Equilibrium spatial pricing in a multi-dimensional setting

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Abstract

Consider a population of citizens uniformly spread over the entire plane, that faces a problem of locating public facilities to be used by its members. The cost of every facility is financed by its usersonly, who in addition face an idiosyncratic private access cost to the facility. We assume that the facilities' cost is independent of location and access costs are linear with respect to the Euclidean distance. We show that the external intervention that covers 0.19% of the facility cost is sufficient to guarantee *secession-proofness* or *no cross-subsidization*, where no group of individuals is charged more than its stand alone cost incurred if it had acted on its own. Moreover, we demonstrate that in this case the Rawlsian access pricing is the only secession-proof allocation.

Keywords: secession-proofness, optimal jurisdictions, Rawlsian allocation, hexagonal partition, cross-subsidization.

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1 Introduction

Consider a society that faces a problem of locating over space one or several public facilities (or public projects as in Mas-Colell, 1980) to serve its members. The facilities, say libraries, could be visited by citizens at some private access cost related to distance between a citizens residence and the nearest facility. Assuming that setting up a facility entails a fixed cost, the following problems arise:

- how many facilities should be built;
- where should the facilities be located;
- how to assign citizens to the facilities; and
- how should the fixed costs of the facilities be covered?

In this paper we examine the case where

- demand for use of services is uniformly distributed over the plane, independently of the cost of services;

- the cost of setting up a facility is independent of location;

- access or transportation cost is proportional to Euclidean distance.

Given the number and location of facilities, the assignment of users to facilities and cost allocation, we assume that citizens-users enjoy a "free entry" option: *any group can build a new facility for their own benefit at the standard fixed cost, and locate it at will.* A threat of free entry leads us imperatively to impose the "secession-proofness" or "core" property: at equilibrium, no coalition of users could benefit from seceding from the proposed arrangement to operate its own facility. Since we may identify the geographical area served by a public facility as a political jurisdiction. Secession-proofness is then a requirement of political stability.[1] The secession-proofness also can be viewed as a "no cross-subsidization" requirement where no group of users is required to contribute more than its stand-alone cost. In other words, the cost allocation should ensure the voluntary participation of any group of citizens.

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[1] Note that one of the authors is a Belgian citizen, familiar with non-overlapping, subject-specific political partitions of a national territory.

The immediate implication of secession-proofness is that the society should minimize its total cost of setting up and operating the facilities, plus covering the access costs. In others words, the secession-proofness yield an *efficient* partition of the society into jurisdictions. The characterization of efficient partitions in this geometric setting is a well-documented problem in mathematics whose solution consists of identical regular hexagons,[2] whose size is calculated as a function of the ratio of fixed costs per facility to access or transportation costs per unit of distance.

However, the area over which total costs per user are minimized is not a hexagon but a disk! Since the plane cannot be partitioned into disks, it helps to explain the first result of this paper that demonstrates that the set of secession-proof allocations is empty. This simply means that it is impossible to allocate facilities' cost over hexagons in efficient partition in order to rule out a threat of secession by all disk-shaped jurisdiction.

Non-existence of secession-proof allocations implies that the stability can be ensured only at some cost. We consider the situation where an external source is willing to finance a fraction δ of the total cost incurred by jurisdictions if they follow the prescribed agreement. Suppose that the total costs (set-up plus operation plus access or transportation) for the jurisdiction-to-be at the proposed equilibrium are subject to the discount factor $1 - \delta$, whereas forming a new jurisdiction to set up and operate an independent facility requires a full non-subsidized cost. Then the allocation will be δ-secession-proof if the savings reaped by the seceding jurisdiction fall short of the subsidy incurred by members of that jurisdiction at the proposed cost allocation.

We then examine the minimal subsidy that can rectify the stability failure. We show that the ratio of total per capita costs to users under a partition into hexagons of optimal size and in an optimally-sized disk is $\delta^* \approx 1.0019$. By using Fubini's Theorem, we demonstrate that the set of δ-secession-proof allocations is non-empty for $\delta \geq \delta^*$. This small figure (less than 0.2%) lends credence to the δ-secession-proofness as a stability concept.

The second result of the paper is the characterisation of the δ^*-secession-proof allocations. It is provided by the so-called "Rawls principle" that requires the minimization of the total cost incurred

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[2] See discussion below.

by the least privileged citizen-user. That principle leads unambiguously to equalize total cost over all citizens-users. A transparent characterization of that principle is as follows:

- citizens-users should be served by the facility closest to their residence;
- subject access to a fee that declines linearly with the residence-to-facility distance;
- adjust access fees so that operators of the facilities break even.

The Rawls principle defines uniquely the δ^*-secession-proof allocation. (For higher values of δ, the set of δ-secession-proof allocations contains other allocations as well.) The Rawlsian policies are often advocated on the basis of *justice* considerations, whereas our result offers a *stability* argument in support of the Rawls principle.

Related Literature. To the best of our knowledge, the related literature in economics deals almost exclusively with the uni-dimensional case. Cremer, de Kerchove and Thisse (1985), Alesina and Spolaore (1997) examine the existence of secession-proof allocations in the case where the population is uniformly spread over a *bounded* interval and the unique cost share rule available for each jurisdiction is the *equal-share* scheme, according to which all citizens in the same jurisdiction make an identical contribution towards the facility cost. Casella (2001) studies a model where individuals are uniformly distributed over the circle. Le Breton and Weber (2003) and Haimanko, Le Breton and Weber (2004) address the existence and characterization of secession-proof cost allocations in the case of general distributions. Bogomolnaia et al. (2005a,b) examine the issue of secession-proof allocations under various notions of stability. Drèze, Le Breton and Weber (2005) prove that when the population is uniformly spread over the entire real line, then the Rawlsian distribution is the unique secession-proof allocation. It is important to note that in the uni-dimensional setting δ^* is equal to zero, i.e., there is no gap between efficiency and optimality and an efficient partition consists of optimal-size intervals. Notice also that our set-up is that of "horizontal differentiation" where individuals display distinct preferences over geographical locations of public facilities. It is in contrast to a large body of literature of "vertical differentiation" in this context, where individuals exhibit identical preferences over quantity or quality attributes of public projects[3]

The paper is organized as follows. The next section contains the model and introduces the definitions needed for our main results that are stated in Section 3. The proofs of all results are relegated to the Appendix.

2 The Model

We consider a society with a continuum of individuals that has to determine a partition into multiple groups (jurisdictions), each choosing a public facility accessible to its members and sharing the facility cost among them. Thus, a solution for the entire population would consist of a set of jurisdictions, chosen facilities and cost allocations for every jurisdiction. The facilities will be located in a multi-dimensional space:

Assumption A.1 — Multidimensionality: The space of facilities' locations is the two-dimensional Euclidean space $X = \Re^2$.

Citizens have idiosyncratic preferences (or accession costs) over possible facilities they could be assigned to. We assume that for every individual the accession cost is represented by the Euclidean distance from her bliss point to the facility in her jurisdiction:

Assumption A.2 — Euclidean accession costs: For every individual located at $l = (l_1, l_2) \in X$, her accession cost to every $t = (t_1, t_2) \in X$ is given by

$$||t - l|| = \sqrt{|t_1 - l_1|^2 + |t_2 - l_2|^2}. \tag{1}$$

This formalization allows us to identify an individual with her location and to characterize the society by the distribution of individuals' locations. We assume that the citizens are uniformly distributed over the entire space X:

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[3] See e.g., Westhoff (1977), Guesnerie and Oddou (1981, 1988), Wooders (1978, 1980), Guesnerie (1995), Weber and Zamir (1985), Greenberg and Weber (1986), Jéhiel and Scotchmer (1997, 2001), Konishi, Le Breton and Weber (1998).

Assumption A.3 — Uniform distribution: The citizens' distribution is given by the two-dimensional Lebesgue measure[4] λ over $\Re^2$.

The area of a measurable[5] set S will be denoted by $\lambda(S)$, i.e., $\lambda(S) = \int\limits_S dt$. In what follows, the null-measured sets with $\lambda(S) = 0$ will be disregarded, so that the qualification "up to a null-set" should be added to almost all our results.

In our set-up, every jurisdiction is a measurable bounded subset of X with positive measure. The collection of such sets will be denoted by $\mathcal{M}(X)$. We assume that the cost of each facility is independent of its location and consists of a fixed cost, independent of the size of a jurisdiction, and a variable cost proportional to the jurisdiction size:

Assumption A.4 — Facility cost: For a facility assigned to a jurisdiction, the cost is given by

$$f(S) = g + \alpha\lambda(S), \tag{2}$$

where $g > 0, \alpha \geq 0$ are two constants.

We now formally introduce the notion of a partition of a measurable subset $S \subset X$:

Definition 2.1: A partition P of a (possiblyinfinite-measured) set S is a jurisdiction structure that consists of sets from $\mathcal{M}(X)$ which are "almost" pairwise disjoint, i.e., $\lambda(T \bigcap T') = 0$ for all $T \neq T'$ in P, and whose union constitutes the entire set S, i.e., $\bigcup_{T \in P} T = S$.

The set of partitions of S is denoted by $\mathcal{P}(S)$. Obviously, if the measure of S is infinite, then every $P \in \mathcal{P}(S)$ consists of an infinite number of jurisdictions.

Now let us turn to the determination of facility choices. For each $S \in \mathcal{M}(X)$ and a location $l \in S$ we denote by $D(S, l)$ the value of total accession cost in S (with respect to location l):

$$D(S, l) = \int\limits_S ||t - l|| dt. \tag{3}$$

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[4] See Halmos (1950), p. 153.

[5] A subset of X is measurable if its intersection with every measurable subset of a finite measure is measurable; hence, we allow for infinite-measured measurable subsets.

Suppose that jurisdiction S forms and chooses a location l. One can derive the total cost of members of S that combines the project cost $f(S)$ and the aggregate access cost, $D(S,l)$. Thus the *per capita* cost of members of S whose location is l is given by

$$K(S,l) = \frac{f(S) + D(S,l)}{\lambda(S)}. \tag{4}$$

Now, consider any measurable set $S \subset X$, finite or infinite-measured. This set can be partitioned into several jurisdictions each choosing a facility location. We define its stand alone per capita cost[6] as the minimum over all possible partitions P of S and sets of location choices $L = \{l(T)\}_{T \in P}$:

$$\tilde{K}(S) = \inf_{P,L} \frac{\sum_{T \in P} \left[D(T, l(T)) + f(T) \right]}{\lambda(S)}. \tag{5}$$

Any partition solving program (**??**) is called $S-$efficient. It is easy to see that efficiency implies that every jurisdiction T would choose the location of the facility in order to minimize the total accession costs of its members. That is, each jurisdiction T would choose the location l that would satisfy the following program:

$$D(T,l) \to \min_{l \in X}. \tag{6}$$

The value of this problem is called "$MAT(T)$" (Minimal Aggregate Transportation cost of the set T).[7] Any solution to (**??**) is called a *central location* of T.[8] We use the following lemma:

Lemma 2.2: For every jurisdiction $T \in \mathcal{M}(X)$, the central location, denoted by $m(T)$, is unique.[9]

Since Lemma 2.2 resolves the locational choice for every jurisdiction, it allows us to reduce the examination of efficiency to partitions only. For this end, denote

$$D(T) = D(T, m(T)), \; K(T) = K(T, m(T)). \tag{7}$$

and for any measurable set $S \subset X$, finite or infinite-measured,

$$\tilde{K}(S) = \inf_{P \in \mathcal{P}(S)} \frac{\sum_{T \in P} \left[D(T) + f(T) \right]}{\lambda(S)}. \tag{8}$$

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[6] Since the total cost could be infinite, an operational definition is needed. It consists of taking limits when the area uniformly approaches S.

[7] A solution to this problem exists. Indeed, the integral in (**??**) is continuous in l, and for $l \to \infty$ the value of a program goes to $+\infty$.

[8] Note that in the unidimensional setting, for every bounded set T, a location is central if and only if it is a median of T.

[9] This statement is esentially multi-dimentional: on the line, there could be an interval of medians.

We have

Definition 2.3: A partition P is S-efficient if it is a solution to (**??**). An X-efficient partition will be simply called an efficient partition.

In what follows, we will focus our analysis on efficient partitions. The characterization of efficient partitions in our geometric setting is a well documented problem in mathematics. The qualitative result (re)discovered by many authors states that there is a unique "shape" of efficient partitions which consists of identical regular hexagons.[10] We have:

Result 2.4: Partition P is efficient if and only if it is comprised of identical regular hexagons, whose stand-alone value is minimal among all regular hexagons.

The size of hexagons in efficient partitions obviously depends upon the value of the fixed component of facility costs: the smaller the cost, the smaller are jurisdictions in an efficient partition. The size of "efficient" hexagons is explicitly derived in the Appendix.

Before turning to the main results of the paper, let us examine cost sharing rules. In every potential jurisdiction $S \in \mathcal{M}(X)$, an S-sharing rule y is a measurable function on S that specifies individual contributions of members of S towards the cost of a facility, $f(S)$, if this jurisdiction forms. We impose the budget-balancedness condition:

Assumption A.5 — Budget balancedness: The total contribution of members of S covers the cost of the facility:

$$\int_S y(t)dt = f(S). \tag{9}$$

When we turn to examination of efficient partitions, it would be useful to consider the notion of *consistent sharing rule*. Since the whole plane is partitioned into identical (hexagonal) jurisdictions, it makes sense to demand that the individuals in identical locations within different jurisdictions bear the same costs. We impose a weak form of consistency that requires that any two individuals in

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[10] See Fejes Toth (1953), Haimovich and Magnanti (1988) as well as Christaller (1933), Lösch (1954), Bollobas and Stern (1972), and Stern (1972) in the economic geography context.

any two different jurisdictions, whose location is identical with respect to their corresponding central points, make the same monetary contribution towards their facilities cost.[11]

Assumption A.6 — Consistent sharing rule: For every efficient partition P^*, every two different (hexagonal) jurisdictions $H, H' \in P^*$ and every two individuals $t \in H, t' \in H'$ satisfying $t - m(S) = t' - m(S')$, we have $y(t) = y(t')$.

The sharing rule y associated with partition P^* determines the following cost allocation for any individual $t \in X$

$$c(t) = y(t) + ||t - m(H^t)||, \tag{10}$$

where $H^t \in P^*$ is the hexagon in P^* that contains t, and $m(H^t)$ is its center.

From now on, we fix one of the (fully equivalent to each other) efficient partitions, name it P^*. The cost sharing rule chosen by the society should satisfy a minimal requirement of *voluntary participation* when no group of individuals should contribute more than the cost incurred if it had acted on its own. Thus, the formation of jurisdictions and the choice of cost allocations within each of them rules out the emergence of a potentially seceding group, that can benefit all its members. Formally,

Definition 2.5: Let a cost allocation c be given. A set $S \in \mathcal{M}(X)$ is *prone to secession* if

$$c(S) = \frac{1}{\lambda(S)} \int_S c(t)dt > K(S). \tag{11}$$

A cost allocation c is *secession-proof* if no set $S \in \mathcal{M}(X)$ is prone to secession. The set of secession-proof cost allocations on X will be denoted by $\mathcal{A}$.

Next definition introduces the allocations that satisfy the *Rawls principle* by minimizing the total cost of the most disadvantaged individual in each jurisdiction. It implies the *cost equalization* across the entire society:

Definition 2.6: A cost allocation r is called *Rawlsian* if the value $r(t)$ is constant within each $H \in P^*$, and, hence, on X. That is, for every $t, t' \in X$ we have $r(t) = r(t')$.

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[11] This assumption simplifies calculus of the proof, while it is not essential for the main result.

3 Results

We are now in position to state the main results of the paper. First, we demonstrate that under our assumptions, a secession-proof allocation fails to exist.

Proposition 3.1: Suppose that assumptions A.1-A.6 hold. Then the set of secession-proof allocations $\mathcal{A}$ is empty.

In absence of secession-proof allocations, we will turn to the search for a solution which is the "closest" to be secession-proof. For instance, we may assume that there is a fixed per capita cost of a secession by any subgroup $S \subset X$; alternatively, one can consider government intervention which subsidizes a certain fraction of a total cost to every citizen to prevent the formation of groups prone to secession. Both approaches are essentially equivalent and yield the following definition of *δ-secession-proofness*:

Definition 3.2: Let $\delta > 0$ be given. A cost allocation c is *δ-secession-proof* if for all $S \in \mathcal{M}(X)$ the following inequality holds:

$$(1 - \delta)c(S) \leq K(S). \tag{12}$$

The set of δ-secession-proof allocations on X will be denoted by $\mathcal{A}(\delta)$.

In other words, if individuals follow the prescribed agreement, then the δ-part of their total cost is covered "from outside". If, however, a jurisdiction wants to secede, then their members will have to bear costs on their own.

This definition relaxes the constraints which determine secession-proof allocations and, obviously, if δ is large enough then the set $\mathcal{A}(\delta)$ is nonempty. Moreover, if $\mathcal{A}(\delta)$ is nonempty for some δ, it is also the case for all $\delta' > \delta$. This allows us to derive the threshold value δ^* defined by

$$\delta^* = \inf\{\delta > 0 | \ \mathcal{A}(\delta) \neq \emptyset\}. \tag{13}$$

It will be shown that the set $\mathcal{A}(\delta^*)$ is itself nonempty. The value δ^* therefore can represent the *cost of stability*, which is the minimal per-capita subsidy which sustains secession-proofness. We can now state our main result:

Proposition 3.3: Under Assumptions A.1-A.6,

(i) $\delta^* \approx 0.0019$;

(ii) The set $\mathcal{A}(\delta^*)$ is a singleton, containing only the Rawlsian allocation.

> That is, the cost of stability δ^* is very small. Moreover, the only δ^*-secession-proof allocation is Rawlsian.

The statement of this proposition requires an explanation. Consider a hexagon H, which is an element of an efficient partition. Obviously, this hexagon is not optimal in terms of per capita cost of its members and the value $K(H)$ exceeds

$$\min_{S \in \mathcal{M}(X)} K(S). \tag{14}$$

In fact, no hexagon represents a solution for (**??**). Unsurprisingly, jurisdictions with the minimal per capita total cost are discs. Denote by $K(B)$ the value of the problem in (**??**). We then show that the cost of stability δ^* is given by

$$\delta^* = 1 - \frac{K(B)}{K(H)}, \tag{15}$$

which, since $K(B) < K(H)$, is obviously positive. Thus, the cost gap between an efficient hexagon and an optimal disc necessitates the government intervention and subsidization of efficient partitions. It is important to point out that this feature does not appear in the uni-dimensional setting where efficient and optimal jurisdictions are intervals of the same size and the cost of stability is equal to zero (see Drèze, Le Breton and Weber (2005)).

As for the uniqueness of a Rawlsian allocation, this is a bit tricky (with the formal proof in the Appendix), but the general idea is the following one. First of all, under a secession-proof allocation for minimal δ^*, every optimal disc should be charged the same; otherwise there would exist discs charged more than their stand-alone cost. Second, every set which is *almost* optimal disc is characterized, via Lemma A.2, by a per capita cost which differs from the minimal cost in the second order. This would allow us to claim that in a very thin ring inscribed into any optimal disc, the measure of agents who are charged less than average is tiny even compared to the measure of the ring. As the ring could be chosen arbitrarily thin, and such rings cover the whole plane, we conclude that the measure of those charged less than average is just zero.

4 Appendix

Proof of Lemma 2.2: Let $S \in \mathcal{M}(X)$ be given and assume that S has two different central points, m and m'. Let L be the straight line connecting m and m' and denote $S' = S \setminus L$ and $\bar{m} = \dfrac{m+m'}{2}$. Obviously then m and m' are central points of S' as well and $D(S) = D(S')$. Then for every $t \in S'$ we have

$$\frac{1}{2}\left(||t-m|| + ||t-m'||\right) > ||t-\bar{m}|| \tag{16}$$

and, since $\lambda(S) = \lambda(S') > 0$, this implies that

$$\int\limits_{S'} ||t-\bar{m}||dt < \frac{1}{2}\left(\int\limits_{S'} ||t-m||dt + \int\limits_{S'} ||t-m'||dt\right). \tag{17}$$

However, by (**??**), the right-hand side of (**??**) is equal to $D(S) = D(S')$, a contradiction to m and m' being central points of S'. □

Before proceeding with the proof of Propositions 3.1 and 3.3, we need some notation and preliminary results. From now on we shall assume, without loss of generality, that the variable component of facility costs α in Assumption A.4 is equal to zero.

Denote by B_a^l the disc with the center at $a \in X$ and the radius $l > 0$.

Lemma A.1: A set S is a solution of (**??**) if and only if $S = B_a^{l^*}$, where $a \in X$ and the value of l^* is given by

$$l^* = \left(\frac{3g}{\pi}\right)^{\frac{1}{3}} \approx 0.985 g^{\frac{1}{3}} \tag{18}$$

Moreover, the per capita cost in such a disc, $K(B)$, is equal to l^*.

From now on, the disc with the optimal size l^* and the center a will be referred to as simply B_a. Sometimes, if the center of the disc is not important, we will use the notation B^l for $l > 0$.

Proof: Take a set S that solves (**??**). Denote the disc of radius l with the center at $m(S)$ by B^l. There trivially exist l_1, l_2 with $0 \le l_1 \le l_2 < \infty$ such that, both $B^{l_1} \setminus S$ and $S \setminus B^{l_2}$ are null-sets, and two sets, $B^l \setminus S$ and $S \setminus B^l$ have a positive measure for all $l \in (l_1, l_2)$. We claim that l_1 and l_2 coincide, i.e., $S = B^{l_1} = B^{l_2}$.

Indeed, if not, take $l_3 = (2l_1 + l_2)/3$ and $l_4 = (l_1 + 2l_2)/3$. Then both $\lambda(S \setminus B^{l_4})$ and $\lambda(B^{l_3} \setminus S)$ are positive numbers. We can shift a positive mass of individuals from $S \setminus B^{l_4}$ to $B^{l_3} \setminus S$ so that the newly created set $\tilde{S}$ has the same measure as S. However,

$$D(\tilde{S}) = \int\limits_{\tilde{S}} ||p - m(\tilde{S})||dp \leq \int\limits_{\tilde{S}} ||p - m(S)||dp < D(S), \tag{19}$$

a contradiction to S being a solution of (**??**).

It is left to derive l^* and $K(B)$. Notice that for every disc B^l, the total access cost $D(B^l) = \frac{2\pi l^3}{3}$. Since the area of B^l is πl^2, the average cost within B^l is $K(B^l) = \frac{g}{\pi l^2} + \frac{2l}{3}$. It is straightforward to verify that the last expression attains its minimum at $l^* = \left(\frac{3g}{\pi}\right)^{\frac{1}{3}}$, yielding the minimal average cost $K(B) = l^*$.□

We will also utilize the lemma that evaluates the average cost of jurisdictions that are "close" to optimal discs:

Lemma A.2: Let $\gamma > 0$ and set S is located between two discs with the same center, $B_a^{l^*-\gamma}$ and $B_a^{l^*}$, i.e. $B_a^{l^*-\gamma} \subset S \subset B_a^{l^*}$. Then $K(S)$, the aggregate average cost over S, differs from the aggregate average cost over optimal disc $K(B)$ only in a second order term:

$$K(S) < l^* + \frac{4}{l^*}\gamma^2. \tag{20}$$

Proof: Let $\tilde{S} \subset B_a^{l^*} \setminus B_a^{l^*-\gamma}$. In our estimations below we take into account three observations: the optimal l^* satisfies $g = \frac{\pi l^{*3}}{3}$, the total transportation cost within S increases(at least, do not decrease) if we replace the $m(S)$ by a, and that the distance between any point in $\tilde{S}$ to a is bounded from above by l^*. Denote $z = \dfrac{3}{\pi}\lambda(\tilde{S})$. We have:

$$K(S) = \frac{g + D(S)}{\lambda(S)} \leq \frac{g + \int\limits_S ||a-t||dt}{\lambda(S)} \leq \frac{g + D(B_a^{l^*-\gamma}) + l^*\lambda(\tilde{S})}{\lambda(B_a^{l^*-\gamma}) + \lambda(\tilde{S})} = \frac{(l^*)^3 + 2(l^*-\gamma)^3 + zl^*}{3(l^*-\gamma)^2 + z} <$$

$$\frac{3(l^*)^3 - 6(l^*)^2\gamma + 6l^*\gamma^2 + zl^*}{3(l^*-\gamma)^2 + z} = \frac{3l^*(l^*-\gamma)^2 + zl^*}{3(l^*-\gamma)^2 + z} + \frac{3l^*\gamma^2}{3(l^*-\gamma)^2 + z} =$$

$$l^* + \frac{3l^*\gamma^2}{(l^*-\gamma)^2 + z/3} \leq l^* + \frac{3l^*\gamma^2}{\left(\sqrt{3}l^*\right)^2/2^2} = l^* + \frac{4}{l^*}\gamma^2, \tag{21}$$

as for γ small enough we have $l^* - \gamma > \dfrac{\sqrt{3}}{2}l^*$.□

Lemma A.3: Let H be a hexagon in an efficient partition. Then the per capita cost in H is given by

$$K(H) = \frac{\sqrt{3}}{2}\left(\frac{2}{3} + \ln\sqrt{3}\right)^{\frac{2}{3}} g^{\frac{1}{3}} \approx g^{\frac{1}{3}}. \tag{22}$$

Proof: Consider a regular hexagon H_l, where l denotes the distance between the center $m(H_l)$ and a midpoint of its side. The total access cost in H_l is

$$\begin{aligned} D(H_l) &= 12\int_0^l \int_0^{\frac{x}{\sqrt{3}}} \sqrt{x^2+y^2}dxdy = 6\int_0^l \left[y\sqrt{x^2+y^2} + x^2\ln\left(y + \sqrt{x^2+y^2}\right)\right]_0^{\frac{x}{\sqrt{3}}} dx \\ &= 6\int_0^l \left[\tfrac{x}{\sqrt{3}}\sqrt{x^2 + \tfrac{x^2}{3}} + x^2\ln\left(\tfrac{x}{\sqrt{3}} + \sqrt{x^2+\tfrac{x^2}{3}}\right) - x^2\ln x\right] dx \\ &= 6\int_0^l x^2\left[\tfrac{2}{3} + \ln\sqrt{3}\right]dx = 2l^3\left[\tfrac{2}{3} + \ln\sqrt{3}\right]. \end{aligned} \tag{23}$$

Since the area of H_l is $2\sqrt{3}l^2$, the average cost per citizen in jurisdiction H_l is given by

$$K(H_l) = \frac{g}{2\sqrt{3}l^2} + \frac{l}{\sqrt{3}}\left[\frac{2}{3} + \ln\sqrt{3}\right], \tag{24}$$

which attains its minimum at the efficient hexagon H, i.e., when

$$l = \tilde{l} = \left(\frac{2}{3} + \ln\sqrt{3}\right)^{-\frac{1}{3}} g^{\frac{1}{3}}. \tag{25}$$

It is easy to verify that then the per capita average cost $K(H) = K(H_{\tilde{l}})$ is given by (**??**) which at the same time represents the average cost of the whole plane X under an efficient partition. □

Take the efficient partition P^* of X. For every positive integer N, consider a subset G_N of P^* that consists of N^2 adjacent hexagons (see Figure 1). Let the sequence $\{G_N\}_{N=1,\dots,\infty}$ be nested, i.e., each G_N is imbedded into G_{N+2} "symmetrically", such that the set $G_{N+2} \setminus G_N$ is a "hexagonal ring" comprised of $4N+4$ regular hexagons. We have the following result:

Lemma A.4: For every $a \in G_N$, the disc B_a is contained in G_{N+2}.

Proof: Denote by $\bar{l}$ the side of a hexagon in partition P^*. Since the minimal width of the hexagonal ring F_N is equal to $\bar{l}$, it suffice to demonstrate that $\bar{l} > l^*$. Note that $\bar{l} = \frac{2}{\sqrt{3}}\tilde{l}$, where $\tilde{l}$ is the distance between the center of the efficient hexagon and the middle point of one of its sides, which has been derived in (**??**). Thus,

$$\bar{l} = \frac{2}{\sqrt{3}} \left(\frac{2}{3} + \ln\sqrt{3} \right)^{-\frac{1}{3}} g^{\frac{1}{3}}, \tag{26}$$

which, by (**??**), exceeds the value l^*. □

Let the efficient partition P^* be endowed with the sharing rule y, that generates cost allocation c, and H is a (hexagonal) jurisdiction in P^*. Denote by λ^H the Lebesgue measure of H and by λ^B the Lebesgue measure of an optimal disc.

For every $a \in X$ denote by the value $\varphi(a)$ the aggregated cost incurred by the members of the disc B_a:

$$\varphi(a) = c(B_a) = \int\limits_{B_a} c(t)dt \tag{27}$$

Define $\bar{\varphi}$ as the aggregated cost incurred by the allocation c on all discs of optimal size whose centers belong to the hexagon H:

$$\bar{\varphi} := \int\limits_{H} \varphi(a)da. \tag{28}$$

Note that, due to the consistency assumption A.6, the value $\bar{\varphi}$ is invariant to a choice of a hexagon in P^*. We need the following result:

Lemma A.5:

$$\bar{\varphi} = I, \quad \text{where} \quad I := \lambda^B \int\limits_{H} c(t)dt. \tag{29}$$

Proof: Define the function $\Psi(a,t)$ on $G_N \times G_{N+2} \subset \Re^4$ by

$$\Psi(a,t) = \begin{cases} c(t), & \text{if} \quad t \in B_a; \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

We will integrate the function $\Psi(a,t)$ over the set $G_N \times G_{N+2}$. According to Fubini's theorem (Halmos (1950), p.148), two different orders of integration yield the same result. First, we integrate with respect to t and then to a. By (**??**) and (**??**) we have

$$\int\limits_{G_N} \left[\int\limits_{G_{N+2}} \Psi(a,t)dt \right] da = \int\limits_{G_N} \left[\int\limits_{B_a} c(t)dt \right] da = \int\limits_{G_N} \varphi(a)da = N^2 \int\limits_{H} \varphi(a)da = N^2\bar{\varphi}. \tag{31}$$

Before integrating in the reverse order, note that the following duality property

$$\{a | t \in B_a\} \equiv B_t \tag{32}$$

holds for every $t \in X$. This is a simple consequence of the symmetry of the distance $||t-p||$ as a function of two arguments, and the circle B_t being the set of points p for which $||p-t|| = ||t-p|| \leq l^*$. Take a point $t \in G_{N-2}$. By Lemma A.4, $B_t \subset G_N$, and

$$\int\limits_{G_N} \Psi(a,t)da = \int\limits_{B_t} c(t)da = c(t) \int\limits_{B_t} da = \lambda^B c(t). \tag{33}$$

We have:

$$\int\limits_{G_{N+2}} \left[\int\limits_{G_N} \Phi(a,t)da \right] dt = \int\limits_{G_{N-2}} \left[\int\limits_{G_N} \Phi(a,t)da \right] dt + L_N, \tag{34}$$

where

$$L_N := \int\limits_{G_{N+2} \backslash G_{N-2}} \left[\int\limits_{G_N} \Phi(a,t)da \right]. \tag{35}$$

By using (**??**), (**??**) and Lemma A.4, the first term in (**??**) can be presented as:

$$\int\limits_{G_{N-2}} \left[\int\limits_{G_N} \Phi(a,t)da \right] dt = \int\limits_{G_{N-2}} \lambda^B c(t)dt = (N-2)^2 I. \tag{36}$$

Fubini's theorem allows us to rewrite (**??**) as

$$N^2\bar{\varphi} = (N-2)^2 I + L_N = N^2 I + L_N - 4(N-1)I. \tag{37}$$

Let us estimate the absolute value of the last two terms. Since for any $t \in G_{N+2}$, hence, for any $t \in G_{N+2} \setminus G_{N-2}$, we have that $\int\limits_{G_N} \Phi(a,t)da = \int\limits_{G_N \bigcap B_t} c(t)da \leq \int\limits_{B_t} c(t)da = \lambda^B c(t)$, it follows that

$$|L_N - 4(N-1)I| \leq |L_N| + 4(N-1)I \leq 4(N-1)I + \int\limits_{G_{N+2} \setminus G_{N-2}} \lambda^B c(t)dt = (12N-4)I < 12NI. \quad (38)$$

Thus,

$$|N^2\bar{\varphi} - N^2 I| \leq 12NI, \ \text{ or } \ |\bar{\varphi} - I| \leq \frac{12I}{N}. \quad (39)$$

Since N can be made arbitrarily large, it immediately yields the desired equality $\bar{\varphi} = I$.□

Proof of Proposition 3.1: It is a corollary of Proposition 3.3.

Proof of Proposition 3.3: Let us show first that

$$\delta^* = 1 - \frac{K(B)}{K(H)}, \quad (40)$$

which, by Lemmas A.1 and A.3, can be calculated as

$$\delta^* = 1 - \frac{2}{\pi^{\frac{1}{3}} 3^{\frac{1}{6}} (\frac{2}{3} + \ln\sqrt{3})^{\frac{2}{3}}} \approx 0.0019. \quad (41)$$

We will demonstrate that the set of δ-secession-proof allocations is empty if and only if $\delta < \delta^*$.

Consider a δ-secession-proof allocation c. The budget balancedness assumption A.5 implies that the value of I, determined by (**??**), is equal to $\lambda^B \lambda^H K(H)$, and by Lemma A.4. so is the value of $\bar{\varphi}$. Hence, there exists $a \in H$ such that $\varphi(a) \geq \lambda^B K(H)$. On the other hand, the stand alone aggregate cost in B_a is $\lambda^B K(B)$. Since c is δ-secession-proof, Definition 3.2 implies that $(1-\delta)\lambda^B K(H) \leq \lambda^B K(B)$, or $\delta \geq 1 - \frac{K(B)}{K(H)}$.

Let us show that Rawlsian allocation is δ-secession-proof whenever $\delta \geq \delta^*$. Indeed, since $r(t) = K(H)$ for every $t \in X$, then for $S = B^{l^*}$ — an optimal disc we observe that $(1-\delta)K(H) \leq K(B)$. Now, for any $S \in \mathcal{M}(X)$ we have $K(S) \geq K(B)$ and therefore $(1-\delta)K(H) \leq K(B) \leq K(S)$.

To complete the proof of the proposition, it remains to demonstrate that the Rawlsian allocation is the only one to be δ^*-secession-proof. That is, the allocation that assigns every individual in X a contribution $K(H)$ is the only δ^*-secession-proof. For this end, consider an arbitrary δ^*-secession-proof allocation $c(\cdot)$ and estimate the number of individuals whose contribution is "substantially" below the level $K(H)$.

Take a positive number $\varepsilon > 0$. Consider first an arbitrary ring $B_a \setminus B_a^{l^*-\gamma}$ and evaluate the measure of individuals t whose cost contribution $c(t)$ satisfies $c(t) < K(H) - \varepsilon$. Denote this set by U, and consider the set $S = B_a \setminus U$, for which, by Lemma A.2, we have $K(S) < l^* + \frac{4}{l^*}\gamma^2$. On the other hand,

$$c(S) = c(B^{l^*}) - c(U) \geq \lambda^B K(H) - \lambda(U)K(H) + \lambda(U)\varepsilon = \lambda(S)K(H) + \lambda(U)\varepsilon. \tag{42}$$

The δ^*-secession-proofness of $c(\cdot)$ implies that the average per capita contribution in group S, adjusted by $1-\delta^*$, does not exceed its stand-alone value, $K(S)$:

$$(1-\delta^*)\frac{c(S)}{\lambda(S)} = (1-\delta^*)(K(H) + \frac{\lambda(U)}{\lambda(S)}\varepsilon) \leq K(S) < l^* + \frac{4}{l^*}\gamma^2. \tag{43}$$

Since $K(B) = l^* = (1-\delta^*)K(H)$, we have:

$$\lambda(U) \leq \frac{4\lambda(S)}{l^*(1-\delta^*)\varepsilon}\gamma^2 < \frac{4\pi(l^*)^2}{l^*(1-\delta^*)\varepsilon}\gamma^2 = W\gamma^2, \tag{44}$$

where W is a constant which is independent of γ.

Now consider the rectangular Q with sides of $2l^*$ and l^* centered at the origin. For any small positive number γ, denote by $R[i,\gamma]$ the ring $B_{p_i} \setminus B_{p_i}^{l^*-\gamma}$ centered at the point $p_i = (i\gamma, 0)$, where i is any (positive or negative) integer. For large enough positive integer N we have the following inclusion:

$$Q \subset \bigcup_{i=-N}^{N} R\left[i, \frac{l^*}{N}\right]. \tag{45}$$

Indeed, it is easy to see that $\forall x \in Q$ there exist at least one i such that $x \in B_{p_i}$, and at least one j such that $x \notin B_{p_j}$. Hence, there exist such i and $j = i \pm 1$ that the two statements

$$x \in B_{p_i}; \quad x \notin B_{p_j} \tag{46}$$

hold simultaneously. As obviously $B_{p_i}^{l^*-\gamma} \subset B_{p_j}$ for $j = i \pm 1$, we have that $x \in B_{p_i} \setminus B_{p_i}^{l^*-\gamma} = R[i,\gamma]$, with $\gamma = \frac{l^*}{N}$.

Denote by U and U_i for $i = -N, \ldots, -1, 0, 1, \ldots, N$, the sets of individuals in Q and $R\left[i, \frac{l^*}{N}\right]$, respectively, who contribute less than $K(H) - \varepsilon$ under the allocation c. By utilizing (**??**), we have $\lambda(U_i) \leq W\frac{(l^*)^2}{N^2}$. Thus, since $U \subset \bigcup_{i=-N}^{N} U_i$, we have

$$\lambda(U) \leq (2N+1)W\frac{(l^*)^2}{N^2} < \frac{3}{N}W(l^*)^2. \tag{47}$$

Since N can be chosen arbitrarily large, (**??**) implies that $\lambda(U) = 0$. Note that this argument actually implies that for any rectangular with the sides of $2l^*$ and l^*, the Lebesgue measure of the set of individuals who contribute less than $l^* - \varepsilon$ under the allocation c has the zero measure.

Finally, consider an arbitrary hexagon H from the efficient partition P^*. It is contained in the union of several rectangulars with the sides of $2l^*$ and l^*. Hence, the measure of the set of all individuals in H who contribute less than $K(H) - \varepsilon$ is zero. But the set of all individuals in H who contribute less than $K(H)$ is a union of the sets of individuals in H who contribute less than $K(H) - 1/n$ for $n = 1, 2, \ldots$, and as the countable union of null-sets, this set has zero measure. Hence, the set of those contributing more than $K(H)$ has zero measure as well, otherwise budget balancing is violated.

Recalling that allocations are consistent, we conclude that every $t \in X$ contributes $K(H)$, implying that the only δ^*-secession-proof allocation is Rawlsian.$\square$

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