Continuity of beliefs in network games

Willemien Kets*

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Abstract

Social networks are important for determining economic outcomes. Networks are typically large and complex, so that individuals may not know the exact structure of the network they belong to. This paper studies the role of beliefs in games on networks in which players have incomplete information about the network structure. Suppose players are located on a network, and play a fixed game with their neighbors. Players have a common prior on the network structure, and, in addition, they have some local information. This paper asks under what conditions on two priors it is the case that for any network game in which players hold one of those priors, for any equilibrium of that game, there is an approximate equilibrium in the game in which players hold the other prior such that ex ante payoffs are close in both equilibria. We show that in order for this to hold, both the distribution of player types and the correlation between neighbor types should be similar under both priors.

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*Address: Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: w.kets@uvt.nl. Tel: +31-13-4662478. Fax: +31-13-4663280. An early version of this paper has circulated under the title "Strategic convergence of beliefs in network games". I thank Mia Deijfen, Andrea Galeotti, John Geanakoplos, Sanjeev Goyal, Wouter Kager, Atsushi Kajii, Stephen Morris, Scott Page, Rajiv Sethi, Dolf Talman, Eric van Damme, Remco van der Hofstad, and Mark Voorneveld for stimulating discussions, as well as Sanjeev Goyal, Sebastian Gryglewicz, Ruud Hendricx, Adriaan Soetevent, Dolf Talman, Marco van der Leij, and Mark Voorneveld for helpful comments and suggestions. Part of this work was completed during a visit to the Santa Fe Institute in July 2006. Financial support from the James S. McDonnell Foundation Grant for Robustness in Social Processes is gratefully acknowledged.

1. Introduction

There is extensive empirical evidence that social structures are important in determining economic outcomes.[1] Social structures can be modeled using networks. Agents are then identified with vertices in a network, with links or edges between vertices representing the social relations between agents. Social networks are typically large, and often have a complex structure (Albert and Barabási, 2002; Newman, 2003). A natural assumption is then that individuals do not have full information on the network structure.[2]

This paper studies games in which players are located on a network, and have incomplete information on the network structure. In these so-called *network games of incomplete information*, players play a fixed game with their neighbors, the agents they are connected with in the network. Players have a common prior on the network, and in addition, each player has some local information on the network structure. A player's type is his *degree*, i.e., the number of connections he has in the network. Payoffs to a player only depend on his own type and action, and on the actions and types of his neighbors in the network. This defines a *network belief system*. A network belief system is a probability space consisting of a class of networks (with a suitably defined σ-field) and the common prior defined on this class, giving for each network in the class the probability that this network is realized.[3] The current paper asks how sensitive players' strategic behavior is to the specification of the network belief system. That is, are game-theoretical predictions for a game with a given network belief system also valid for the corresponding game with another network belief system? Which features of the network belief systems determine whether the answer to this question is positive?

More precisely, take two priors, μ and μ', defined on a class of networks. We ask under what conditions on these priors it is the case that for any network game of incomplete information in which players have beliefs μ, for any equilibrium in that game, there is an approximate equilibrium in the associated game with prior μ' such that ex ante expected payoffs are close in both equilibria. We thus study the lower hemicontinuity of the

[1] For instance, see Coleman et al. (1966), Granovetter (1994), Glaeser, Sacerdote, and Scheinkman (1996), Topa (2001), and Conley and Udry (2005).

[2] Indeed, sociological research has shown that individuals only have local knowledge of the network they belong to. For instance, Friedkin (1983) finds that individuals typically only observe their direct contacts and the contacts of their contacts.

[3] A network belief system is called a random network model in probability theory and statistical physics. We prefer to use the term 'network belief system' rather than 'random network model' as the latter term is often used for specific models.

correspondence of (interim) approximate equilibria (cf. Monderer and Samet, 1996; Kajii and Morris, 1998). Our main result (Theorem 4.1) states that the strategic behavior of players is sensitive to the degree distribution, as well as to the correlation between the degrees of neighbors. Hence, the essential features of a network belief system in terms of game-theoretical predictions are the degree distribution and the degree correlation.

The motivation for this question is twofold. Firstly, this question is relevant for applications. Empirical work on social networks has shown that social networks are characterized by different properties.[4] Some of these properties relate to the local environment of a player. For instance, an important property of networks is the distribution of the number of direct contacts that people have. Other properties are defined on a larger scale. The clustering coefficient of a network, for instance, quantifies the extent to which friends of your friends are also your friends. Another example is the degree correlation, i.e., the correlation in the number of contacts people have. The literature on network games of incomplete information discussed below mostly focuses on network belief systems that are characterized fully by the distribution of the number of contacts individuals have, the so-called random network models with a given asymptotic degree distribution (see Section 2.1 for a discussion). When the number of individuals in the network grows large, the clustering coefficient is zero in these models and there is no correlation in the number of contacts people have. However, many social networks are characterized by considerable clustering and by positive degree correlation.[5] Is it possible to translate game-theoretical predictions from the class of network belief systems with a zero clustering coefficient and degree correlation to a class of belief systems with positive clustering and nonzero degree correlation? This paper shows that the answer is no. Even if the degree distribution in both classes of network belief systems is the same, it will not be the case that in any network game of incomplete information, for each equilibrium in a game with a network belief system from one of the classes, there is an approximate equilibrium in the corresponding game with a network belief system from the other class such that ex ante payoffs are similar in both equilibria. The reason is that, in addition to the degree distribution, the degree correlation of a network belief system is crucial for the game-theoretical predictions: it is not guaranteed that equilibria are similar under two network belief systems if these two network belief systems differ in terms of degree distribution or degree correlation.

Secondly, the current study is of theoretical interest. We study continuity of beliefs in a network setting. Continuity of beliefs in general Bayesian games has been studied by

[4] See Newman (2003) and Jackson (2005) for a survey.

[5] See e.g. Newman and Park (2003), Palla et al. (2005) and Copič, Jackson, and Kirman (2005).

Monderer and Samet (1996) and Kajii and Morris (1998).[6] We study a similar question as Kajii and Morris (1998),[7] but we restrict attention to the class of network games of incomplete information. We are able to substantially weaken the conditions of Kajii and Morris by exploiting the symmetry of the probability space and the payoff functions and the fact that payoffs only depend on the actions and types of the direct neighbors of a player. That this can be done is not obvious. While payoffs only depend directly on the actions of neighbors in our setting, actions and beliefs of those further away in the network may have a considerable effect on the payoffs of a player, through the effect on the neighbors of those players and the neighbors of the neighbors of those players, and so on. There is thus a tension between the local nature of the payoffs and the interdependencies intrinsic to the network setting. By formulating the problem in terms of a "reduced" probability space, we are able to resolve this tension: rather than to the full network belief system, our conditions for continuity refer to the probability space of the types of a player and his direct neighbors. This allows us to show that a necessary and sufficient condition is that two network belief systems are similar in terms of the degree distribution and degree correlation they induce.

This work thus contributes to the literature on network games of incomplete information.[8] An important result from the literature on network games of incomplete information is that the assumptions on the information players have can have critical effect on the results. For instance, Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006) analyze a local public goods game under different informational assumptions. They obtain qualitatively different results under different informational assumptions. The message of the current paper is that the assumptions on the beliefs of players are also very important: the game-theoretical predictions may crucially hinge on the specification of the network belief system.

[6]These authors study lower hemicontinuity of the Bayesian-Nash equilibrium correspondence, as we do. Milgrom and Weber (1985) study upper hemicontinuity of the correspondence.

[7]Like we do, Kajii and Morris (1998) study a setting in which the state space and players' information partitions are fixed, and the common prior varies. By contrast, Monderer and Samet (1996) consider the case in which the common prior is fixed, but information partitions vary.

[8]This literature started with the early papers of Kirman (1986), Kirman et al. (1986) and Ioannides (1990). More recently, Galeotti et al. (2006) provide a general framework for analyzing strategic interactions on networks both under complete and incomplete information about the network structure; López-Pintado (2004), Sundararajan (2006), Galeotti and Vega-Redondo (2005), Jackson and Yariv (2005), and Ioannides and Soetevent (2006) develop specific models. See Jackson (2005) and Vega-Redondo (2006) for a survey of the literature.

The outline of this paper is as follows. Network belief systems are introduced in Section 2. Network games of incomplete information are discussed in Section 3, where we also discuss random networks with a given asymptotic degree distribution in some detail. The main result is presented in Section 4. Section 5 concludes. Proofs that are not included in the main text can be found in Appendix A.

2. Network belief systems

In our framework, players are located on a network. An (unweighted and undirected) *network* g is a triple consisting of a finite, nonempty set of *vertices* and a finite, nonempty set of *edges* or *links*, and a relation that associates with each edge two vertices (not necessarily distinct), called its *endpoints*. Players are thus identified with vertices, with edges representing the relations between players. A network may contain *multiple edges*, i.e., distinct edges that connect the same pair of vertices, or *self-loops*, edges whose endpoints are equal. A network is *simple* if it does not contain multiple edges or self-loops. In simple networks, an edge is determined by its endpoints, so that we can denote an edge as an unordered pair of vertices. Let g be a simple network with vertex set V and edge set E. If $\{i, j\} \in E$, where $i, j \in V$, then i and j are *neighbors* in network g.

We focus on a setting where the network is not fixed, but instead drawn from a class of networks according to some probability distribution. Let $n \in \mathbb{N}, n \geq 2$, and let $V = \{1, \dots, n\}$ be a vertex set with n vertices. Let $\mathcal{G}$ be the set of all *simple* networks with vertex set V, and let $\mathscr{F}_{\mathcal{G}}$ be the set of all subsets of $\mathcal{G}$. Let ν be a probability measure on $(\mathcal{G}, \mathscr{F}_{\mathcal{G}})$. The probability space $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu)$ defines a *network belief system.*[9]

We define some random variables that will be useful in the following. The edge set is now a random variable. More specifically, let $P_V = \{S \subseteq V : |S| = 2\}$ be the set of all possible links given vertex set V and let Q_V be the set of all subsets of P_V. The edge set is then a measurable function $E : \mathcal{G} \to Q_V$, with $E(g) \in Q_V$ the edge set of network g. We are particularly interested in the local environment of vertices. Fix $i \in V$ and $g \in \mathcal{G}$. Define

$$N_i(g) := \{j \in V \mid \{i, j\} \in E(g)\}$$

[9]Of course, one could alternatively take $\mathcal{G}$ to be the set of all networks with n vertices, and set $\nu(g) = 0$ if g is not simple. For notational simplicity and to avoid measurability problems, we have chosen the present approach.

to be the *neighborhood* of i in network g. Let

$$D_i(g) := |N_i(g)|$$

be the number of neighbors of i in g. We refer to D_i as the *degree* of i. The degree sequence of a network is a vector of the degrees of the vertices in a nonincreasing order. Formally, let $g \in \mathcal{G}$. Define

$$\begin{aligned} N_1 &:= V \\ j(1) &:= \max\{j \in N_1 : D_j(g) \geq D_k(g) \text{ for all } k \in N_1\} \end{aligned}$$

and for all $\ell \in \{2, \dots, n\}$:

$$\begin{aligned} N_\ell &:= N_{\ell-1} \setminus \{j(\ell-1)\} \\ j(\ell) &:= \max\{j \in N_\ell : D_j(g) \geq D_k(g) \text{ for all } k \in N_\ell\} \end{aligned}$$

The degree sequence of g is then $(D_{j(\ell)})_{\ell \in \{1,\dots,n\}}$. We can also consider the degrees of the neighbors of a given player. The *neighbor degree profile* for a player $i \in V$ in a network $g \in \mathcal{G}$ is the vector of the degrees of the neighbors of i in g, in a nonincreasing order. Formally, for $t = 0$, set $\Omega_K^t = (0)$. For $t \in \{1, \dots, n-1\}$, let

$$\Omega_K^t := \{(k_1, \dots, k_t) \in \{1, \dots, n-1\}^t \mid k_1 \geq k_2 \geq \dots \geq k_{t-1} \geq k_t\}.$$

Let

$$\Omega_K := \bigcup_{t \in \{0,\dots,n-1\}} \Omega_K^t,$$

and let $\mathscr{F}_K$ be the set of all subsets of Ω_K. For $i \in V$, we can define the function $K_i : \mathcal{G} \to \Omega_K$, with $K_i(g)$ the neighbor degree profile of i in $g \in \mathcal{G}$, as follows. Let $g \in \mathcal{G}$, with degree sequence $(D_{j(\ell)})_{\ell \in \{1,\dots,n\}}$. If $D_i(g) = 0$, then set $K_i(g) := (0)$. If $D_i(g) > 0$, let

$$K_i(g) := (D_{j(\ell)})_{\ell \in \{1,\dots,n\}, j(\ell) \in N_i(g)}.$$

Example 2.1. Let $n = 4$, and consider the vertex set $V = \{1, 2, 3, 4\}$, and let the network be g. Suppose that $E(g) = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$. This gives rise to the network in Figure 1. The degree of 1 is $D_1(g) = 2$, the neighborhood of 1 is $N_1(g) = \{2, 3\}$ and the neighbor degree profile of 1 is $K_1(g) = (2, 2)$ since players 2 and 3 both have two neighbors. The degree sequence of g is $(2, 2, 2, 2)$. ◁

Throughout this paper, we make the following assumption on network belief systems:

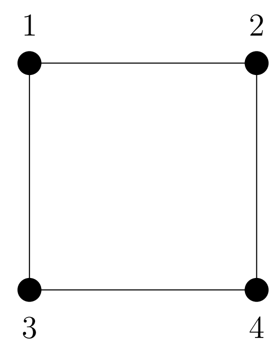

Figure 1: The network of Example 2.1.

Assumption 1. (Exchangeability) The degrees $D_1, D_2, \ldots$ are exchangeable, that is, for any $k \in \mathbb{N}$, $i_1, \ldots, i_k \in V$, the random variables $D_{i_1}, D_{i_2}, \ldots, D_{i_k}$ have the same joint distribution as $D_{\pi(i_1)}, D_{\pi(i_2)}, \ldots, D_{\pi(i_k)}$ for any permutation $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$. In particular, for all $i, j \in V$, for all $\theta \in \{0, \ldots, n-1\}$,

$$\nu(\{g \in \mathcal{G} \mid D_i(g) = \theta\}) = \nu(\{g \in \mathcal{G} \mid D_j(g) = \theta\}),$$

i.e., the probability that a vertex has a certain degree is the same for each vertex. ◁

Network belief systems $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu)$ with this property exist. For instance, let ν be the uniform distribution on the finite set $\mathcal{G}$.

It will be convenient to define a probability distribution for neighbor degree profiles. By Assumption 1, we only need to consider the probability distribution of the neighbor degree profile of a fixed player, as the distribution of the neighbor degree profile is the same for all players. Let $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu)$ be a network belief system, and recall that the sample space of neighbor degree profiles is Ω_K, with associated σ-field $\mathscr{F}_K$. Define the probability distribution μ_ν of the neighbor degree profile of player 1 by

$$\forall F \in \mathscr{F}_K : \quad \mu_\nu(F) := \nu(\{g \in \mathcal{G} \mid K_1(g) \in F\}). \tag{2.1}$$

With some common, minor abuse of notation, define $\mu_\nu(t)$, $t \in \{0, \ldots, n-1\}$, to be the marginal distribution $\mu_\nu(t) := \mu_\nu(\Omega_K^t)$, and let, for $F \in \mathscr{F}_K$ and $t \in \{0, \ldots, n-1\}$ such that $\mu_\nu(t) > 0$, $\mu_\nu(F \mid t) := \mu_\nu(F \mid \Omega_K^t)$. The probability distribution $(\mu_\nu(t))_{t \in \{0, \ldots, n-1\}}$ giving for each $t \in \{0, \ldots, n-1\}$ the probability $\mu_\nu(t)$ that player 1 has degree t is the *degree distribution* of the network belief system $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu)$. Henceforth, we will omit the reference to player 1 when we speak about D_1 or K_1. This should not create any problems, as by Assumption 1, the distribution of the neighbor degree profile is the same for all players.

In the next section, we discuss a class of network belief systems that has been widely studied in the literature on network games of incomplete information, the class of random network models with a given asymptotic degree distribution.

2.1. *Example: Random networks with a given asymptotic degree distribution*

Let ξ be a probability distribution with support the set of nonnegative integers $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and assume that ξ has a finite mean, i.e., assume that

$$\sum_{t \in \mathbb{N}_0} t\xi(t) < \infty. \tag{2.2}$$

Let $n \in \mathbb{N}, n \geq 2$, and let ξ be a probability distribution on $\mathbb{N}_0$ such that (2.2) is satisfied. A random network (model) with n players and asymptotic degree distribution ξ is a network belief system $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_{\xi}^{(n)})$, where $\mathcal{G}^{(n)}$ is the set of all simple networks on vertex set $V^{(n)} := \{1, \ldots, n\}$, $\mathscr{F}_{\mathcal{G}}^{(n)}$ is the set of all subsets of $\mathcal{G}^{(n)}$, and the probability measure $\nu_{\xi}^{(n)}$ on $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)})$ is defined indirectly as follows.[10] For each vertex $i \in V^{(n)}$, draw a number D_i independently from the distribution ξ, and attach D_i "half-edges" to i. Then, we form edges by selecting pairs of half-edges uniformly at random (without replacement) and connecting them, until no half-edges are left.[11] The probability that an edge selected uniformly at random from the set of all edges in such a network has a vertex $i \in V^{(n)}$ as one of its endpoints is proportional to D_i. Similarly, the probability that two vertices $i, j \in V^{(n)}$ are the endpoints of an edge selected uniformly at random from the edge set of such a random graph is proportional to $D_i D_j$. However, the network constructed in this way may not be simple, i.e., it may contain multiple edges between a given pair of players, or edges from a given player to himself (self-loops). Let g be a network obtained by the above procedure. For each vertex $i \in V^{(n)}$, we erase all self-loops at i. Furthermore, if two vertices i, j, with $i \neq j$, are connected by multiple edges, we merge these edges into a single edge. We thus obtain a simple network $\hat{g}$ corresponding to the network g. The degrees of the vertices in the simple network $\hat{g}$ corresponding to the original network g is

[10] We use the erased configuration model here (Britton, Deijfen, and Martin-Löf, 2006). Applications often refer to the so-called configuration model (Bender and Canfield, 1978; Bollobás, 1980). For that model, the degree distribution is exactly the given distribution xi, while in the erased configuration model, the degree distribution only converges to ξ when n grows large. However, the configuration model assigns *strictly* positive probability to networks that contain multiple edges or self-loops. In game-theoretic applications, this is of course problematic.

[11] Of course, this only works if the total number of half-edges is even; if the sum $\sum_{i \in V^{(n)}} D_i$ is odd, replace D_1 by $D_1 + 1$. The results we use are derived for i.i.d. sequences of n random variables in the limit $n \to \infty$. These results are also valid if we change the degree of a single vertex.

given by $\hat{D}_1, \ldots, \hat{D}_n$, with

$$\hat{D}_i = D_i - 2S_i - M_i \tag{2.3}$$

for all $i \in V^{(n)}$, where S_i is the number of self-loops of vertex i in the original network g, and M_i is the number of multiple edges of i in g.

This random construction procedure, which always produces a simple network, defines, for given $n \in \mathbb{N}$ and distribution ξ on $\mathbb{N}_0$ such that (2.2) is satisfied, the network belief system $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_\xi^{(n)})$. Note that this network belief system satisfies the exchangeability assumption (Assumption 1). The degree distribution of this network belief system is *not* equal to ξ; however, its degree distribution comes arbitrarily close to ξ when n grows large, as we show now. For $t \in \mathbb{N}_0$, define the random variable $\hat{P}^{(n)}(t) : \mathcal{G} \to \{0, 1, \ldots, n\}$ by

$$\hat{P}^{(n)}(t) := \frac{1}{n} \sum_{i \in V^{(n)}} I[\hat{D}_i = t],$$

where $I[A]$ is the indicator function of the event A. For $\hat{g} \in \mathcal{G}$ and $t \in \mathbb{N}_0$, $\hat{P}^{(n)}(t)(\hat{g})$ thus counts the number of vertices with degree t in the simple network $\hat{g}$. We refer to $(\hat{P}^{(n)}(t))_{t \in \mathbb{N}}$ as the *empirical degree distribution* of the network belief system $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_\xi^{(n)})$. Note that by definition, for any simple network $\hat{g}$, $\hat{P}^{(n)}(t)(\hat{g}) = 0$ for $t > n - 1$.

Taking expectations, we obtain the degree distribution of the network belief system $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_\xi^{(n)})$. That is, if we write $\nu_\xi^{(n)}(t) := \nu_\xi^{(n)}\left(\{\hat{g} \in \mathcal{G}^{(n)} \mid D_1(\hat{g}) = t\}\right)$,

$$t \in \mathbb{N}_0 : \quad \nu_\xi^{(n)}(t) := \mathbb{E}_\xi^{(n)}\left[\hat{P}^{(n)}(t)\right],$$

where $\mathbb{E}_\xi^{(n)}[\cdot]$ is the expectation with respect to the probability measure $\nu_\xi^{(n)}$. To characterize the degree distribution of the network belief system, we use Theorem 7.3 for the empirical degree distribution of Van der Hofstad (2006):

Theorem 2.2. Van der Hofstad (2006, Th. 7.3)
Let ξ be a probability distribution on $\mathbb{N}_0$ such that (2.2) is satisfied. For each $n \in \mathbb{N}$, let $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_\xi^{(n)})$ be the network belief system defined above. Then, the empirical degree distribution of the network belief system converges in probability to ξ:

$$\forall \varepsilon > 0 : \lim_{n \to \infty} \nu_\xi^{(n)} \left(\left\{ \hat{g} \in \mathcal{G}^{(n)} \middle| \sum_{t \in \mathbb{N}} |\hat{P}^{(n)}(t)(\hat{g}) - \xi(t)| \geq \varepsilon \right\} \right) = 0.$$

Proof. See Van der Hofstad (2006). □

A simple corollary of this result gives us a characterization of the degree distribution of the network belief system:

Corollary 2.3. *Let ξ be a probability distribution on $\mathbb{N}_0$ such that* (2.2) *is satisfied. For each $n \in \mathbb{N}$, let $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_\xi^{(n)})$ be the network belief system defined above. Then,*

$$\lim_{n \to \infty} \sum_{k \in \mathbb{N}_0} |\nu_\xi^{(n)}(t) - \xi(t)| = 0.$$

Proof. See Appendix A. □

In words, when the number of vertices n grows large, the degree distribution $(\nu_\xi^{(n)}(t))_{t \in \mathbb{N}_0}$ of the network belief system $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_\xi^{(n)})$ comes arbitrarily close to the asymptotic distribution ξ.

These results give us information about the distribution of the degrees of the individual vertices in the network belief system, but nothing about the correlation between the degrees of neighboring vertices. We now consider the degree correlation of the network belief system, i.e., the correlation in the degrees of the neighbors of player 1. Formally, let ξ be a distribution with support in the nonnegative integers with a finite mean, i.e., ξ satisfies (2.2). For $n \in \mathbb{N}$, $n \geq 2$, let $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_\xi^{(n)})$ be the network belief system defined above. Let the neighbor degree profile K_1 of player 1 be defined as before. By Proposition A.3.1 of Van der Hofstad et al. (2005), the degrees of neighboring vertices are asymptotically independent in the network belief system defined above:

Proposition 2.4. *Let ξ be a probability distribution on $\mathbb{N}_0$ such that* (2.2) *is satisfied. For each $n \in \mathbb{N}$, let $(\mathcal{G}^{(n)}, \mathscr{F}_{\mathcal{G}}^{(n)}, \nu_\xi^{(n)})$ be the network belief system defined above. Then, for all $t \in \mathbb{N}$, for all $(\theta_1, \ldots, \theta_t) \in \mathbb{N}^t$,*

$$\lim_{n \to \infty} \mu_\xi^{(n)}(\{\hat{g} \in \mathcal{G}^{(n)} \mid K_1(\hat{g}) = (\theta_1, \ldots, \theta_t)\}) = \frac{1}{\sum_{\theta \in \mathbb{N}_0} \theta \xi(\theta)} \prod_{\ell=1}^{t} \theta_\ell \xi(\theta_\ell),$$

where the factor θ_ℓ in the product stems from the fact that one is θ times more likely to have a neighbor of degree θ than to have a neighbor of degree 1.

Hence, in the limit of a large number of vertices, the correlation between the degrees of neighboring nodes vanishes for random networks with a given degree asymptotic distribution.

3. Network games of incomplete information

A network game of incomplete information is a standard Bayesian game where the states of nature are networks drawn from a network belief system $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu)$ and the players' types are their degrees: each player knows how many neighbors he has. Formally, let $n \in \mathbb{N}, n \geq 2$. A *network game of incomplete information* is a Bayesian game

$$\langle N, \mathcal{G}, (A_i)_{i\in N}, (\Theta_i)_{i\in N}, (\tau_i)_{i\in N}, \nu, (u_i)_{i\in N} \rangle,$$

where $N = \{1, \ldots, n\}$ is the set of players and $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu)$ (with $\mathscr{F}_{\mathcal{G}} = 2^{\mathcal{G}}$) a network belief system on vertex set $V = N$. Each player $i \in N$ has a nonempty, finite set A_i of pure strategies or *actions*. If the state of nature/network is $g \in \mathcal{G}$, player i's private information is his degree: the set of *types* or signals is $\Theta_i = \{0, \ldots, n-1\}$ and the *signal function* $\tau_i : \mathcal{G} \to \Theta_i$ assigns to each network $g \in \mathcal{G}$ the degree $\tau_i(g) := D_i(g)$ of player i. Finally, each player $i \in N$ has a von Neumann-Morgenstern *utility function* $u_i : (\times_{i\in N} A_i) \times \mathcal{G} \to \mathbb{R}$.

Recall that by Assumption 1, the signals of players are exchangeable. Furthermore, we make the following assumptions on the action sets and the payoff functions.

Assumption 2. (Identical actions) All players have the same set of actions: there exists a nonempty, finite set A such that $A_i = A$ for all $i \in N$. ◁

Assumption 3. (Local interactions) There is a set of functions $\mathcal{V} := \{v_t : A \times A^t \times \Theta^t \to \mathbb{R} \mid t \in \Theta\}$ such that for all $i \in N$, $a \in A^n$, $g \in \mathcal{G}$,

$$u_i(a, g) = \begin{cases} v_{\tau_i(g)}(a_i, (a_j)_{j\in N_i(g)}, (\tau_j(g))_{j\in N_i(g)}) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

That is, the utility of a player $i \in N$ in network $g \in \mathcal{G}$ only depends on the types and actions of his neighbors and himself when he has a positive degree. ◁

In the following, we refer to $\mathcal{V}$ as a set of *local payoff functions*.

Assumption 4. (Symmetric interactions) Payoffs depend on neighbors' actions and types in a symmetric way. Formally, for all $t \in \Theta \setminus \{0\}$, $a \in A$, $(a_1, \ldots, a_t) \in A^t$, $(\theta_1, \ldots, \theta_t) \in \Theta^t$, and all permutations $\pi : \{1, \ldots, t\} \to \{1, \ldots, t\}$,

$$v_t(a, (a_1, \ldots, a_t), (\theta_1, \ldots, \theta_t)) = v_t(a, (a_{\pi(1)}, \ldots, a_{\pi(t)}), (\theta_{\pi(1)}, \ldots, \theta_{\pi(t)})).$$

◁

Assumptions 1-4 are fairly general; they are satisfied in e.g. Sundararajan (2006), Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006) and Ioannides and Soetevent (2006).

Strategies are defined in the usual way. For $i \in N$, a (mixed) *strategy* for player i is a function $\sigma_i : \Theta \to \Delta(A)$. The probability that action $a_i \in A$ is played under strategy σ_i

by player $i \in N$ given that he has type $t_i \in \Theta$ is denoted by $\sigma_i(a_i|t_i)$. A *strategy profile* is a function $\sigma = (\sigma_i)_{i\in N}$, with σ_i a strategy of player i for each $i \in N$. We extend utility functions to mixed strategies in the usual way. For strategy profile $\sigma = (\sigma_j)_{j\in N}$ and $i \in N$, we write σ_{-i} to denote the strategy profile $\sigma = (\sigma_j)_{j\in N\setminus\{i\}}$ of the opponents of i. We say that a strategy profile σ is *symmetric* if $\sigma_i(\cdot) = \sigma_j(\cdot)$ for all $i, j \in N$.

We can now define expected payoffs. First, for notational simplicity, for $g \in \mathcal{G}$ and $t \in \Theta$, define

$$\nu(g|t) := \nu(g|\{g' \in \mathcal{G}|D_i(g') = t\}),$$

and

$$\nu(t) := \nu(\{g \in \mathcal{G}|D_i(g) = t\}).$$

For player $i \in N$, the *interim expected payoff* of action $a_i \in A$, given that he receives signal $t_i \in \Theta$ when the other players play according to the strategy profile σ_{-i} and players' prior is ν is given by

$$\begin{aligned}\varphi_i(a_i, \sigma_{-i}; t_i, \nu) &= \sum_{g\in\mathcal{G}} \nu(g|t_i)u_i(a_i, \sigma_{-i}, g)\\ &= \sum_{g\in\mathcal{G}} \nu(g|t_i)v_{t_i}(a_i, \sigma_{N_i(g)}, \tau_{N_i(g)}),\end{aligned}$$

where we have defined $\sigma_{N_i(g)} := (\sigma_j(\tau_j(g)))_{j\in N_i(g)}$ and $\tau_{N_i(g)} := (\tau_j(g))_{j\in N_i(g)}$. Similarly, the *ex ante expected payoff* of a player $i \in N$ of the strategy profile σ when the prior is ν is

$$\begin{aligned}\Phi_i(\sigma; \nu) &= \sum_{g\in\mathcal{G}} \nu(g)u_i(\sigma, g)\\ &= \sum_{t_i\in\Theta} \nu(t_i) \sum_{a_i\in A} \sigma_i(a_i|t_i)\varphi_i(a_i, \sigma_{-i}; t_i, \nu).\end{aligned}$$

Definition 3.1. Let $\varepsilon \geq 0$, and let $n \in \mathbb{N}, n \geq 2$. Consider a network game of incomplete information with n players and prior ν. A strategy profile $\sigma = (\sigma_i, \sigma_{-i})$ is an (interim) *ε-equilibrium* of the game if for all players $i \in N$, for all $t_i \in \Theta$ with $\nu(t_i) > 0$, all $a_i \in A$ with $\sigma_i(a_i|t_i) > 0$,

$$\varphi_i(a_i, \sigma_{-i}; t, \nu) \geq \varphi_i(b_i, \sigma_{-i}; t, \nu) - \varepsilon. \tag{3.1}$$

for all $b_i \in A$. That is, in an ε-equilibrium, a player can gain at most ε from unilateral deviation. An ε-equilibrium is *symmetric* if it is in symmetric strategies. ◁

A Bayesian-Nash equilibrium is a 0-equilibrium. By standard arguments, a symmetric Bayesian-Nash equilibrium for a network game of incomplete information exists:[12]

[12] See e.g. Becker and Damianov (2006).

Proposition 3.2. *Let $n \in \mathbb{N}, n \geq 2$, and let $\Gamma = \langle N, \mathcal{G}, (A)_{i\in N}, (\Theta)_{i\in N}, (\tau_i)_{i\in N}, \nu, (u_i)_{i\in N}\rangle$ be a network game of incomplete information with n players and prior ν. Then there exists a symmetric Bayesian-Nash equilibrium of the game Γ.*

When players follow a symmetric strategy, we can simplify the expressions for players' expected payoffs considerably. Recall that the sample space of neighbor degree profiles is Ω_K, with associated σ-field $\mathscr{F}_K$, and that the distribution of the neighbor degree profile of player 1 is $\mu_\nu(\cdot)$. A symmetric strategy profile σ with $\sigma_i(\cdot) = \sigma_j(\cdot)$ for all $i, j \in N$ can be denoted by $\tilde{\sigma} := (\tilde{\sigma}_t)_{t\in\Theta}$, with $\tilde{\sigma}_t(a) = \sigma_i(\cdot|t)$ for some $i \in N$ the probability that a player of type $t \in \Theta$ takes action $a \in A$. Let σ be a symmetric strategy profile, and let $\tilde{\sigma} = (\tilde{\sigma}_t)_{t\in\Theta}$, with, for all $t \in \Theta$, $\tilde{\sigma}_t = \sigma_i(\cdot|t)$ for some $i \in N$. For $t \in \Theta$ and type profile $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_t) \in \Omega_K$, we write $\tilde{\sigma}_{\boldsymbol{\theta}}$ to denote $(\tilde{\sigma}_{\theta_1}, \ldots, \tilde{\sigma}_{\theta_t})$. Then, for $i \in N$, $t \in \Theta$ such that $\mu_\nu(t) > 0$, and $a \in A$, we define

$$\begin{aligned} \tilde{\varphi}(a, \tilde{\sigma}; t, \mu_\nu) &:= \varphi_i(a, \sigma_{-i}; t, \nu) \\ &= \sum_{\boldsymbol{\theta}\in\Omega_K^t} \mu_\nu(\boldsymbol{\theta}|t) v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \theta), \end{aligned} \tag{3.2}$$

to be the interim expected payoff of any player of type t of action a when players play according to the symmetric strategy profile σ and the distribution of the neighbor degree profile is μ_ν. Similarly, for $i \in N$ and symmetric strategy profile σ, we define

$$\begin{aligned} \tilde{\Phi}(\tilde{\sigma}; \mu_\nu) &:= \Phi_i(\sigma; \nu) \\ &= \sum_{t\in\Theta} \mu_\nu(t) \sum_{a\in A} \tilde{\sigma}_t(a) \tilde{\varphi}(a, \tilde{\sigma}; t, \mu_\nu) \end{aligned} \tag{3.3}$$

to be the ex ante expected payoff of any player when players play according to the symmetric strategy profile σ and the distribution of the neighbor degree profile is μ_ν.

Some remarks are in order. In (3.2) and (3.3), respectively, interim and ex ante expected payoffs are defined solely in terms of the probability distribution μ_ν, defined on the "reduced" space Ω_K, rather than in terms of the prior ν, which is defined on the full state space $\mathcal{G}$. This has two notable implications. Firstly, all network games of incomplete information with the same action set, the same set of local payoff functions and the same probability distribution for the neighbor degree profile give the same expected payoffs when players follow a symmetric strategy. We can thus define classes of network games of incomplete information by specifying the action set, the set of local payoff functions and the probability distribution of the neighbor degree profile. This we will use in the next section.

Secondly, the current representation of expected payoffs allows us to analyze play in terms of overlapping "local" games, consisting of a player and his neighbors. In network games of incomplete information, each player plays a game with his neighbors in each state, who in turn interact with their neighbors, who in turn interact with their neighbors, and so on. In the next section, we show that each of these local games can nevertheless be viewed in isolation for our purposes.

4. Continuity of beliefs

Our objective is to define a "measure" of similarity of network belief systems such that if two network belief systems are similar according to this measure, then, for each network game of incomplete information, for each symmetric equilibrium of the game in which beliefs are given by one of these network belief systems, there exists an approximate equilibrium of the game with beliefs given by the other network belief system, such that ex ante payoffs are similar under both equilibria. If that is the case, then, for all possible payoff functions, can obtain approximately the same payoffs (ex ante) under both belief systems: from the players' (ex ante) perspective, the two network belief systems are very similar. At the same time, we do not want to make the conditions on network belief systems to be similar any stricter than necessary – when we say that two network belief systems are similar if and only if they are identical, the above holds trivially. We thus want to define a measure that guarantees that the above holds, but that is no stricter than necessary. In this section, we define such a measure. To get a feeling for the issues that play a role, consider the local investment game described in the next section.

4.1. Local investment

Let $k_1, k_2 \in \mathbb{N}, k_1 > k_2 > 1$. Let $n = (k_1+1)(k_2+2) + (k_2+1)(k_2+2)$, and let $\mathcal{G}$ be the set of all simple networks with n players, with $\mathscr{F}_{\mathcal{G}}$ its associated σ-field. The set of player types is thus $\Theta = \{0, 1, \ldots, (k_1+1)(k_2+2) + (k_2+1)(k_2+2) - 1\}$. We consider two different network belief systems with n players, $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu_{inf})$ and $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu_{cont})$:

Infection The network belief system $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu_{inf})$ puts equal probability on all networks with n players that consist of

- $k_1 + 2$ players of degree k_1, each linked with k_1 players of degree 1, and
- $k_2 + 2$ players of degree k_2, each linked with k_2 players of degree 1.

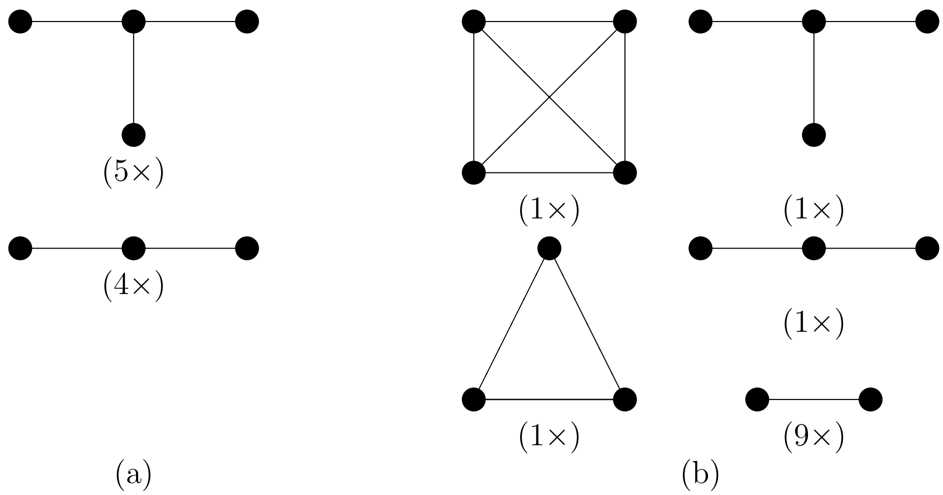

Figure 2: (a) A network that is realized with positive probability under $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu_{inf})$. (b) A network that is realized with positive probability under $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu_{cont})$. The numbers in brackets below the components give the multiplicity of those components in the network. For instance, there are 5 "star components" in the network in (a), consisting of a central player with degree 3 and three players of degree 1.

Containment The network belief system $(\mathcal{G}, \mathscr{F}_{\mathcal{G}}, \nu_{inf})$ puts equal probability on all networks with n players that consist of

- $k_1 + 1$ players of degree k_1 that are connected to each other,
- $k_2 + 1$ players of degree k_2 that are connected to each other,
- a player of degree k_1 who is linked with k_1 players of degree 1,
- a player of degree k_2 who is linked with k_2 players of degree 1, and
- $1/2(k_1(k_1+1)+k_2(k_2+1))$ players of degree 1, each linked with one of the other players of degree 1.

The name we have chosen for the two different belief systems will become clear when we discuss the possible equilibria of the local investment game under both network belief systems. Note that all networks that are realized with positive probability under a given network belief system are isomorphic: they only differ in the labels of the players. In Figure 2(a) and (b), we show two networks that are realized with positive probability when $k_1 = 3$ and $k_2 = 2$ under the "infection" and "containment" network belief system, respectively.

Note that these two network belief systems are identical in terms of their degree (type) distribution: in both network belief systems, there are $k_1 + 2$ players with degree k_1, $k_2 + 2$ players with degree k_2, and $k_1(k_1+2)+k_2(k_2+2)$ players with degree 1. However, they differ considerably in terms of the correlation between neighbor types. In particular, consider the conditional probability that a player with degree 1 is linked with a player of degree k_1. Denote the event that player 1 is connected to at least one player of degree $t \in \Theta$ by A_t. Then, the conditional probability that a player with degree 1 is linked with a player of degree k_1 is

$$\nu_{inf}(A_{k_1}|1) = \frac{k_1(k_1+2)}{k_1(k_1+2)+k_2(k_2+2)}$$

and

$$\nu_{cont}(A_{k_1}|1) = \frac{k_1}{k_1(k_1+2)+k_2(k_2+2)}$$

for the infection network belief system and the containment network belief system, respectively.

Consider the following game. Let N be a set of n players, and suppose that the players are located on a network, drawn from the set of all simple networks with n players according to either ν_{inf} or ν_{cont}. Each player has two actions, S and R. Action S is the safe action: it always gives a payoff of 0. The payoffs of the risky action R depend on a player's type, the prevailing network and the actions of other players. Let Θ_0 be a nonempty subset of $\mathbb{N}$. For player $i \in N$, the payoffs to action R in network g when the action profile of i's opponents is a_{-1} are given by

$$u_i(R, a_{-i}, g) = \begin{cases} -1 & \text{if } \tau_i(g) \in \Theta_0 \\ \nu_{cont}(A_3|1) & \text{if } \tau_i(g) \notin \Theta_0 \text{ and } a_j = R \text{ for all } j \in N_i(g) \\ -(1-\nu_{cont}(A_3|1)) & \text{otherwise.} \end{cases}$$

One interpretation of this game is that players need to decide whether to invest (play R) or not (play S). For players of type $t \in \Theta_0$, action R always gives a negative payoff, independent of the network or other players' actions. For other types, investment is risky: if all neighbors invest, a player earns a positive payoff if he has a type in $\Theta^{(n)} \setminus \Theta_0$, but if there is a neighbor who opts out, he receives a negative payoff.

We compare the infection and containment network belief systems in terms of their predictions for this game. Suppose that $\Theta_0 = \{k_1\}$. First note that, under the containment network belief system, there is a symmetric equilibrium $\tilde{\sigma}$ in which players invest if they they have type $t \neq k_1$, and play S if they have the 'bad' type $t = k_1$. Consider a player who has type $t = 1$. The safe action always gives a payoff of 0. When all players with type

$t \in \{1, k_2\}$ play R, and players with type $t = k_1$ play S, the interim expected payoffs of action R to a player of type $t = 1$ are

$$\begin{aligned}
\tilde{\varphi}(R, \tilde{\sigma}; 1, \mu_{cont}) &= (1 - \mu_{cont}(A_3|1))\nu_{cont}(A_3|1) - \mu_{cont}(A_3|1)\,(1 - \nu_{cont}(A_3|1)) \\
&= 0
\end{aligned}$$

Hence, a player with type $t = 1$ cannot gain by deviating if all players play according to this strategy. Now consider a player of type $t = k_2$. Given other players strategies, playing R always gives him a payoff of $\nu_{cont}(A_3|1) > 0$. Hence, it is optimal for him to play R. Finally, players with type $t = k_1$ have a dominant strategy to play S.

Now consider the infection network belief system. If $\varepsilon < \nu_{inf}(A_3|1) - \nu_{cont}(A_3|1)$, then in *any* symmetric ε-equilibrium, all types will choose the safe action with probability 1. To see this, first note that, as under the containment network belief system, players with degree $t = k_1$ will always play the safe action. Interim expected payoffs to a player with type $t = 1$ of playing R are then

$$\begin{aligned}
\tilde{\varphi}(R, \tilde{\sigma}; 1, \mu_{inf}) &= (1 - \mu_{inf}(A_3|1))\nu_{cont}(A_3|1) - \mu_{inf}(A_3|1)\,(1 - \nu_{cont}(A_3|1)) \\
&= -(\nu_{inf}(A_3|1) - \nu_{cont}(A_3|1)) \\
&< -\varepsilon
\end{aligned}$$

Hence, in any symmetric ε-equilibrium, players with type $t = 1$ will play S. Then, in any symmetric ε-equilibrium, the conditional probability that a player with type $t = k_2$ has at least one neighbor that plays the safe action is 1. Given other players' strategies, playing R gives a player with type $t = k_2$ interim expected payoffs of

$$\begin{aligned}
\tilde{\varphi}(R, \tilde{\sigma}; 2, \mu_{inf}) &= -\,(1 - \nu_{cont}(A_3|1)) \\
&\leq -\,(\nu_{inf}(A_3|1) - \nu_{cont}(A_3|1)) \\
&< -\varepsilon
\end{aligned}$$

Hence, in any symmetric ε-equilibrium, players with type $t = k_2$ play S.

Hence, while under the containment network belief system, there is a symmetric equilibrium in which only the 'bad' type $t = k_1$ chooses the safe action, and all other types invest, under the infection network belief system, other player types are 'infected': if ε is sufficiently small, in any symmetric ε-equilibrium, all types will choose the safe action. This example suggests that, while the payoffs are determined by a player's degree, the degree correlation of network belief systems also plays an important role. Of course, this stylized example is only suggestive. In the next section, we make precise what we mean by game-theoretical 'outcomes' being 'similar' or 'different', and we show that the degree distribution and the degree correlation of network belief systems are crucial for game-theoretical predictions to be similar.

4.2. Results

Before establishing our main result, we need some more definitions. Fix the player set N, and consequently the measurable space $(\mathcal{G}, \mathscr{F}_{\mathcal{G}})$, the type set Θ, and for each $i \in N$ the signal functions $\tau_i(\cdot)$. Let $\mathcal{M}_{\mathcal{G}}$ be the set of all probability distributions on the measurable space $(\mathcal{G}, \mathscr{F}_{\mathcal{G}})$, and let $\mathcal{M}_K$ be the set of all probability distributions on the measurable space $(\Omega_K, \mathscr{F}_K)$ of neighbor degree profiles. For each $\nu \in \mathcal{M}_{\mathcal{G}}$, the associated probability distribution $\mu_\nu \in \mathcal{M}_K$ is defined unambiguously by (2.1). Let $\mu \in \mathcal{M}_K$ be *feasible* if there exists a $\nu \in \mathcal{M}_{\mathcal{G}}$ such that $\mu = \mu_\nu$. Define

$$\mathcal{A}_K := \{\mu \in \mathcal{M}_K \mid \exists \nu \in \mathcal{M}_{\mathcal{G}} : \mu = \mu_\nu\}$$

to be the set of all distributions μ that are feasible/attainable. For a finite action set A, for a set of local payoff functions $\mathcal{V} := \{v_t : A \times A^t \times \Theta^t \mid t \in \Theta\}$ and for $\mu \in \mathcal{A}_K$, define $\mathscr{G}(A, \mathcal{V}, \mu)$ to be the set of all network games of incomplete information

$$\langle N, \mathcal{G}, (A)_{i\in N}, (\Theta)_{i\in N}, (\tau_i)_{i\in N}, \nu, (u_i)_{i\in N}\rangle$$

such that

(a) Players utility is given by $\mathcal{V}$. That is, for each $i \in N$,

$$u_i(a, g) = v_{\tau_i(g)}(a_i, (a_j)_{j\in N_i(g)}, (\tau_j(g))_{j\in N_i(g)})$$

for $a \in A^n$ and $g \in \mathcal{G}$.

(b) It holds that $\nu \in \{\bar{\nu} \in \mathcal{M}_{\mathcal{G}} \mid \mu = \mu_{\bar{\nu}}\}$.

Let $\mu \in \mathcal{A}_K$ be a feasible prior. For finite action set A and set of local payoff functions $\mathcal{V}$, consider the class $\mathscr{G}(A, \mathcal{V}, \mu)$ of network games of incomplete information. As all network games of incomplete information with the same action set, the same set of local payoff functions and the same prior over the neighbor degree profile give the same expected payoffs under symmetric strategy, the set of symmetric (approximate) Bayesian-Nash equilibria coincides for all games in the set $\mathscr{G}(A, \mathcal{V}, \mu)$. For $\varepsilon \geq 0$, we denote by $\mathcal{N}^\varepsilon(A, \mathcal{V}, \mu)$ the set of symmetric ε-equilibria of games in $\mathscr{G}(A, \mathcal{V}, \mu)$. Hence, the set $\mathcal{N}^0(A, \mathcal{V}, \mu)$ denotes the set of symmetric Bayesian-Nash equilibria of games in the set $\mathscr{G}(A, \mathcal{V}, \mu)$.

For finite action set A and set of local payoff functions $\mathcal{V}$, define

$$\mathscr{G}(A, \mathcal{V}) := \{\Gamma \in \mathscr{G}(A, \mathcal{V}, \mu) \mid \mu \in \mathcal{A}_K\}$$

to be the set of all possible network games of incomplete information with that action set and set of payoff functions. Then, for such a class of games, we can define its *bound* M by:

$$M := \max_{t\in\Theta;\boldsymbol{\theta}\in\Theta^t;a,a'\in A^{t+1}} |v_t(a, \boldsymbol{\theta}) - v_t(a', \boldsymbol{\theta})|. \tag{4.1}$$

This maximum exists, as the signal set Θ and the action set A are finite. Assume that there exists $t \in \Theta, \boldsymbol{\theta} \in \Theta^t$ and $a \in A^{t+1}$ such that $v_t(a, \boldsymbol{\theta}) = 0$.

We want to formalize the idea that two feasible network belief systems are "close" or similar in a strategic sense if for each network game of incomplete information, for each symmetric equilibrium of the game in which beliefs are given by one of these network belief systems, there exists an approximate equilibrium of the game with beliefs given by the other network belief system, such that ex ante payoffs are similar under both equilibria. To that end, let μ, μ' be two feasible priors, and let A be a finite action set. Let $\mathcal{V}$ be a set of local payoff functions. For $\varepsilon > 0$, define

$$\chi(\mu, \mu'; A, \mathcal{V}, \varepsilon) := \sup_{\tilde{\sigma}\in\mathcal{N}^0(A,V,\mu)} \inf_{\tilde{\sigma}'\in\mathcal{N}^\varepsilon(A,V,\mu')} |\tilde{\Phi}(\tilde{\sigma}, \mu) - \tilde{\Phi}(\tilde{\sigma}', \mu')|,$$

and let

$$\chi^*(\mu, \mu'; A, \mathcal{V}, \varepsilon) := \max\{\chi(\mu, \mu'; A, \mathcal{V}, \varepsilon), \chi(\mu', \mu; A, \mathcal{V}, \varepsilon)\}.$$

Hence, for two feasible priors and a given action set and set of local payoff functions, $\chi^*(\cdot)$ is a measure of the difference in outcomes under μ and μ' when players play according to a symmetric strategy. That is, for a given $\varepsilon > 0$, $\chi^*(\mu, \mu'; A, \mathcal{V}, \varepsilon)$ gives the absolute difference in ex ante payoffs when players have beliefs μ or μ' and we allow for ε-equilibria. To obtain $\chi(\mu, \mu'; A, \mathcal{V}, \varepsilon)$ for given $\mu, \mu' \in \mathcal{A}_K$, given $A, \mathcal{V}$ and $\varepsilon > 0$, for each symmetric equilibrium under μ, we find a symmetric ε-equilibrium under μ' which minimizes the (absolute) difference in ex ante payoffs under both equilibria, and then we look for the symmetric equilibrium under μ' which maximizes this difference. This formalizes the idea that for *each* symmetric equilibrium of the game in which beliefs are given by one prior, there exists *some* approximate equilibrium of the game with the other prior, such that ex ante payoffs are similar under both equilibria. To obtain a symmetric measure, we do the same for μ and μ' with their roles reversed. Finally, when ε increases, the set of approximate equilibria weakly increases, as more and more symmetric strategies will satisfy the equilibrium criterion, and the (absolute) difference in ex ante payoffs will decrease weakly. Hence, the interesting case is when ε becomes very small.

The local investment game of the previous section suggests that not only the ex ante probability that a player has a certain type matters for the outcomes in a network game with different network belief systems, but that also the correlation between neighbor types plays an important role.

Define

$$S_{\mu,\mu'} := \{t \in \Theta \mid \mu(t) > 0 \text{ and } \mu'(t) > 0\}$$

to be the set of types that occur with positive probability under both μ and μ'. Then, for two feasible priors $\mu, \mu' \in \mathcal{A}_K$, define

$$d_1(\mu,\mu') := \max_{t\in\Theta} |\mu(t) - \mu'(t)| \tag{4.2}$$

$$d_2(\mu,\mu') := \max_{F\in\mathscr{F}_K, t\in S_{\mu,\mu'}} |\mu(F|t) - \mu'(F|t)| \tag{4.3}$$

We can combine (4.2) and (4.3) to obtain

$$d^*(\mu,\mu') := \max\{d_1(\mu,\mu'), d_2(\mu,\mu')\}.$$

The function $d^*(\cdot)$ is nonnegative and symmetric, and it holds that $d^*(\mu,\mu') = 0$ if and only if $\mu = \mu'$. [13] However, $d^*(\cdot)$ need not satisfy the triangle inequality and is therefore not a metric. However, the function $d^*(\cdot)$ generates a topology in the following sense (cf. Kajii and Morris, 1998): a generalized sequence of feasible priors $(\mu^q)_{q\in\mathcal{Q}}$ with $\mathcal{Q}$ a directed index set with partial order $\succ$ converges to a feasible prior μ if and only if for all $\varepsilon > 0$, there exists $Q \in \mathcal{Q}$ such that $d^*(\mu^q,\mu) < \varepsilon$ for all $q \succ Q$.

We are now ready to state our main result.

Theorem 4.1. *Let μ be a feasible prior, and let $(\mu^q)_{q\in\mathcal{Q}}$ be a generalized sequence of feasible priors, where $\mathcal{Q}$ is a directed index set with partial order $\succ$. For all finite action sets A and all sets of local payoff functions $\mathcal{V}$, for all $\varepsilon > 0$, we have that*

$$(d^*(\mu^q,\mu) \to 0) \iff (\chi^*(\mu^q,\mu; A, \mathcal{V}, \varepsilon) \to 0).$$

The proof of Theorem 4.1 follows from the Proposition 4.4 and Lemma 4.5. Proposition 4.4 shows that if $d^*(\mu,\mu')$ is small for two feasible priors μ, μ', then for any network game

[13] The latter claim follows from Lemma 4.2 below.

of incomplete information, for any symmetric equilibrium of the game in which players hold prior μ, there exists an approximate equilibrium in the associated game with prior μ' such that ex ante payoffs are similar. We then show in Lemma 4.5 that both $d_1(\cdot)$ and $d_2(\cdot)$ need to be small for this result to hold. Before proceeding to Proposition 4.4 and Lemma 4.5, we first state Lemma 4.2, that will be used at various stages.

Lemma 4.2. *Let $\mu \in Ak$, and let $(\mu^q)_{q\in\mathbb{N}} \in (\mathcal{A}_K)^\infty$. For $\mathcal{T} \subseteq \Theta$, assume that there exists $c > 0$ such that $\mu^q(t) \geq c$ for all $t \in \mathcal{T}$ uniformly over q. Then,*

$$\lim_{q\to\infty} \max_{F\in\mathscr{F}_K} |\mu(F) - \mu^q(F)| = 0$$

if and only if

$$\lim_{q\to\infty} \max_{t\in\Theta} |\mu(t) - \mu^q(t)| = 0$$

and

$$\lim_{q\to\infty} \max_{F\in\mathscr{F}_K} \max_{t\in\Theta} |\mu(F|t) - \mu^q(F|t)| = 0.$$

Proof. See Appendix A. □

We now proceed to Proposition 4.4. Proposition 4.4 uses Lemma 4.3.

Lemma 4.3. *Let $\mu, \mu' \in \mathcal{A}_K$, and let $\delta \in [0, 1]$. Let A be a finite action set and let $\mathcal{V}$ be a set of local payoff functions. Let the bound $M \in \mathbb{R}$ be defined as in (4.1). Let $\Gamma \in \mathscr{G}(A, \mathcal{V}, \mu)$ and $\Gamma' \in \mathscr{G}(A, \mathcal{V}, \mu')$ be two network games of incomplete information. If $\tilde{\sigma}$ is a symmetric equilibrium of Γ and $d_2(\mu, \mu') \leq \delta$, then there exists a symmetric $3\delta M$-equilibrium $\tilde{\sigma}'$ of Γ', with $\tilde{\sigma}'_t = \tilde{\sigma}_t$ for all $t \in S_{\mu,\mu'}$.*

Proof. Set $\tilde{\sigma}'_t = \tilde{\sigma}_t$ for $t \in S_{\mu,\mu'}$. For $t \notin S_{\mu,\mu'}$, take $\tilde{\sigma}'_t$ such that $(\tilde{\sigma}'_t)_{t\notin S_{\mu,\mu'}}$ is a symmetric equilibrium of the reduced game where each player with a type $t \in S_{\mu,\mu'}$ is required to play $\tilde{\sigma}'_t = \tilde{\sigma}_t$. Such an equilibrium exists by Proposition 3.2. By construction, $\tilde{\sigma}'_t$ is a best response to $\tilde{\sigma}'$ for $t \notin S_{\mu,\mu'}$. It thus remains to show that $\tilde{\sigma}'_t$ is a $3\delta M$-best response for a type $t \in S_{\mu,\mu'}$. First, let

$$S_{\mu'} := \{t \in \Theta \mid \mu'(t) > 0\}$$

be the set of types that have positive probability under μ'. Also, let H be the event that a player interacts with at least one player with a type that has positive probability under μ' but not under μ, i.e.,

$$H = \{\boldsymbol{\theta} \in \Omega_K \mid \exists \ell : \theta_\ell \in S_{\mu'} \setminus S_{\mu,\mu'}\},$$

and let H^c be the complement (relative to Ω_K) of H. By definition, $\mu(H|t) = 0$ for all $t \in S_{\mu,\mu'}$, and

$$|\mu(E|t) - \mu'(E|t)| \leq \delta$$

for all $E \in \mathscr{F}_K$ and $t \in S_{\mu,\mu'}$. Hence,

$$\forall t \in S_{\mu,\mu'} : \quad \mu'(H \mid t) \leq \delta. \tag{4.4}$$

Let $t \in S_{\mu,\mu'}$, and $a, b \in A$ with $\tilde{\sigma}'_t(a) > 0$. Consider the difference

$$\begin{aligned}\tilde{\varphi}(a, \tilde{\sigma}'; t, \mu') - \tilde{\varphi}(b, \tilde{\sigma}'; t, \mu') = \\ \sum_{\boldsymbol{\theta} \in H} \mu'(\boldsymbol{\theta}|t) \left[v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right] + \\ \sum_{\boldsymbol{\theta} \in H^c} \mu'(\boldsymbol{\theta}|t) \left[v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right].\end{aligned} \tag{4.5}$$

By (4.4), recalling that the bound on the class $\mathscr{G}(A, \mathcal{V})$ of games is M, the first sum in (4.5) is at least $-\delta M$. To evaluate the second sum, first note that the neighbors of a player with neighbor degree profile $\boldsymbol{\theta} \in H^c$ play according to $\tilde{\sigma}$. As a lies in the support of the symmetric equilibrium $\tilde{\sigma}$ of Γ, and $\mu(\boldsymbol{\theta}|t) = 0$ for all $\boldsymbol{\theta} \in H$, it follows that

$$\begin{aligned}\sum_{\boldsymbol{\theta} \in H^c} \mu(\boldsymbol{\theta}|t) \left[v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right] &\geq \sum_{\boldsymbol{\theta} \in H} \mu(\boldsymbol{\theta}|t) \left[v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right] \\ &= 0\end{aligned}$$

Define $G_t := \{\boldsymbol{\theta} \in H^c \mid \mu(\boldsymbol{\theta}|t) - \mu'(\boldsymbol{\theta}|t) > 0\}$ and let G_t^c be the complement of G_t relative to H^c. For notational simplicity, define

$$V_{\mu,\mu'}(a, b; \tilde{\sigma}) := \left| \sum_{\boldsymbol{\theta} \in H^c} \left(\mu(\boldsymbol{\theta}|t) - \mu'(\boldsymbol{\theta}|t) \right) \left(v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right) \right|.$$

By the triangle inequality, and using that $d_2(\mu, \mu') \leq \delta$, it follows that

$$\begin{aligned}V_{\mu,\mu'}(a, b; \tilde{\sigma}) &\leq \left| \sum_{\boldsymbol{\theta} \in G_t} \left(\mu(\boldsymbol{\theta}|t) - \mu'(\boldsymbol{\theta}|t) \right) \left(v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right) \right| + \\ &\qquad \left| \sum_{\boldsymbol{\theta} \in G_t^c} \left(\mu(\boldsymbol{\theta}|t) - \mu'(\boldsymbol{\theta}|t) \right) \left(v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right) \right| \\ &= \sum_{\boldsymbol{\theta} \in G_t} \left(\mu(\boldsymbol{\theta}|t) - \mu'(\boldsymbol{\theta}|t) \right) \left| \left(v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right) \right| \\ &\qquad \sum_{\boldsymbol{\theta} \in G_t^c} \left(\mu'(\boldsymbol{\theta}|t) - \mu(\boldsymbol{\theta}|t) \right) \left| \left(v_t(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - v_t(b, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right) \right| \\ &\leq 2\delta M\end{aligned}$$

Combining these results gives

$$\varphi(a, \tilde{\sigma}'; t, \mu') - \varphi(b, \tilde{\sigma}'; t, \mu') \geq -3\delta M.$$

□

Proposition 4.4. *Let $\mu, \mu' \in \mathcal{A}_K$, and let $\delta \in [0,1]$. Let A be a finite action set and let $\mathcal{V}$ be a set of local payoff functions. Let the bound $M \in \mathbb{R}$ be as defined in* (4.1)*. Let $\Gamma \in \mathscr{G}(A, \mathcal{V}, \mu)$ and $\Gamma' \in \mathscr{G}(A, \mathcal{V}, \mu')$ be two network games of incomplete information. Suppose that $d^*(\mu, \mu') \leq \delta$. If $\tilde{\sigma}$ is a symmetric equilibrium of Γ, then there exists a symmetric $3\delta M$-equilibrium $\tilde{\sigma}'$ of Γ' such that*

$$|\tilde{\Phi}(\tilde{\sigma}; \mu) - \tilde{\Phi}(\tilde{\sigma}'; \mu')| \leq 3n\delta M,$$

where n is the number of players.

Proof. Define

$$G := \{\boldsymbol{\theta} \in \Omega_K \mid \mu(\boldsymbol{\theta}) - \mu'(\boldsymbol{\theta}) > 0\},$$

and let G^c be the complement of G. Also, define the function $\ell : \Omega_K \to \Theta$ by $\ell(\boldsymbol{\theta}) = t$ for $\boldsymbol{\theta} \in \Omega_K^t$.

Let $\tilde{\sigma}$ be any symmetric equilibrium of Γ. By Lemma 4.3, there exists a $3\delta M$-equilibrium $\tilde{\sigma}'$ of Γ' such that $\tilde{\sigma}'_t = \tilde{\sigma}_t$ for $t \in S_{\mu,\mu'}$. By Lemma 4.2,

$$\max_{F \in \mathscr{F}_K} |\mu(F) - \mu'(F)| \leq \delta n, \tag{4.6}$$

and thus

$$\begin{aligned}
|\tilde{\Phi}(\tilde{\sigma}; \mu') - \tilde{\Phi}(\tilde{\sigma}; \mu)| &\leq \left| \sum_{\boldsymbol{\theta} \in G} (\mu(\boldsymbol{\theta}) - \mu'(\boldsymbol{\theta})) \sum_{a \in A} \tilde{\sigma}_{\ell(\boldsymbol{\theta})}(a) v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right| + \\
&\qquad \left| \sum_{\boldsymbol{\theta} \in G^c} (\mu(\boldsymbol{\theta}) - \mu'(\boldsymbol{\theta})) \sum_{a \in A} \tilde{\sigma}_{\ell(\boldsymbol{\theta})}(a) v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right| \\
&= \sum_{\boldsymbol{\theta} \in G} (\mu(\boldsymbol{\theta}) - \mu'(\boldsymbol{\theta})) \sum_{a \in A} \tilde{\sigma}_{\ell(\boldsymbol{\theta})}(a) |v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta})| + \\
&\qquad \sum_{\boldsymbol{\theta} \in G^c} (\mu'(\boldsymbol{\theta}) - \mu(\boldsymbol{\theta})) \sum_{a \in A} \tilde{\sigma}_{\ell(\boldsymbol{\theta})}(a) |v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta})| \\
&\leq 2n\delta M. \qquad (4.7)
\end{aligned}$$

where we have used the triangle inequality in the first line. For $\mu_1, \mu_2 \in \mathcal{A}_K$, define

$$F_{\mu_1} := \{\boldsymbol{\theta} \in \Omega_K \mid \mu_1(\ell(\boldsymbol{\theta})) > 0\},$$

and

$$F_{\mu_1,\mu_2} := \{\boldsymbol{\theta} \in \Omega_K \mid \mu_1(\ell(\boldsymbol{\theta})) > 0, \mu_2(\ell(\boldsymbol{\theta})) > 0\}.$$

Then, by (4.6), as $\mu(F_{\mu'} \setminus F_{\mu,\mu'}) = 0$ by definition,

$$\mu'(F_{\mu'} \setminus F_{\mu,\mu'}) \leq n\delta.$$

Recalling that $\tilde{\sigma}'_t = \tilde{\sigma}_t$ for t such that $\mu(t) > 0$ and $\mu'(t) > 0$, this gives us

$$\begin{aligned}
&|\tilde{\Phi}(\tilde{\sigma}'; \mu') - \tilde{\Phi}(\tilde{\sigma}; \mu')| \leq \\
&\quad \left| \sum_{\boldsymbol{\theta} \in F_{\mu,\mu'}} \mu'(\boldsymbol{\theta}) \left[\sum_{a \in A} \tilde{\sigma}'_{\ell(\boldsymbol{\theta})}(a) v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}'_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - \sum_{a \in A} \tilde{\sigma}_{\ell(\boldsymbol{\theta})}(a) v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right] \right| + \\
&\quad \left| \sum_{\boldsymbol{\theta} \in F_{\mu'} \setminus F_{\mu,\mu'}} \mu'(\boldsymbol{\theta}) \left[\sum_{a \in A} \tilde{\sigma}'_{\ell(\boldsymbol{\theta})}(a) v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}'_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - \sum_{a \in A} \tilde{\sigma}_{\ell(\boldsymbol{\theta})}(a) v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right] \right| \\
&\quad = \sum_{\boldsymbol{\theta} \in F_{\mu'} \setminus F_{\mu,\mu'}} \mu'(\boldsymbol{\theta}) \left| \sum_{a \in A} \tilde{\sigma}'_{\ell(\boldsymbol{\theta})}(a) v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}'_{\boldsymbol{\theta}}, \boldsymbol{\theta}) - \sum_{a \in A} \tilde{\sigma}_{\ell(\boldsymbol{\theta})}(a) v_{\ell(\boldsymbol{\theta})}(a, \tilde{\sigma}_{\boldsymbol{\theta}}, \boldsymbol{\theta}) \right| \\
&\quad \leq M \sum_{\boldsymbol{\theta} \in F_{\mu'} \setminus F_{\mu,\mu'}} \mu'(\boldsymbol{\theta}) \\
&\quad \leq n\delta M
\end{aligned} \tag{4.8}$$

where we have used the triangle inequality in the first line. Combining (4.7) and (4.8) gives

$$|\tilde{\Phi}(\tilde{\sigma}; \mu) - \tilde{\Phi}(\tilde{\sigma}'; \mu')| \leq 3n\delta M.$$

□

We now establish that if either $d_1(\mu, \mu')$ or $d_2(\mu, \mu')$ is large for two feasible priors μ, μ', then ex ante payoffs can be very different under μ and μ'.

Lemma 4.5. *Fix $\delta \in [0, 1]$, and let $\mu, \mu' \in \mathcal{A}_K$. If*

$$\max_{t \in \Theta} |\mu(t) - \mu'(t)| > \delta$$

or

$$\max_{t \in S_{\mu,\mu'}, F \in \mathscr{F}_K} |\mu(F|t) - \mu'(F|t)| > \delta$$

there exists a network game of incomplete information $\Gamma \in \mathscr{G}(A, \mathcal{V}, \mu)$ for some A and $\mathcal{V}$ with bound 1 and an equilibrium $\tilde{\sigma}$ of Γ such that for any $\Gamma' \in \mathscr{G}(A, \mathcal{V}, \mu')$, for any symmetric δ-equilibrium $\tilde{\sigma}'$ of Γ' it holds that

$$|\tilde{\Phi}(\tilde{\sigma}; \mu) - \tilde{\Phi}(\tilde{\sigma}'; \mu')| > \delta.$$

Proof. By Lemma 4.2, if either

$$\max_{t\in\Theta}|\mu(t) - \mu'(t)| > \delta$$

or

$$\max_{t\in S_{\mu,\mu'}, F\in\mathscr{F}_K} |\mu(F|t) - \mu'(F|t)| > \delta$$

then, for $c > 0$,

$$\max_{F\in\mathscr{F}_K} |\mu(F|t) - \mu'(F|t)| > \left(1 + \frac{1}{c}\right)\delta$$

Hence, there exists a nonempty set of types $T \subseteq \Theta$, and, for each $t \in T$, an event $F_t \in \mathscr{F}_K$ such that $|\mu(F_t|t) - \mu'(F_t|t)| > \delta$. Let A be a finite action set and consider the set $\mathcal{V} := \{v_t A \times A^t \times \Theta^t \mid t \in \Theta\}$ of local payoff functions with

$$v_t(a, a^{(t)}, \boldsymbol{\theta}) = \begin{cases} 1 & \text{if } t \in T \text{ and } \boldsymbol{\theta} \in F_t, \\ 0 & \text{otherwise} \end{cases}$$

A game in the set $\mathscr{G}(A, \mathcal{V})$ is bounded by 1, and $|\tilde{\Phi}(\tilde{\sigma}; \mu) - \tilde{\Phi}(\tilde{\sigma}'; \mu')| > \delta$ for any two symmetric strategy profiles $\tilde{\sigma}$ and $\tilde{\sigma}'$, where $\tilde{\Phi}(\cdot)$ is the ex ante expected payoff under $\mathcal{V}$. □

We can now prove Theorem 4.1.

Proof. (If) Let A be a finite action set and let $\mathcal{V}$ be a set of local payoff functions. Let $M \in \mathbb{R}$ be as defined in (4.1). By Proposition 4.4, for $\varepsilon \geq 5Md^*(\mu, \mu^q)$, $\chi^*(\mu, \mu^q; A, \mathcal{V}, \varepsilon) \leq 3nd^*(\mu, \mu^q)$. Hence, for all finite action sets A and sets of local payoff functions $\mathcal{V}$ and for all $\varepsilon > 0$, if $d^*(\mu, \mu^q) \to 0$, then $\chi\chi^*(\mu, \mu^q; A, \mathcal{V}, \varepsilon) \to 0$.

(Only if) For $\delta \in [0, 1)$, if $d_1(\mu, \mu') > \delta$ or $d_2(\mu, \mu') > \delta$, then, by Lemma 4.5, there exists a finite action set A and set of local payoff functions $\mathcal{V}$ such that there exists a symmetric equilibrium $\tilde{\sigma}$ for a game $\Gamma\mathscr{G}(A, \mathcal{V}, \mu)$ such that in any δ-equilibrium $\tilde{\sigma}'$ of a game $\Gamma' \in \mathscr{G}(A, \mathcal{V}, \mu')$, we have $|\tilde{\Phi}(\tilde{\sigma}, \mu) - \tilde{\Phi}(\tilde{\sigma}', \mu')| > \delta$. □

Some remarks are in order. Firstly, an important question is how strong our conditions are. On the one hand, not very strong: the conditions we obtain for the class of network games of incomplete information are much weaker than those of Kajii and Morris (1998) for general Bayesian games with countable type sets. Kajii and Morris give two conditions. Firstly, priors need to be close in the weak topology, and, in addition, with high ex ante probability, it needs to be approximate common knowledge that players' posterior beliefs are close. It can be shown that only the first condition is relevant in the current context, as the type space is finite in our case.[14] Hence, Kajii and Morris thus require that network belief systems are close in the weak topology. By contrast, we require only that the degree distribution and the degree correlation of network belief systems are close. That is, rather than requiring closeness on a global scale – for the network belief system as a whole – we require only closeness on a more local scale, the "reduced" probability space. Hence, while it is not very surprising that we obtain weaker conditions than Kajii and Morris, given that we restrict attention to a subclass of this general class, the main value of our approach is that we exploit the local nature of the payoffs and the symmetry of the probability space and the payoff functions so that we can compare network belief systems on a local scale, rather than on a global scale. "Global" closeness is a very restrictive condition, while our conditions only require network belief systems to be close on a local scale.

On the other hand, our conditions can be rather restrictive. As discussed before, the literature on network games of incomplete information mostly focuses on games on random networks with a given asymptotic degree distribution (see Section 2.1). Theorem 4.4 essentially states that results from random network models with a given asymptotic degree distribution cannot be translated directly to network belief systems with a different (asymptotic) degree distribution (as in that case $d_1(\cdot)$ would remain large when the number of players grows large) or to network belief systems in which player types are not (asymptotically) independent (as then $d_2(\cdot)$ would remain large).

A second remark is that is straightforward to generalize the results to more general classes of local payoff functions when some restrictions on the network belief system are satisfied. We have assumed that a player's payoffs only depend on the actions and types of his direct neighbors, in a symmetric way (Assumption 3 and 4). Alternatively, we could assume that payoffs depend on the actions of types of a players' neighbors up to the kth degree for some $k \in \mathbb{N}$, again in a symmetric way. Formally, for $k \in \mathbb{N}$, $i \in N$ and $g \in \mathcal{G}$, the set $N_i^k(g)$ of kth degree neighbors of player i in network g is defined inductively as follows. Let $N_i^1(g) := N_i(g)$, and for $k \in \{2, 3, \ldots\}$, let $N_i^k(g) := \{N_j(g) | j \in N_i^{k-1}(g)\} \setminus N_i^{k-1}(g)$. To rule out cases where a player is a kth degree neighbor of himself, or where a given player is both a kth and an ℓth degree neighbor of another player, where $k, \ell \in \mathbb{N}, k \neq \ell$, we need to restrict $\mathcal{G}$ to be a subset of the set of all acyclic networks.[15] We can then define a degree profile for a player's neighbors up to the kth degree analogously to our

[14] As uniform convergence of prior probabilities implies uniform convergence of conditional probabilities on a finite event, weak convergence is necessary and sufficient in Bayesian games with a finite type space.

[15] An acyclic network is a network without any cycles, i.e., a network such that there is no sequence $(v_1, v_2, \ldots, v_\ell)$ of vertices for $\ell \in \mathbb{N}, \ell \geq 2$ such that $\{v_i, v_{i+1}\}$ is an edge for all $i \in \{1, \ldots, \ell\}$ and $\ell = 1$.

definition of a player's neighbor degree profile. For the restricted state space $\mathcal{G}$, results remain essentially unchanged if we replace the distribution of the neighbor degree profile by the distribution of the degree profile of neighbors up to kth degree. Note that even as k approaches the number of players n, our conditions remain weaker than those of Kajii and Morris (1998), as we essentially require weak convergence of degree profiles, while Kajii and Morris require weak convergence on the state space when the type set is finite.

5. Conclusions

Social networks are ubiquitous, and they may have a large effect on economic outcomes. We study a setting in which players are located on a social network and play a fixed game with their neighbors. Given the complexity of social networks, it is important to study whether game-theoretical predictions are robust in some sense to assumptions on players' beliefs and information. We have studied the robustness of game-theoretical predictions to assumptions on players' beliefs for the class of network games of incomplete information. In this class of games, players have incomplete information on the network structure: they have a common prior, and in addition, they know the number of contacts they have, i.e., their degree. We have asked what the conditions are on two priors such that for any network game of incomplete information in which players hold one of these priors, for any equilibrium in that game, there is an approximate equilibrium in the associated game with where players hold the other prior such that ex ante expected payoffs are close. Our main result (Theorem 4.1) states that the strategic behavior of a player is sensitive to the degree distribution, as well as to the correlation in the degrees of neighbors. Hence, the essential features of a prior in terms of game-theoretical predictions are the degree distribution and the degree correlation.

The present work can be extended in several directions. Firstly, a natural direction for future work would be to consider different classes of payoff functions. In the class of games we consider, payoffs only depend on the types and actions of a player and his neighbors in the network. In many settings, other properties of the network structure also affect payoffs. Such settings have been studied by some authors for the case that the network structure is common knowledge,[16] but not for the case when there is uncertainty about the network.

Another natural direction is to consider an alternative way to model players' beliefs over different networks. In the current framework, the number of players is common knowledge.

[16] See e.g. Ballester, Calvó-Armengol, and Zenou (2006).

The argument of Myerson (1998) that this is not always a natural assumption in general games holds a fortiori for network games. In general games, players typically interact with all other players, while in network games, players only interact with a small subset of the population. Hence, it is natural to assume that players do not know the size of the full network. In such "network games with population uncertainty" (cf. Myerson, 1998), players' type sets are countably infinite. In that case, our conditions for beliefs to be close will not be sufficient (see the discussion in Section 4 and Kajii and Morris (1998)). It is possible to show that in that case, priors need to be close in the weak topology, and, in addition, it needs to be the case that with high ex ante probability, a player has a type such that his posterior beliefs are close under both priors, and he interacts with high probability only with types for which posteriors are close, and that with high probability interact only with…, etcetera.[17] The local investment game we discussed in Section 4.1 gives some hint for why this is the case. In that game, players need to choose whether to invest or not. For some types, investing is never profitable, but for others, investment is profitable if all their neighbors invest. We have seen that even if a player has a type such that investing is profitable, and he is quite sure that his neighbor has a type such that investment is profitable, he might not invest, as he may think that his neighbor thinks that the neighbor of his neighbor thinks…that his neighbor has a type such that investment is not profitable. However, note that this "contagion" stems from correlation between players' types, not from the physical link between two given players, as in e.g. Morris (2000). In our setting, the actual network structure is irrelevant, it is the beliefs induced by the network belief system that count. This implies for instance that contagion can "jump" from component to component. In Section 4.1, we saw an example of a network belief system in which there were network components in which no player invests, even if all players in that component are connected to types for which investment could be profitable.

To establish our results, we have applied ideas and concepts from the literature on higher order beliefs in the setting of network games of incomplete information. There are other important questions in the setting of network games of incomplete information that can be answered using ideas from this literature. One important question is how sensitive game-theoretical predictions are to the assumptions on players' information about the network

[17]These conditions mirror the conditions of Kajii and Morris (1998) for general Bayesian games. This latter condition is the network analogon of the requirement of Kajii and Morris for general Bayesian games that with high ex ante probability, it needs to be approximate common knowledge that players' posterior beliefs are close. The condition of Kajii and Morris can be translated to a network context by replacing the belief operator they use by a variant of the neighborhood operator of Morris (1997, 2000).

structure. As in much of the literature on network games of incomplete information, we have assumed that players only know their degree. Indeed, Friedkin (1983) argues that the "observational horizon" of individuals is limited: they only know their local environment in the network. But how local is local? Some people may know only their direct friends, while others may know some of the friends of their friends, etcetera. As this observational horizon is hard to measure empirically and may vary considerably among agents in the network, it is important to investigate the robustness of predictions to informational assumptions in network games. Also, such a study may give insights under what conditions individuals have an incentive to learn about the network structure. Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2006) have studied this issue in a specific setting, but these issues have not been studied systematically. The results of Galeotti et al. (2006) indicate that informational assumptions are far from innocuous. The link with the literature on higher order beliefs may also be helpful here. The present results suggest that such robustness questions are important to study in network games of incomplete information. Moreover, this paper shows how one can utilize ideas from the literature on higher order beliefs to study such issues.

Appendix A Proofs

Proof of Corollary 2.3
Using $\mu_\xi^{(n)}(t) = \mathbb{E}_\xi^{(n)}[\hat{P}^{(n)}(t)]$, we find that

$$\sum_{t\in\mathbb{N}_0} |\mu_\xi^{(n)}(t) - \xi(t)| \leq \mathbb{E}_\xi^{(n)} \left[\sum_{t\in\mathbb{N}_0} |\hat{P}^{(n)}(t) - \xi(t)| \right]. \tag{A.1}$$

By Theorem 2.2, the random variable $\sum_{t\in\mathbb{N}_0} |\hat{P}^{(n)}(t) - \xi(t)|$ converges in probability to 0. Also, this random variable is bounded:

$$\begin{aligned} \sum_{t\in\mathbb{N}_0} |\hat{P}^{(n)}(t) - \xi(t)| &\leq \sum_{t\in\mathbb{N}_0} |\hat{P}^{(n)}(t) + \xi(t)| \\ &= 2, \end{aligned}$$

where the last equality follows from the definitions of ξ and $\hat{P}^{(n)}$. Hence, by dominated convergence,

$$\lim_{n\to\infty} \mathbb{E}_\xi^{(n)} \left[\sum_{t\in\mathbb{N}_0} |\hat{P}^{(n)}(t) - \xi(t)| \right] = 0,$$

and thus, from (A.1),

$$\lim_{n\to\infty}\sum_{t\in\mathbb{N}_0}|\mu_\xi^{(n)}(t)-\xi(t)|=0.$$

□

Proof of Lemma 4.2

(If) Suppose that $\lim_{q\to\infty}\max_{F\in\mathscr{F}_K}|\mu(F)-\mu(F)|=0$. Then, it is easy to see that for all $q\in\mathbb{N}$,

$$\max_{t\in\mathcal{T}}|\mu(t)-\mu(t)| \;\le\; \max_{F\in\mathscr{F}_K}|\mu(F)-\mu(F)|,$$

and thus

$$\lim_{q\to\infty}\max_{t\in\mathcal{T}}|\mu(t)-\mu(t)|=0.$$

Also, for all $q\in\mathbb{N}$,

$$\begin{aligned}\max_{t\in\mathcal{T}}\max_{F\in\mathscr{F}_K}|\mu(F|t)-\mu^q(F|t)| &\le \max_{t\in\mathcal{T}}\max_{F\in\mathscr{F}_K}\frac{1}{\mu^q(t)}|\mu(F,\Omega_K^t)-\mu^q(F,\Omega_K^t)|+\\ &\qquad \max_{t\in\mathcal{T}}\max_{F\in\mathscr{F}_K}\frac{\mu(F|t)}{\mu^q(t)}|\mu(t)-\mu^q(t)|\\ &\le \left(\frac{2}{c}\right)\cdot\max_{F\in\mathscr{F}_K}|\mu(F)-\mu^q(F)|\end{aligned}$$

and thus

$$\lim_{q\to\infty}\max_{t\in\mathcal{T}}\max_{F\in\mathscr{F}_K}|\mu(F|t)-\mu^q(F|t)|=0.$$

(Only if) Suppose that

$$\lim_{q\to\infty}\max_{t\in\mathcal{T}}|\mu(t)-\mu^q(t)|=0 \tag{A.2}$$

and

$$\lim_{q\to\infty}\max_{t\in\mathcal{T}}\max_{F\in\mathscr{F}_K}|\mu(F|t)-\mu^q(F|t)|=0. \tag{A.3}$$

Fix $\varepsilon>0$. For each $F\in\mathscr{F}_K$ and each $q\in\mathbb{N}$, it holds that

$$\begin{aligned}|\mu(F)-\mu^q(F)| &= |\sum_{t\in\mathcal{T}}[\mu(F|t)-\mu^q(F|t)]\,\mu(t)+\sum_{t\in\mathcal{T}}\mu^q(F|t)[\mu(t)-\mu^q(t)]|\\ &\le \sum_{t\in\mathcal{T}}|\mu(F|t)-\mu^q(F|t)|\mu(t)+\sum_{t\in\mathcal{T}}\mu^q(F|t)|\mu(t)-\mu^q(t)|\end{aligned}$$

By (A.2) and (A.3), there exists $Q\in\mathbb{N}$ such that for all $t\in\mathcal{T}$, $q>Q$ implies that

$$|\mu(F|t)-\mu^q(F|t)|<\varepsilon\cdot\left(\frac{c}{1+c}\right)$$

and

$$|\mu(t) - \mu^q(t)| < \varepsilon \cdot \left(\frac{c}{1+c}\right).$$

Hence, for $q > Q$,

$$\begin{aligned}
|\mu(F) - \mu^q(F)| &\leq \varepsilon \cdot \left(\frac{c}{1+c}\right) \left[\sum_{t \in \mathcal{T}} \mu(t) + \sum_{t \in \mathcal{T}} \mu^q(F|t)\right] \\
&\leq \varepsilon \cdot \left(\frac{c}{1+c}\right) \cdot \left[1 + \frac{1}{c} \sum_{t \in \mathcal{T}} \mu^q(F|t)\mu^q(t)\right] \\
&\leq \varepsilon \cdot \left(\frac{c}{1+c}\right) \cdot \left[1 + \frac{1}{c}\right] \\
&= \varepsilon
\end{aligned}$$

and hence $\lim_{q \to \infty} |\mu(F) - \mu^q(F)| = 0$. As this holds for all $F \in \mathscr{F}_K$, we have

$$\lim_{q \to \infty} \max_{F \in \mathscr{F}_K} |\mu(F) - \mu^q(F)| = 0.$$

□

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