S and equal to 0 outside S.

Definition 1 *A hedonic game is a pair $(N; \succeq)$, where $\succeq = (\succeq_i)_{i\in N}$ and $\succeq_i$ is a reflexive, complete, and transitive binary relation on $\mathcal{N}(i)$.*[4]

Definition 2 *Let $(N, \succeq)$ be a hedonic game. Given $\pi \in \Pi(N)$, a coalition $T \in \mathcal{N}$ blocks π if $T \succ_i \pi(i)$ for each $i \in T$. A core-partition is a partition π^* that is blocked by no coalition.*

Let us first recall the usual notion of balanced family (see Bondareva, 1963; Shapley, 1967). A family of coalitions $\mathcal{B} \subset \mathcal{N}$ is balanced if for each $S \in \mathcal{B}$ there exists a balancing weight $\lambda_S \in \mathbb{R}_+$ such that $\sum_{S\in\mathcal{B}} \lambda_S \mathbf{1}^S = \mathbf{1}$. The next definition is due to Bogomolnaia and Jackson (2002).

Definition 3 *Let $(N, \succeq)$ be a hedonic game. The game is ordinally balanced if for each balanced family $\mathcal{B}$ there exists a partition $\pi \in P(N)$ such that for each $j \in N$, there is $S \in \mathcal{B}(j)$ such that $\pi(j) \succeq_j S$.*

The following result provides a sufficient condition for core-partition existence of a hedonic game. It can be seen as the counterpart of Scarf 's Theorem (1967) in the hedonic game setting.

Theorem 1 (Bogomolnaia and Jackson (2002)) *Let $(N, \succeq)$ be a hedonic game. The game admits a core-partition if it is ordinally balanced.*

[4] Strict preference relation and the indifference relation are denoted by $\succ_i$ and $\sim_i$, respectively, $S \succ_i T \Leftrightarrow [S \succeq_i T$ and $T \not\succeq_i S]$ and $S \sim_i T \Leftrightarrow [S \succeq_i T$ and $T \succeq_i S]$.

The condition is not necessary as shown in the following example due to Banerjee et al. (2001).

$$\{1,2\} \succ_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1\}$$
$$\{1,2\} \succ_2 \{2,3\} \succ_2 \{1,2,3\} \succ_2 \{2\}$$
$$\{1,3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}$$

It is an easy matter to check that the above hedonic game admits a unique core-partition: $\pi^* = \{\{1,2\},\{3\}\}$. Consider the balanced family $\mathcal{B} = \{\{1,2\},\{2,3\},\{1,3\}\}$ (with weights $\lambda_S = \frac{1}{2}$ for each $S \in \mathcal{B}$). Then for any of the five possible partitions of the game, at least one player strictly prefers the worst coalition he belongs to in $\mathcal{B}$ to his one in π. For instance if $\pi = \{\{1\},\{2,3\}\}$, one has $\{1,3\} \succ_1 \{1\}$. Thus, the game is not ordinally balanced.

Our objective in this paper is to provide a necessary and sufficient condition for the existence of a core-partition in a hedonic game. Our result is based on a refinement of the usual notion of balanced family. The refinement involves the notion of pivotal distribution.

Definition 4 *A family $I = (I(S))_{S\in\mathcal{N}}$ is called pivotal distribution if, for each $S \in \mathcal{N}$, $\emptyset \neq I(S) \subset S$. The set of all pivotal distributions is denoted by $\mathcal{I}$.*

The new notions of I-balanced family and pivotally balanced game can be defined now.

Definition 5 *Given a pivotal distribution $I \in \mathcal{I}$. A family of coalitions $\mathcal{B} \subset \mathcal{N}$ is I-balanced if the family $(I(S))_{S\in\mathcal{B}}$ is balanced.*

The usual notion of balanced family coincides with the particular case of I-balanced family for a full pivotal distribution, i.e. $I(S) = S$ for each $S \in \mathcal{N}$.

Remark 1 *There is another equivalent formulation for an I-balanced family. Given a pivotal distribution $I \in \mathcal{I}$, a family of coalitions $\mathcal{B} \subset \mathcal{N}$ is I-balanced if for each $S \in \mathcal{B}$ there exists $\lambda_S \in \mathbb{R}_+$ such that $\sum_{S\in\mathcal{B}} \lambda_S \mathbf{1}^{I(S)} = \mathbf{1}$. It leads back to a b-balanced family of coalitions à la Billera (1970), where $b_S = \mathbf{1}^{I(S)}$ for each $S \in \mathcal{N}$ and $b = \mathbf{1}$ (See Section 4, p.10).*

Definition 6 *Let $(N, \succeq)$ be a hedonic game. The game is pivotally balanced if there exists a pivotal distribution $I \in \mathcal{I}$ satisfying: for each I-balanced family $\mathcal{B}$, there exists a partition $\pi \in P(N)$ such that, for each $j \in N$, there is $S \in \mathcal{B}(j)$ with $\pi(j) \succeq_j S$.*

Observe that the notion ordinally balanced game coincides to the notion pivotally balanced game where the pivotal distribution is full and given by

$I(S) = S$ for each $S \in \mathcal{N}$.

Let us come back to the example given by Banerjee et al. (2001). Consider the following pivotal distribution I where $I(\{123\}) = \{1,2\}$, $I(\{1,2\}) = \{1,2\}$, $I(\{1,3\}) = \{1\}$, $I(\{2,3\}) = \{2\}$, $I(\{i\}) = \{i\}$ for each $i = 1,...,3$. First note that the problematic family $\mathcal{B} = \{\{1,2\},\{2,3\},\{1,3\}\}$ is not any more I-balanced. Indeed, $\sum_{S\in\mathcal{B}(3)} \lambda_S \mathbf{1}_3^{I(S)} = 0$ for all $(\lambda_S)_{S\in\mathcal{B}}$. In this example all I-balanced families include necessarily the singleton $\{3\}$ since it is the only coalition S that satisfies $3 \in I(S)$. Then consider the partition $\pi = \{\{1,2\},\{3\}\}$. Any I-balanced family is not preferred by the agents 1 or 2 since $\{1,2\}$ is a maximal element for both players 1 and 2. Since the worst coalition for player 3 in any I-balanced family is $\{3\}$, this coalition cannot be preferred by player 3 given that $\pi(3) = \{3\}$. Thus, the game is pivotally balanced with respect to I.

The above example sheds light on two important features of pivotal balancedness. The pivotal distribution allows to select families that are not balanced in the usual sense and conversely to eliminate balanced families that are not suitable. The example shows that the balanced family $\mathcal{B} = \{\{1,2\},\{2,3\},\{1,3\}\}$ is not I-balanced, I being given in the example. Conversely, the family $\mathcal{B}' = \{\{1,3\},\{2,3\},\{3\}\}$ is clearly not balanced but I-balanced for the pivotal distribution I (consider the weights $\lambda_{\{1,3\}} = \lambda_{\{2,3\}} = \lambda_{\{3\}} = 1$). The second interesting feature of the example is the way the pivotal distribution I is explicitly constructed to obtaining a pivotally balanced game with respect to I. In the example consider the core partition of the game, $\pi^* = \{\{1,2\},\{3\}\}$, then I is actually given by, for each $S \in \mathcal{N}$, $I(S) = \{i \in S \mid \pi^*(i) \succeq_i S\}$. Necessarily the sets $I(S)$, $S \in \mathcal{N}$, are non-empty since π^* is a core-partition. Thus, one associates to each coalition the set of players in the coalition who are not worse off in the original partition π^*. This idea is central in the proof of the next result.

The following theorem is the central result of the paper.

Theorem 2 *Let $(N, \succeq)$ be a hedonic game. The game admits a core-partition if and only if it is pivotally balanced.*

The proof is given in Section 4 as we need a representation in terms of a NTU game for the *if part* of the proof (see Theorem 3).

So far we have considered the solution with coalition structure, which can potentially prescribe any of the partitions of the player set. But the usual *core* solution where only the grand coalition N is a potential candidate to being the solution can be defined as well in the setting of a hedonic game. [5] It follows by

[5] The ultimate goal of a hedonic game, namely the formation of groups, is however altered in that case.

definition that the core of a hedonic game $(N, \succeq)$ is non-empty if and only if N is a core-partition of $(N, \succeq)$. One can state straightforwardly the associated characterization for that solution.

Proposition 1 *Let $(N, \succeq)$ be a hedonic game. The game admits a non-empty core if and only if there exists a pivotal distribution $I \in \mathcal{I}$ satisfying: for each I-balanced family $\mathcal{B}$ and $j \in N$, there is $S \in \mathcal{B}(j)$ such that $N \succeq_j S$.*

The proof of Proposition 1 is omitted since it is a direct adaptation of Theorem 2's proof.

The above characterizations follow the stream of cooperative game theory where core-like solutions are analyzed throughout a balancedness condition (e.g. Bondareva, 1963; Billera, 1970; Bonnisseau and Iehlé, 2003; Kaneko and Wooders, 1982; Predtetchinski and Herings, 2004; Reny and Wooders, 1996; Scarf, 1967; Shapley, 1967; Zhou, 1994). The balanced game conditions in the literature are more or less refined, depending on the class of games into consideration, but they all rely on the same principle of balanced family of coalitions which generalize the notion of partition. Unsurprisingly our results are also in this line but they contrast with the prior literature in the specific role attributed to sub-groups of players in the coalitions. This feature is intimately linked with the intrinsic nature of the hedonic game.

Though the manipulation of balancedness notions may appear difficult at first glance, these notions have been widely exploited in economic or game theories. Obviously, the more precise is the description the larger is the number of results that are potentially seizable. As we obtain a characterization, one can reasonably expect that Theorem 2 serves as a tool box to exhibit new results and evaluate stability in the class of hedonic games. The next section is precisely devoted to the manipulation of our new balancedness concept.

3 About sufficient conditions

We review different properties that guarantee core-partition existence. Given our main result, each of them can be restated as a condition of pivotal balancedness. We describe two cases: consecutiveness and top coalition property, where the pivotal is explicitly constructed and the game is shown to be pivotally balanced. Note that we only deduce here pivotally balanced game property from the conditions of consecutiveness and top coalition but not for their weak counterparts. To proceed with the review we follow the contributions of Bogomolnaia and Jackson (2002) and Banerjee et al. (2001).

In Bogomolnaia and Jackson (2002), the authors identify two classes of conditions: ordinally balanced game condition, as presented in Theorem 1, and two consecutiveness properties. We have already seen that ordinal balancedness coincides with I-balancedness in the particular case where $I(S) = S$ for any $S \in \mathcal{N}$.

Turn now to consecutiveness and weak consecutiveness properties. An ordering of players is a bijection $f : N \to N$. A coalition $S \in \mathcal{N}$ is consecutive with respect to an ordering of players f, if $f(i) < f(j) < f(k)$ with $i, k \in S$ implies $j \in S$. A hedonic game is consecutive if there exists an ordering of players f such that $S \succ_i \{i\}$ for some i implies that S is consecutive with respect to f. A hedonic game $(N, \succeq)$ is weakly consecutive if there exists an ordering of players f such that whenever a partition $\pi \in \Pi(N)$ is blocked by some coalition T, there exists T' that is consecutive with respect to f that blocks π.

Proposition 2 *Let $(N, \succeq)$ be a consecutive hedonic game. The game is pivotally balanced.*

Proof.

Let f be the ordering given by assumption. For each $S \in \mathcal{N}$, let $I(S) = S$ if S is consecutive (with respect to f) and $I(S) = \{i\}$ for some $i \in S$ otherwise. One can show that the game is pivotally balanced with respect to I.

Let $\mathcal{B}$ be an I-balanced family, i.e. $\mathcal{B}' := (I(S)_{S \in \mathcal{B}}$ is a balanced family of consecutive coalitions. From a well known argument taken from Greenberg and Weber (1986, Proposition 1, p.109), we know that any balanced family of consecutive coalitions contains a partition.[6] Hence, one deduces that $\mathcal{B}'$ contains a partition $\pi \in \Pi(N)$. Let $i \in N$, if $\pi(i) \in \mathcal{B}$ then there exists obviously $S \in \mathcal{B}(i)$ such that $\pi(i) \succeq_i S$ since there exists $S \in \mathcal{B}(i)$ such that $S = \pi(i)$. If $\pi(i) \notin \mathcal{B}$ then by construction of $\mathcal{B}'$ it holds necessarily that $\pi(i) = \{i\}$ and that there is $S \in \mathcal{B}(i)$ such that S is not consecutive. By assumption on the game we know that $\pi(i) = \{i\} \succeq_i S$ if S is not consecutive. Thus, we have demonstrated that the game is pivotally balanced. □

Given Bogomolnaia and Jackson (2002, Proposition 1, p.211), observe that there exist games which are not consecutive, neither weakly consecutive, but

---

[6] In Greenberg and Weber (1986), the authors state this preliminary result to show the non-emptiness of the core with coalition structure by applying Scarf's result on balanced games. The result is then used to show the existence of Tiebout equilibrium.

for which pivotally balancedness holds true.

3.2 Top-coalitions

Banerjee et al. (2001) introduce the top coalition properties. They are also sufficient conditions for core-partition existence in a hedonic game. Let $U \in \mathcal{N}$ be a coalition, a non-empty subset $S \subset U$ is a top coalition of U if for any $i \in S$ and any $T \subset U$ with $i \in T$ we have $S \succeq_i T$. A hedonic game $(N, \succeq)$ satisfies the top coalition property if for any coalition $U \in \mathcal{N}$, there exists a top-coalition of U.

Banerjee et al. (2001) also defined a weak top coalition property. We refer the reader to Banerjee et al. (2001, Definition 14) for further details. Note however that the construction of the pivotal distribution would be based on the same principle as the argument below.

Proposition 3 *Let $(N, \succeq)$ be a hedonic game satisfying the top coalition property. The game is pivotally balanced.*

Proof.

To obtain pivotal balancedness from top coalition property, we follow Banerjee et al. (2001) and we define π_1 a top coalition of N, π_2 a top coalition of $N \setminus \pi_1$, ..., π_k a top coalition of $N \setminus \cup_{\{k'<k\}} \pi_{k'}$ and so on. The procedure yields eventually a partition $\pi = (\pi_1, ..., \pi_K) \in \Pi(N)$ in $K \leq |N|$ steps. For each $S \in \mathcal{N}$, let k^S be the smallest number such that $\pi_{k^S} \cap S \neq \emptyset$ and set $I(S) = \pi_{k^S} \cap S$. Then, the game is pivotally balanced with respect to that distribution.

Indeed, let $\mathcal{B}$ be an I-balanced family. By way of contradiction, suppose that the game is not pivotally balanced. Let $i \in N$ be such that for each $S \in \mathcal{B}(i)$, $S \succ_i \pi(i)$. Then necessarily $S \not\subset N \setminus \cup_{\{k'<k\}} \pi_{k'}$ for k such that $\pi_k = \pi(i)$, otherwise π_k is not a top coalition of $N \setminus \cup_{\{k'<k\}} \pi_{k'}$. Thus, there exists $j \in S \cap \pi_{k'}$ with $k' < k$. Then, $i \notin I(S)$ from the construction of I. One deduces that for each $S \in \mathcal{B}(i)$, $i \notin I(S)$ which is in contradiction with the fact that $\mathcal{B}$ is I-balanced. Then, the game is pivotally balanced. □

Given Bogomolnaia and Jackson (2002, Proposition 1, p.211), observe that there exist games which do not satisfy top coalition property, neither weak top coalition property, but for which pivotally balancedness holds true.

4 NTU games representation and proof of Theorem 2

To prove Theorem 2, we use the NTU game representation of a hedonic game. Let $(N, \succeq)$ be a hedonic game. Following a strategy set up by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002), let us a define an associate NTU-hedonic game. First, we define a utility profile consistent with $(N, \succeq)$, for each $i \in N$, let $u_i : \mathcal{N}(i) \to \mathbb{R}$ be such that, for each $S, T \in \mathcal{N}(i)$, $u_i(S) \geq u_i(T)$ iff $S \succeq_i T$.[7] For each $S \in \mathcal{N}$, let $V_S = \{x \in \mathbb{R}^N \mid x_i \leq u_i(S) \text{ for all } i \in S\}$ be the set of feasible payoffs of S, and $V = \bigcup_{\pi \in \Pi(N)} \{x \in \mathbb{R}^N \mid x_i \leq u_i(\pi(i)) \text{ for all } i \in N\}$ be the set of payoffs such that there exists a partition for which the payoffs are feasible.[8] The family of payoff sets $((V_S)_{S \in \mathcal{N}}, V)$ is called NTU-hedonic game and is denoted $V(N, \succeq)$. The core of $V(N, \succeq)$ is the set $\partial V \setminus \text{int} \cup_{S \in \mathcal{N}} V_S$.[9]

To complete our representation in terms of NTU games, let $i \in \mathcal{I}$ be a pivotal distribution and define another NTU game $V^I(N, \succeq)$, called Billera's game in the sequel, where $V^I = V$ and, for each $S \in \mathcal{N}$, $V^I_S = \{x \in \mathbb{R}^N \mid \sum_{i \in S} \mathbf{1}_i^{I(S)} x_i \leq \sum_{i \in S} \mathbf{1}_i^{I(S)} u_i(S)\}$. This is an instance of Billera (1970)'s pseudo hyperplane game, $\{V^B, H^{b_S, v_S}, S \in \mathcal{N}\}$, where $V^B = V^I$ and, for each $S \in \mathcal{N}$, $b_S = \mathbf{1}^{I(S)}$ and $v_S = \sum_{i \in S} \mathbf{1}_i^{I(S)} u_i(S)$. The core of $V^I(N, \succeq)$ can be restated as the set of payoffs $x \in V^I$ such that, for each $S \in \mathcal{N}$, $\sum_{i \in S} \mathbf{1}_i^{I(S)} x_i \geq \sum_{i \in S} \mathbf{1}_i^{I(S)} u_i(S)$.

The following theorem connects the core-partition of hedonic games with the core of the games $V(N, \succeq)$ and $V^I(N, \succeq)$.

Theorem 3 *Let $(N, \succeq)$ be a hedonic game. The following propositions are equivalent:*

(i) The game admits a core-partition.
(ii) The game is pivotally balanced.
(iii) The NTU-hedonic game $V(N, \succeq)$ has a non-empty core.
(iv) Billera's game $V^I(N, \succeq)$ has a non-empty core for some pivotal distribution $I \in \mathcal{I}$.[10]

[7] Such an utility profile always exists since the number of coalitions is finite. Note also that we need only one utility profile, but in general the profile is not uniquely defined.

[8] Usually to define the game in partition structure, V_N is redefined as V and the set V is omitted. Our formulation here is equivalent and used for convenience. In Bonnisseau and Iehlé (2003), the distinction between V and V_N has deeper implications.

[9] For any set $Y \subset \mathbb{R}^N$, ∂Y and int Y will denote respectively its boundary and interior.

[10] The last assertion concerning Billera's game is given for sake of completeness. We actually build our proof on the notion of an NTU game rather than on the notion

We will need the following notion of b-balancedness due to Billera (1970). For each $S \in \mathcal{N}$, let $b_S \in \mathbb{R}^N \setminus \{0\}$ such that $b_S^i \in \mathbb{R}_+$ if $i \in S$ and $b_S^i = 0$ otherwise, and let $b \in \mathbb{R}^N_{++}$. A family of coalitions $\mathcal{B} \subset \mathcal{N}$ is b-balanced if for each $S \in \mathcal{B}$, there exists $\lambda_S \in \mathbb{R}_+$ such that $\sum_{S\in\mathcal{B}} \lambda_S b_S = b$. Given the hedonic game $(N, \succeq)$, the associate NTU-hedonic game $V(N, \succeq)$ is b-balanced if for any b-balanced family $\mathcal{B} \subset \mathcal{N}$, $\cap_{S\in\mathcal{B}} V_S \subset V$. The result of Billera (1970, Theorem 1) implies that any b-balanced NTU-hedonic game has a non-empty core. The result is used in the proof of Theorem 3.

Proof.

$(i) \Rightarrow (ii)$ Let us consider a core-partition π^* of $(N, \succeq)$. For each $S \in \mathcal{N}$, define $I(S) = \{i \in S \mid \pi^*(i) \succeq_i S\}$. Necessarily the sets $I(S)$, $S \in \mathcal{N}$, are non-empty since π^* is a core-partition. Next, suppose that the game is not pivotally balanced with respect to $I = (I_S)_{S\in\mathcal{N}}$. Thus, there exist a family $\mathcal{B} \subset \mathcal{N}$ and $\lambda_S \in \mathbb{R}_+$ for each $S \in \mathcal{B}$ with $\sum_{S\in\mathcal{B}} \lambda_S \mathbf{1}^{I(S)} = \mathbf{1}$, and $k \in N$ such that for all $S \in \mathcal{B}(k)$, $S \succ_k \pi^*(k)$. From the construction of the pivotal distribution, one deduces that $k \notin I(S)$ for all $S \in \mathcal{B}(k)$. Hence, $\mathcal{B}$ is not I-balanced since $(\sum_{S\in\mathcal{B}(k)} \lambda_S \mathbf{1}^{I(S)})_k = 0 < 1$. It leads to a contradiction. □

$(ii) \Rightarrow (iii)$ Suppose that (ii) holds true and let $I \in \mathcal{I}$ be the associated pivotal distribution. We show first that $V(N, \succeq)$ is b-balanced, the vectors $(b_S)_{S\in\mathcal{N}}$ being given by $(\mathbf{1}^{I(S)})_{S\in\mathcal{N}}$ and $b = \mathbf{1}$. Let $\mathcal{B}$ be a b-balanced family of coalitions and $x \in \cap_{S\in\mathcal{B}} V_S$. From Remark 1, $\mathcal{B}$ is I-balanced. From (ii), there exists a partition π^* such that such that for all $i \in N$, there is $S \in \mathcal{B}(i)$ such that $u_i(S) \leq u_i(\pi^*(i))$. Since $x \in V_S$ for all $S \in \mathcal{B}$, it holds that for all $i \in N$ and $S \in \mathcal{B}(i)$, $x_i \leq u_i(S)$. Then, one gets $x_i \leq u_i(\pi^*(i))$ for all $i \in N$, i.e. $x \in V$. Hence, the game $V(N, \succeq)$ is b-balanced à la Billera (1970). From Theorem 1 in Billera (1970), the game $V(N, \succeq)$ has a non-empty core. □

$(iii) \Rightarrow (i)$ Obvious, from the construction of the game $V(N, \succeq)$. □

$(iii) \Rightarrow (iv)$ Let x be in the core of $V(N, \succeq)$. For each $S \in \mathcal{N}$, define $I(S) = \{i \in S \mid x_i \geq u_i(S)\}$. Since x is in the core of $V(N, \succeq)$, the set $I(S)$ is non-empty for each $S \in \mathcal{N}$. By definition $x \in V^I$, furthermore, for each $S \in \mathcal{N}$, $\sum_{i\in S} \mathbf{1}_i^{I(S)} x_i = \sum_{i\in I(S)} x_i \geq \sum_{i\in I(S)} u_i(S) = \sum_{i\in S} \mathbf{1}_i^{I(S)} u_i(S)$. Thus, x belongs to the core of $V^I(N, \succeq)$. □

$(iv) \Rightarrow (iii)$ Billera (1970, Lemma 5). □

---

of Billera's game which is of rare usage.

5 Concluding remarks

Three remarks conclude the paper.

1. For a TU game (Transferable Utility) Bondareva (1963) and Shapley (1967) proved that the core is non-empty if and only if the game is balanced, using the notion of balanced family; for an NTU game Predtetchinski and Herings (2004) and Bonnisseau and Iehlé (2003) introduced the general notion of Π-balancedness to characterize the core non-emptiness (Scarf's balanced game condition is sufficient but not necessary for non-emptiness of the core of an NTU game). While the hedonic game may be viewed as an NTU game, here we do not appeal to the general notion of Π-balancedness. The associated NTU-hedonic game admits indeed a very particular geometrical structure. Thus the characterization builds on the intermediary concept of pivotal balancedness, which is especially coined in the implication $(i) \Rightarrow (ii)$ in Theorem 3 and its proof (it coincides with the *only if* part of the proof of Theorem 2).

2. To clarify the use of b-balancedness in our proof, let us recall that Billera (1970, Theorem 1) establishes a generalization of Scarf (1967)'s non-emptiness result for an NTU game by replacing the condition of balanced game by the weaker condition of b-balanced game. In the proof of Theorem 3 we make full use of the result to obtain the implication $(ii) \Rightarrow (iii)$ in Theorem 3 (it coincides with the *if part* of the proof of Theorem 2). For that step, Scarf's result is not sufficiently strong while Billera's result fits well. Of course $(ii) \Rightarrow (iii)$ can be demonstrated through the recent contributions of Predtetchinski and Herings (2004) or Bonnisseau and Iehlé (2003), but Billera's weaker result is sufficient here.

3. In Kaneko and Wooders (1982, Theorem 2.7), the authors provide a characterization for the core of a partitioning game (either TU or NTU). Their result shows interesting links with balancedness conditions. In addition, the feasible payoffs of any coalition are somehow re-calibrated with respect to a partitioning rule, this draw the resulting game close to a simple coalition formation game as can be a hedonic game. To the best of our knowledge, the connections of their result with the more recent stream of hedonic game literature has not been done, but we let for future research the comparison with Theorem 3 since that goes beyond the scope of our paper. [11]

[11] This remark also applies to the prior contribution of Greenberg and Weber (1986) on consecutive games.

Acknowledgments

I thank an Associate Editor and a Referee for helpful comments. I am indebted to Jean-Marc Bonnisseau for his guidance on balancedness issues. This work has been partially elaborated during a visit to Universita Federico II, I wish to thank the Dipartimento di Matematica-Statistica for its hospitality. I also thank Elena del Mercato for stimulating discussions. All errors are mine.

References

Aumann, R., Drèze, J., 1974. Cooperative games with coalition structures. International Journal of Game Theory 3, 217–237.

Banerjee, S., Konishi, H., Sonmez, T., 2001. Core in a simple coalition formation game. Social Choice and Welfare 18 (1), 135–153.

Billera, L., 1970. Some theorems on the core of an n-person game without side payments. SIAM Journal on Applied Mathematics 18(3), 567–579.

Bogomolnaia, A., Jackson, M., 2002. The stability of hedonic coalition structures. Games and Economic Behavior 38 (2), 201–230.

Bondareva, O., 1963. Some applications of linear programming methods to the theory of cooperative games. Problemy Kibernetica 10, 119–139, in Russian.

Bonnisseau, J.-M., Iehlé, V., 2003. Payoff-dependent balancedness and cores. Cahier de la MSE, Université Paris 1. Forthcoming in *Games and Economic Behavior*.

Burani, N., Zwicker, W., 2003. Coalition formation games with separable preferences. Mathematical Social Sciences 45, 27–52.

Dimitrov, D., Borm, P., Hendrickx, R., Sung, S., 2006. Simple priorities and core stability in hedonic games. Social Choice and Welfare 26(2), 421–433.

Drèze, J., Greenberg, J., 1980. Hedonic coalitions: optimality and stability. Econometrica 48 (4), 987–1003.

Gale, D., Shapley, L.S., 1962. College admissions and the stability of marriage. American Mathematical Monthly 69, 9–15.

Greenberg, J., Weber, S., 1986. Strong Tiebout equilibrium under restricted preferences domains. Journal of Economic Theory 38, 101–117.

Kaneko, M., Wooders, M., 1982. Cores of partitioning games. Mathematical Social Sciences 3, 313–328.

Pápai, S., 2004. Unique stability in simple coalition formation games. Games and Economic Behavior 48, 337–354.

Pápai, S., 2005. Hedonic coalition formation and individual preferences. Working paper, University of Notre Dame.

Predtetchinski, A., Herings, P., 2004. A necessary and sufficient condition for non-emptiness of the core of a non-transferable utility game. Journal of Economic Theory 116, 84–92.

Reny, P., Wooders, M., 1996. The partnered core of a game without side payments. Journal of Economic Theory 70, 298–311.

Scarf, H., 1967. The core of an n-person game. Econometrica 35, 50–69.

Scarf, H., Shapley, L.S., 1974. On cores and indivisibility. Journal of Mathematical Economics 1, 23–37.

Shapley, L.S., 1967. On balanced sets and cores. Naval Research Logistics Quarterly 14, 453–460.

Zhou, L., 1994. A new bargaining set of an N-person game and endogenous coalition formation. Games and Economic Behavior 6, 512–526.