Stochastically Stable Recontracting with Multiple Indivisible Goods*

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Abstract

We consider multiple-type housing markets in which agents have strict preferences. For such markets the core can both be empty as well as multi-valued. We study a dynamic coalitional recontracting process as introduced by Serrano and Volij (2004). Its set of recurrent classes is indeed always a non-empty solution concept. We show that each core allocation is a singleton recurrent class, and provide examples of non-singleton recurrent classes consisting of blocking-cycles of individually rational allocations. However, in contrast to many standard applications of dynamic Markov processes, our first main result shows that stochastic stability does not serve as a proper selection device among recurrent classes.

To obtain sharp predictions on final allocations, we propose a method to explicitly compute the limit invariant distribution of the dynamic recontracting process. The approach exploits the interplay of coalitional stability *and* accessibility that determines the probability distribution over final allocations. Our main results show (i) How the limit invariant distribution serves as a selection device among recurrent classes, (ii) That some core and Walrasian allocations may be the worse predictors of final allocations, but that (iii) Blocking-cycles may emerge as a powerful prediction for final allocations even if the core is non-empty.

Keywords: coalitional stability, coalitional accessibility, indivisible goods, limit invariant distribution, stochastic stability.

JEL classification: D63, D70

1 Introduction

Consider the classical Shapley and Scarf (1974)'s housing markets with strict preferences. One of the most important requirement of coalitional stability of allocations is the one imposed by the core. It is well-known that the strong core (henceforth, core) of such markets display some remarkable properties. Among others, it is always non-empty, singleton and it coincides with the unique Walrasian allocation (Roth and Postlewaite (1977)). However, there are two problems with using the core as solution concept and as predictor of outcomes in a market. First, the properties underlined above are not robust to relaxation of the model to indifference in preferences or to multiple type of indivisible goods. By allowing such relaxations, the core can be empty or multivalued. The second problem is that the way core allocations are reached

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is left unspecified. These two issues underline a possible lack of predictions of the core as solution concept. To address both problems, we suggest to study a dynamic coalitional trading procedure in the spirit of "Edgeworth recontracting" –namely, a well-defined Markov process.

The two key concepts of such procedures are that of (coalitional) *accessibility* and *stability* of allocations. Indeed, specific questions we are interested in relate to the evolution of trades over time. In particular, (i) Will we obtain the core as the set of long-run predictions of the process? (ii) Are some core allocations easier to reach than others? (iii) Can non-core allocations be easier to reach than core allocations?

Serrano and Volij (2004) already used such a Markov process for Shapley-Scarf economies.[1] At every point in time, a stochastic process selects a coalition of agents that is allowed to agree upon a new allocation of the coalition's initial endowment. Agents are myopic and do not take into account the future recontracting process.[2]. When the blocking notion used is weak-blocking, Serrano and Volij (2004) show that the unique core allocation is also the unique recurrent class of the system. When indifference in preferences are possible, this result breaks down. The authors provide examples where (i) The set of stochastically stable states coincides with the set of core allocations, (ii) Requiring stochastic stability selects certain core allocations that are not necessarily Walrasian, and (iii) The set of stochastically stable states overlaps with the set of core allocations but contains a cycle of non-core allocations.

We extend Serrano and Volij (2004) analysis to markets in which there are several type of indivisible commodities –e.g. houses and cars– (Moulin (1995) and Konishi et al. (2001)) and where agents have strict preferences.[3]

We first underline sharp differences between the one-type model and ours. While the set of recurrent classes of the system overlaps with but can be larger than the core, our first main result shows that invoking stochastic stability as an equilibrium selection device does not help. In contrast to many standard applications of dynamic Markov processes, the set of stochastically stable allocations contains each allocation of each recurrent class. Thus, as every core allocation forms a singleton recurrent class, predictions are still inconclusive in the case of a multi-valued core. Furthermore, the emergence of blocking cycles as recurrent classes rather enriches the set of solutions than boils it down to a sharp prediction. Stochastic stability may reduce the set of recurrent classes only if preferences are not strict or if the notion of strong-blocking is used.[4] Hence, in a dynamic recontracting process, it may turn out to be insufficient to compare the relative stability of the various recurrent classes –i.e. to analyze the number of mistakes by agents needed for the process to get from one recurrent class to another. It underlines some non-robustness of the results in the one-type case. We also argue how results may be sensitive to some modeling details of the recontracting process.

Next, we propose a new method to recover predictions on final allocations. The approach underlines some dynamic features of stability and accessibility of allocations that are absent

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[1] Pioneering papers on Edgeworth recontracting are Feldman (1974) and Green (1974). They show the equivalence between the set of recurrent classes of some Markov process and the core. However, their results cannot be applied to Shapley-Scarf economies.

[2] The dynamic coalitional recontracting process is formally introduced in section 4. In the spirit of the literature on evolutionary game theory –see e.g. Kandori et al. (1993) and Young (1993)– it consists in the repetition over time of a one-shot exchange economy

[3] Since we want to compare our findings with Serrano and Volij (2004), we use a dynamic recontracting process identical to theirs.

[4] If one considers instead strong-blocking, then stochastic stability may provide a selection among the recurrent classes even if preferences are strict. In this paper, we however restrict ourselves to weak-blocking.

both from the core's static coalitional stability and from the stochastic stability requirements.

Starting with a result by Freidlin and Wentzell (2004), we develop a method to compute the limit invariant distribution of dynamic recontracting processes to finally receive answers to our initial research questions. As the limit invariant distribution is the unique probability distribution over allocations to which the recontracting process is converging to in the long-run, it indeed offers *always* a sharp prediction for the final allocation of the economy. Moreover, the limit invariant distribution is the result of an interplay of coalitional stability and accessibility. The more difficult it is for the process to exit a certain allocation after it has been reached –*stability*– *and* the easier it is to reach this allocation via coalitional recontracting from *any other* allocation –*accessibility*–, the higher is the respective value of the limit invariant distribution (*i.e.,* the probability that the system ends up in a certain allocation in the long-run).

We illustrate the computational techniques and the intuition behind our solution-concept with several examples. Our main results are as follows. We show (i) How the limit invariant distribution serves as a selection/prediction device among core allocations, (ii) That some core and Walrasian allocations may be the worse predictors of final allocations, but that (iii) Blocking-cycles may emerge as a powerful prediction for final allocations even if the core is non-empty.

The remainder of the paper is organized as follows. In Section 2 we define multiple-type housing markets. In Section 3 we discuss some basic results for multiple-type housing markets and introduce example. The dynamic recontracting process is introduced and characterized through its recurrent classes in Section 4. Finally, we introduce the perturbed process in Section 5 and then derive its set of stochastically stable states and the limit invariant distribution. Section 6 concludes with some remarks on the robustness of our findings and on the relation between the different solution concepts.

2 Multiple-Type Housing Markets

We mostly follow Miyagawa's (2002) and Klaus's (2005) model and notation of housing markets with multiple types. Let $N = \{1, \ldots, n\}$, $n \geq 2$, be the *set of agents.* There exist $\bar{\ell} \geq 1$ types of indivisible objects. The *set of object types* is denoted by $L = \{1, \ldots, \bar{\ell}\}$ and each agent $i \in N$ is endowed with one object of each type $\ell \in L$, denoted by i. Thus, N also denotes the set of objects of each type.

Allocations An allocation is an assignment of objects such that each agent receives exactly one object of each type. Formally, an *allocation* is a matrix $x = (x_i(\ell))_{i \in N, \ell \in L} \in N^{N \times L}$ such that

(i) For each $i \in N$ and each $\ell \in L$, $x_i(\ell) \in N$ denotes the object of type ℓ that agent i consumes, *e.g.*, if $x_i(\ell) = j$, then agent i receives agent j's endowment of type ℓ, and

(ii) no object of any type is assigned to more than one agent at allocation x. Thus, for each $\ell \in L$, $\cup_{i \in N}\{x_i(\ell)\} = N$.

Let X denote the *set of allocations.* Given $x \in X$ and $\ell \in L$, $x(\ell) = (x_1(\ell), \ldots, x_n(\ell))$ denotes the allocation of type-ℓ objects. Given $x \in X$ and $i \in N$, $x_i = (x_i(1), \ldots, x_i(\bar{\ell}))$ denotes the list of objects that agent i receives at allocation x. We call x_i *agent i's (consumption) bundle.* Note that the set of bundles for each $i \in N$ can be denoted by N^L. We denote each *agent i's endowment* by $e_i = (i, \ldots, i) \in N^L$. Similarly, for any coalition $S \subseteq N$, we denote coalition S's endowment by $e_S = (e_i)_{i \in S}$.

Markets Each agent $i \in N$ has complete and transitive preferences R_i over bundles. Let P_i and I_i be the strict and indifference part associated to R_i. Let $\mathcal{R}$ be *set of linear orders over* N^L. Thus, for each $i \in N$, each $R_i \in \mathcal{R}$, and each $\{x_i, y_i\} \subset N^L$, $x_i \, R_i \, y_i$ implies that $x_i \, P_i \, y_i$ or $x_i = y_i$. Let $\mathcal{R}^N = \times_{i \in N} \mathcal{R}$ be the *set of preference profiles.* Since the set of agents and their endowments remain fixed throughout, $\mathcal{R}^N$ also denotes the *set of multiple-type housing markets.* For $\bar{\ell} = 1$, our model coincides with the classical Shapley and Scarf (1974) housing market model.[5]

Individual Rationality Given $R \in \mathcal{R}^N$, an allocation x is *individually rational* if for each $i \in N$, $x_i \, R_i \, e_i$. Let $IR(R)$ be the set of individually rational allocations for $R \in \mathcal{R}^N$.

To introduce the standard (cooperative) solution concepts for multiple-type housing markets we need some additional notation. The set of all feasible reallocations of objects among the members of coalition $S \subseteq N$ is denoted by,

$$X_S = \{(x_i(\ell))_{i \in S, \ell \in L} \in N^{S \times L} : \text{ for each } \ell \in L, \ \cup_{i \in S}\{x_i(\ell)\} = S\}.$$

Let $y \in X$ and $S \subseteq N$. Then, by $y_S = (y_i)_{i \in S}$ we denote the restriction of allocation y to coalition S. For notational convenience we will also use $X_{-S} \equiv X_{N \setminus S}$ and $y_{-S} \equiv y_{N \setminus S}$.

For each $R \in \mathcal{R}^N$, each $\{x, y\} \subset X$, and each $S \subseteq N$, we say that *y S-blocks x* if $y_S \in X_S$ and for each $i \in S$, $y_i \, R_i \, x_i$ and $y_i \, P_j \, x_i$ for some $j \in S$.

The Core Given $R \in \mathcal{R}^N$, an allocation is in the *(strong) core* if no coalition of agents can improve their welfare by reallocating their endowments among themselves. Formally, an allocation $x \in X$ is a *core allocation* for $R \in \mathcal{R}^N$ if there exist no coalition $S \subseteq N$ and no $y_S \in X_S$ such that y S-blocks x. Let *core*(R) be the set of core allocations for $R \in \mathcal{R}^N$.

Walrasian Allocations Define a price system by $p \equiv (p_\ell)_{\ell \in L} \in \mathbb{R}_+^{n\bar{\ell}}$ such that for all $l \in L$, $p_\ell = (p_\ell(1), \ldots, p_\ell(n)) \in \mathbb{R}_+^n$. Then, given $R \in \mathcal{R}^N$, an allocation x is a *Walrasian allocation* for R if there exists a price system $p \in \mathbb{R}_+^{n\bar{\ell}} \setminus \{\mathbf{0}\}$ such that for each $i \in N$ we have,

(i) that x is affordable, *i.e.*, $\sum_{\ell \in L} p_\ell(i) \geq \sum_{\ell \in L} p_\ell(x_i(\ell))$,

and

(ii) x is a best affordable bundle, *i.e.*, if $y_i \, P_i \, x_i$, then $\sum_{\ell \in L} p_\ell(y_i(\ell)) > \sum_{\ell \in L} p_\ell(i)$.

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[5] Note that instead of considering the whole domain of linear orders $\mathcal{R}$ as our reference domain, we could restrict the domain to the domain of separable preferences $\mathcal{R}^s$ (see Klaus, 2005) or to the domain of additively separable preferences $\mathcal{R}^{as}$ (see Konishi, Quint, and Wako, 2001). However, separability plays no role in our analysis.

Note that the budget inequality (i) can be replaced by a budget equality: this can be easily checked by adding (i) up over all agents $i \in N$ and applying $\sum_{i\in N}\left(\sum_{\ell\in L} p_\ell(i)\right) = \sum_{i\in N}\left(\sum_{\ell\in L} p_\ell(x_i(\ell))\right)$. Let $W(R)$ be the set of Walrasian allocations for $R \in \mathcal{R}^N$.

3 Basic Results and Examples for Multiple-Type Housing Markets

First, we summarize some results for the benchmark case of one object type.

Remark 1. ***The Benchmark Case: Housing Markets with Strict Preferences***
For any housing market with one object type (and strict preferences), Shapley and Scarf (1974) showed that a core allocation always exists. Roth and Postlewaite (1977) proved that the set of core allocations for any housing market with one object type equals the set of Walrasian equilibria and is a singleton. Using the so-called top-trading algorithm (due to David Gale, see Shapley and Scarf, 1974) one can easily calculate the unique core allocation for any housing market with one object type. Furthermore, for any non-core allocation x there exists a coalition S that can block x via the bundles it receives at the core allocation. This particular feature of the core is termed the global dominance property (Roth and Postlewaite, 1977). $\triangle$

As soon as one either relaxes the assumption of strict preferences or one increases the number of object types, existence, single-valuedness, and the global dominance property of the core do not necessarily hold anymore. For markets with $\bar{\ell} \geq 2$, Konishi, Quint, and Wako (2001) demonstrated that the core may be empty or multi-valued –even for additively separable preferences. Moreover (Konishi, Quint, and Wako, 2001, Proposition 3.1), for each $R \in \mathcal{R}^N$, $W(R) \subseteq Core(R)$.

We next introduce several examples that will be analyzed in detail in the next sections.[6]

Example 1. ***An Empty Core***
Consider a multiple-type housing market with two object types and three agents who have the following preferences:

$$(3,1) \succ_1 (1,2) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(2,1) \succ_2 (3,2) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(2,3) \succ_3 (1,3) \succ_3 (3,3) \succ_3 \textit{anything}.$$

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(1,1),(3,2),(2,3)\},$$

$$x^3 = \{(1,2),(2,1),(3,3)\}, \quad x^4 = \{(3,1),(2,2),(1,3)\}.$$

Clearly, x^2 $\{2,3\}$-blocks x^1, x^3 $\{1,2\}$-blocks x^2, x^4 $\{1,3\}$-blocks x^3, and x^2 $\{2,3\}$-blocks x^4. Hence, the core and the set of Walrasian allocations coincide and are empty. $\diamond$

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[6] Exception made of Example 5.

We relegate the computation of the core and the set of Walrasian allocations for all remaining examples to Appendix A.

We continue with an example where $|Core(R)| = W(R) = 1$.

Example 2. ***The Unique Walrasian Allocation Equals the Core Allocation***
Consider a multiple-type housing market with two object types and three agents who have the following preferences:

$$(3,1) \succ_1 (2,2) \succ_1 (1,2) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(2,1) \succ_2 (3,3) \succ_2 (3,2) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(2,3) \succ_3 (1,1) \succ_3 (1,3) \succ_3 (3,3) \succ_3 \textit{anything}.$$

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(1,1),(3,2),(2,3)\},$$

$$x^3 = \{(1,2),(2,1),(3,3)\}, \quad x^4 = \{(3,1),(2,2),(1,3)\},$$

$$x^5 = \{(2,2),(3,3),(1,1)\}.$$

The core and the set of Walrasian allocations coincide and equal $Core(R) = W(R) = \{x^5\}$. ⋄

Konishi, Quint, and Wako (2001, Example 3.3) showed that, even for additively separable preferences, the core may be multi-valued. The next example has multiple core allocations, of which only one is Walrasian. Since in our context separability does not play a role, we introduce alternative examples that are easier to analyze; for instance, our next multiple-type housing market has four individually rational allocations while Konishi, Quint, and Wako's (2001) corresponding example has eleven individually rational allocations.

Example 3. ***Multiple Core Allocations and Unique Walrasian Allocation***
Consider a multiple-type housing market with two object types and three agents who have the following preferences:

$$(1,2) \succ_1 (3,3) \succ_1 (2,3) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(1,3) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(3,1) \succ_3 (2,1) \succ_3 (3,3) \succ_3 \textit{anything}.$$

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(2,3),(1,2),(3,1)\},$$

$$x^3 = \{(1,2),(3,3),(2,1)\}, \quad x^4 = \{(3,3),(1,2),(2,1)\}.$$

The set of Walrasian allocations $W(R) = \{x^3\}$ is a strict subset of $Core(R) = \{x^2, x^3, x^4\}$. ⋄

Next we illustrate that the set of Walrasian allocations may contain multiple allocations.

Example 4. ***Multiple Walrasian Allocations***
Consider a multiple-type housing market with two object types and three agents who have the following preferences:

$$(1,2) \succ_1 (3,3) \succ_1 (2,3) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(1,3) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (2,2) \succ_2 \textit{anything}$$

$$(2,1) \succ_3 (3,1) \succ_3 (3,3) \succ_3 \textit{anything}.$$

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(2,3),(1,2),(3,1)\},$$

$$x^3 = \{(3,3),(1,2),(2,1)\}, \quad x^4 = \{(1,2),(3,3),(2,1)\}.$$

The core equals the set of Walrasian allocations $W(R) = Core(R) = \{x^3, x^4\}$. ◇

In our last example the set of Walrasian allocations is empty while the core is nonempty.

Example 5. ***No Walrasian Allocations and a Multi-Valued Core***
Consider a multiple-type housing market with two object types and three agents who have the following preferences:

$$(1,2) \succ_1 (3,3) \succ_1 (2,3) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(3,2) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(1,3) \succ_3 (3,1) \succ_3 (2,3) \succ_3 (2,1) \succ_3 (3,3) \succ_3 \textit{anything}.$$

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(1,1),(3,2),(2,3)\},$$

$$x^3 = \{(2,3),(1,2),(3,1)\}, \quad x^4 = \{(3,3),(1,2),(2,1)\},$$

$$x^5 = \{(1,2),(3,3),(2,1)\}.$$

The core equals $Core(R) = \{x^2, x^3\}$ and the set of Walrasian allocations is empty. ◇

The last two examples illustrate two features of the relationship between the core and the set of Walrasian allocations not yet recognized in the literature.

Remark 2. ***New Insights on Walrasian Allocations through Examples 4 and 5***
Konishi, Quint, and Wako (2001) proved that the set of Walrasian equilibria is a subset of the core and that it might be empty if the core is also empty. With Example 5 we prove that the set of Walrasian allocations might even be empty if the core is nonempty. Second, with Example 4 we provide a multiple-type housing market where the set of Walrasian allocations contains more than one allocation. △

Moreover, the examples collected in this section demonstrate that the core, or the set of Walrasian allocations, are not necessarily satisfactory solution concept for mutiple-type housing markets.

Remark 3. ***Walrasian and/or Core Allocations as Solutions?***
First, both the core and the set of Walrasian allocations can be empty (and the set of Walrasian allocations might even be empty while the core is nonempty). When the core, and therefore the set of Walrasian allocations, is empty, we do not have predictions to offer as to what will happen in the market. Will agents keep their endowments or will they trade? Second, if the core and the set of Walrasian allocations are multi-valued, then which allocation(s) will result from trade? △

To address the problems mentioned in the last remark, we propose a dynamic recontracting process and characterize market outcomes through the respective set of recurrent classes (Section 4) or stochastically stable states (Section 5.1), and – finally – the limit invariant distribution (Section 5.2).

4 Unperturbed Dynamic Recontracting Processes

We model dynamic recontracting by a *Markov Process* $(X, M(R))$.[7] The state space is given by the set of allocations X and the transition matrix $M(R)$ depicts the following dynamics which depends on the prevailing preference profile R. In every period t the process is in a state $x(t) \in X$ and a coalition of agents is selected at random. The system moves from one state to another when the agents in the selected coalition recontract among themselves, *i.e.*, agree upon a redistribution of their endowments. Agents never make mistakes in the sense that they only recontract if they (weakly) benefit from doing so. As we will allow for mistakes later on, we refer to the dynamic recontracting process discussed in this section as *unperturbed.* The following three assumptions are satisfied at each date

Assumption 1. ***Opportunities to Recontract***
Each coalition $S \subseteq N$ is chosen with positive probability to recontract among themselves. A coalition that has the opportunity to recontract is an *active coalition.*

This rather mild assumption covers most of the common models of coalition formation. In particular, the probability with which a certain coalition has the opportunity to recontract can depend on the size of the coalition, the allocation in period t, or the identities of the agents involved as long as the corresponding probability distribution has full support in the set of coalitions.

Assumption 2. ***Recontracting Behavior***
Given an active coalition $S \subseteq N$, $y_S \in X_S$ is a *weakly improving reallocation for S* if for each $i \in S$, $y_i \; R_i \; x_i(t)$ and $y_j \; P_j \; x_j(t)$ for at least one $j \in S$. Each weakly improving reallocation for the active coalition S is chosen with positive probability, if any.

Agents are myopic in the sense that they agree upon a reallocation of their endowments if it is weakly improving in the subsequent period.[8]. Hence, they ignore the prospect of future reallocations along the path of the recontracting process.

The assumption on which weakly improving reallocation is chosen is again mild. It is only important that any such reallocation is chosen with positive probability. The probability itself can depend on the identities or even the preferences of the agents in the active coalition.

We now fix the allocation for agents that did not participate in the recontracting.

Assumption 3. ***Allocations Resulting from Recontracting***
In period $t+1$, $x(t+1) = x(t)$ if there is no weakly improving allocation for S. Otherwise, members of the active coalition S consume the weakly improving reallocation y_S they agreed

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[7] In Appendix B we define and explain all the terms related to Markov processes used in this and the following sections.

[8] Hence the notion of weak-blocking

upon. All remaining agents either keep their previous allotments (if this is feasible) or receive their endowments.[9] Formally, if there exists a weakly improving reallocation for S, then $x(t+1) = (y_S, x_{-S}(t))$ if $x_{-S}(t) \in X_{-S}$ and $x(t+1) = (y_S, e_S)$ otherwise.

A Markov process $(X, M(R))$ that satisfies Assumptions 1, 2, and 3 is an *unperturbed dynamic recontracting process.* Serrano and Volij (2004, Section 7) consider this specification of the unperturbed recontracting process for Shapley-Scarf economies ($\bar{\ell} = 1$).[10]

An important characteristic of the unperturbed dynamic recontracting process is its "set of recurrent classes." A set $A \subseteq X$ is *a recurrent class* if it is a minimal set of allocations that once entered throughout the process is never abandoned, *i.e.*, for each $x \in A$ and each $x' \notin A$, $M(x, x') = 0$. For each $R \in \mathcal{R}^N$, let $RC(R)$ be the set of recurrent classes of multiple-type housing market R.

Proposition 1. ***Recurrent Classes of a Dynamic Recontracting Process***
For any unperturbed dynamic recontracting process $(X, M(R))$ it holds that

(i) $RC(R) \neq \emptyset$.

(ii) *A recurrent class is a singleton if and only if it is a core allocation.*

(iii) *Each element of a recurrent class is an individually rational allocation.*

(iv) $IR(R) \supseteq RC(R) \supseteq Core(R) \supseteq W(R)$.[11]

Proof. Statement (i) follows from the finiteness of the Markov process (the finiteness of X).

Statement (ii) is a direct implication of the definition of the recontracting process and the properties of the core.

To prove statement (iii), suppose $x(t) \notin IR(R)$. There exists $i \in N$ with $e_i \, P_i \, x_i(t)$. Recall that Assumption 1 implies that there is a positive probability that only agent i (*i.e.,* the singleton coalition $\{i\}$) is allowed to recontract. If $x_i(t)$ is not individually rational for agent i, agent i will recontract with himself and receive his initial endowment. Hence, x cannot form a singleton recurrent class. Moreover, x cannot be part of a non-singleton recurrent class as agent i would have to agree on x_i in some previous period.

Statement (iv): By (iii) each element of a recurrent class has to be individually rational. It is also clear that not every individually rational state has to a be member of a recurrent class. This implies $IR(R) \supseteq RC(R)$. By (ii) each allocation in the core is in $RC(R)$. Hence, $RC(R) \supseteq Core(R)$.[12] Finally, $Core(R) \supseteq W(R)$ follows from Konishi, Quint, and Wako (2001, Proposition 3.1).[13] □

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[9] This assumption turns out to be important in Section 5 where we discuss its impact in detail.

[10] We assume that a coalition can recontract using weak blocking. Alternatively, we could model recontracting by strong blocking. We discuss how this modification affects our results in the Conclusions (Section 6).

[11] Since strictly speaking $RC(R)$ is a set of sets of allocations while $IR(R)$, $Core(R)$, and $W(R)$ are sets of allocations, this set inclusion is stated with a slight abuse of notation. For instance, in Example 2, $RC(R) = \{\{x^5\}, \{x^2, x^3, x^4\}\}$. Hence, $RC(R)$ here has to be interpreted as the set of allocations contained in the recurrent classes.

[12] Example 2 is a multiple-type housing market with $IR(R) \supsetneq RC(R) \supsetneq Core(R)$.

[13] Example 3 is a multiple-type housing market with $Core(R) \supsetneq W(R)$.

Proposition 1 shows that the set of recurrent classes of an unperturbed dynamic recontracting process deserves attention as a solution for multiple-type housing markets: for any multiple-type housing market $R \in \mathcal{R}$, $RC(R)$ is non-empty, consists only of individually rational allocations, and contains all core allocations. Core allocations, however, can never be part of a non-singleton recurrent class. To demonstrate (i) the non-emptiness of $RC(R)$, (ii) the coexistence of singleton recurrent classes (*i.e.*, allocations in the core) and non-singleton recurrent classes, and (iii) the absence of selection between core-allocations, we continue with three of the examples introduced in Section 3.[14]

Example 1 describes a multiple-type housing market without a singleton recurrent class but with a multi-valued (non-core) recurrent class.

Example 1 (continued). ***Empty Core and Non-Singleton Recurrent Class***
Recall that $Core(R) = \emptyset$. By Proposition 1 (iii) only allocations in $IR(R) = \{x^1, x^2, x^3, x^4\}$ are candidates to be elements of a recurrent class. Since x^1 can be blocked by any other state in $IR(R)$ it can never be an element of a recurrent class. Hence, we are left with the allocations x^2, x^3, and x^4. Recall that we have a blocking cycle where x^3 $\{1,2\}$-blocks x^2, x^4 $\{1,3\}$-blocks x^3, and x^2 $\{2,3\}$-blocks x^4. Hence, none of the individually rational allocations x^2, x^3, and x^4 can form a singleton recurrent class. Furthermore, each of the allocations in $\{x^2, x^3, x^4\}$ can be reached from one another through (a sequence of) blocking(s) while once an allocation in $\{x^2, x^3, x^4\}$ is reached no outside allocation can block. Therefore, $\{x^2, x^3, x^4\}$ constitutes the only recurrent class. ◇

Example 2 is a multiple-type housing market with two recurrent classes (one of them equals the core and contains the unique Walrasian allocation, the other contains three non-core allocations).

Example 2 (continued). ***The Set of Recurrent Classes Exceeds the Core***
Recall that $Core(R) = \{x^5\}$. By Proposition 1 (iii) only individually rational allocations in $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ are candidates to be elements of a recurrent class. By Proposition 1 (ii), $Core(R) = \{x^5\}$ is the only singleton recurrent class. Next, recall that the only difference between Examples 1 and 2 is that – loosely speaking – we added allocation x^5 to the agents' preferences such that x^5 is now not only individually rational, but also the unique core allocation. However, none of the allocations x^2, x^3, or x^4 can be blocked by x^5 (or x^1) while there is again the blocking cycle where x^3 $\{1,2\}$-blocks x^2, x^4 $\{1,3\}$-blocks x^3, and x^2 $\{2,3\}$-blocks x^4. Hence, $\{x^2, x^3, x^4\}$ forms (as in Example 1) the only non-singleton recurrent class. ◇

We conclude with a multiple-type housing market where the set of recurrent classes coincides with the set of core-allocations.

Example 3 (continued). ***The Core Equals the Set of Recurrent Classes***
Recall that $Core(R) = \{x^2, x^3, x^4\}$. By Proposition 1 (iii) only allocations in $IR(R) = \{x^1, x^2, x^3, x^4\}$ are candidates to be elements of a recurrent class. Since x^1 can be blocked by any other state in $IR(R)$ it can never be an element of a recurrent class. By Proposition 1 (ii), $\{x^2\}$, $\{x^3\}$ and $\{x^4\}$ are (singleton) recurrent classes and $RC(R) = Core(R)$. ◇

To summarize, even in the case of an empty core, $RC(R)$ offers a prediction for the outcome of a multiple-type housing market (see Example 1). This, however, is achieved at

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[14] The recurrent classes of the other examples are spelled out in Appendix A.

the expense of a weakly larger set of final outcomes whenever the core is non-empty (see Example 2). This closely resembles the situation in the literature on evolutionary selection of Nash equilibria in, *e.g.*, coordination games (see Young, 1993): While every Nash equilibrium of the coordination game is also a singleton recurrent class of the unperturbed learning process, the set of recurrent classes typically exceeds the set of Nash equilibria. In particular the set of recurrent classes is not a selection of Nash equilibria, as it is not a selection of core allocations in Example 3.

5 Perturbed Dynamic Recontracting Processes

We will now apply the methods developed in this literature –and also used in Serrano and Volij (2004)– and perturb the dynamic recontracting process, *i.e.*, we allow agents to make mistakes with a small probability. Adding randomness breaks the possible inertia of the system and guarantees that the system never gets stuck in any state.

To select among the recurrent classes, we follow the standard approach to Markov processes and perturb our dynamic recontracting process by assuming that at any given date, any agent in an active coalition can make a mistake when recontracting with probability $\epsilon > 0$ –making a mistake here means that an agent agrees on a reallocation that makes her worse off. For expositional ease we restrict ourselves to mistakes where an agent agrees upon another *individually rational* allocation that makes her worse off. We do this for three reasons: First, mistakes where agents agree upon non-individually rational allocations considerably increase the state space that has to be taken into account in Subsection 5.2. Second, encountering these mistakes makes it necessary to specify preferences for *all* consumption bundles over non-individually rational allocations. Third and most importantly, none of the general results displayed in Proposition 2 and 3 are sensitive to this restriction. Loosely speaking, agents manage to avoid any mistake that makes them worse off than with their initial endowment but commit mistakes in agreeing upon individually rational allocations.

Assumption 4. *The probability with which a member of an active coalition* $i \in S$ *agrees on a reallocation* y_S *with* $x_i(t)\ P_i\ y_i\ R_i\ e_i$ *equals* $\epsilon > 0$.

In particular, the probability of a mistake does not depend on the allocation or the agents' identities. We call a process that satisfies Assumptions $1-4$ a *perturbed dynamic recontracting process* and denote it by $(X, M^{\epsilon}(R))$. This process is clearly ergodic: as mistakes induce (indirect) transitions between any two individually rational allocations, and non-individually rational allocations remain transient, its unique recurrent class is $IR(R)$. Hence, the process exhibits a unique invariant distribution μ_{ϵ} with support $IR(R)$ that displays the long-run probability distribution over states. Moreover, the perturbation is regular on $IR(R)$, *i.e.,* transition probabilities between any two individually rational allocations are non-zero and polynomials in ϵ. As every transition from an allocation x' to an allocation x'' can involve a maximum of n mistakes (by every agent of the grand coalition) we can denote

$$M^{\epsilon}(x', x'') \equiv \sum_{i=0,\ldots,n} m_i(x', x'')\epsilon^i$$

where $m_i(x', x'')$ captures the probability that a certain (set of) coalition(s) form(s) at allocation x' and agrees upon allocation x'' with i agents making a mistake. Note in particular that $m_0(x', x'') = M(x', x'')$ – the intercept of the polynomial is the respective entry in transition matrix of the unperturbed process.

5.1 Stochastic Stability

Young (1993, Theorem 4 (i)) has shown that the limit invariant distribution $\mu^* \equiv \lim_{\epsilon \to 0} \mu_\epsilon$ of a perturbed Markov process (X, M^ϵ) exists and that it is an invariant distribution of the unperturbed process (X, M) if the perturbation is regular, *i.e.*, if Assumption 4 holds. The support of every invariant distribution of the unperturbed process, however, is a (non-empty) collection of recurrent classes.[15]. Hence, not every recurrent class of the unperturbed process has to be in the support of μ^*, but the support is a collection of recurrent classes. States (or allocations) in the support of μ^* are called stochastically stable allocations (or states). We denote the support of μ^* of an economy R by $SRC(R)$ and refer to it as the set of stochastically stable allocations (or the set of stochastically stable recurrent classes). Hence, stochastic stability operates as a selection device on the set of recurrent classes: $SRC(R) \subseteq RC(R)$. For further reference we summarize as follows.

Lemma 1. ***Young (1993, Theorem 4 (i))***
For any dynamic recontracting process $(X, M(R))$ that fulfills Assumptions 1, 2, and 3 it holds that
$SRC(R) \neq \emptyset$.
$SRC(R) \subseteq RC(R)$.

Next, we will display a general methodology to determine the set of stochastically stable recurrent classes $SRC(R)$, for each $R \in \mathcal{R}$, of our dynamic recontracting process and apply it to examples already introduced in previous sections.

***x*-Trees** Consider the set of directed graphs that have vertex set X. Then, any directed graph is defined by its set of directed edges. We denote a directed edge from $x' \in X$ to $x'' \in X$ by $[x', x'']$ and interpret it as *x'' is the outcome of recontracting that started from x'*. Note that the irreducibility of a perturbed Markov process implies that for any directed edge $[x', x'']$ we have that $M^\epsilon(x', x'') > 0$.

To characterize the stochastically stable states of a Markov process (X, M) we consider the set of (directed) *spanning trees* or *x-trees* T_x for every $x \in X$. An x-tree T_x is a directed graph such that for every $y \in X$ with $x \neq y$ there is exactly one (cycle-free) sequence of edges (a directed *path*) from y to x. Denote by $\mathcal{T}_x$ the set of all x-trees. As the perturbed process is irreducible, $\mathcal{T}_x$ is non-empty for all $x \in X$.

Stochastic Potential Let $[x', x'']$ be an edge in an x-tree $T_x \in \mathcal{T}_x$. The *edge-resistance* $r(x', x'')$ is the minimum number of mistakes needed to get directly from x' to x'', *i.e.*, the minimal number of agents that are worse off through recontracting when actively participating in a blocking of allocation x' that results in allocation x''. Formally, $r(x', x'') = \min\{r \geq 0 \mid \infty > \lim_{\epsilon \to 0} \epsilon^{-r} M^\epsilon(x', x'') > 0\}$. Finally, the *stochastic potential* of $x \in X$, denoted by $\gamma(x)$, is the minimal sum of edge-resistances over all x-trees, *i.e.*, $\gamma(x) = \min_{T_x \in \mathcal{T}_x} \sum_{[x', x''] \in T_x} r(x', x'')$. We will refer to an x-tree $\widetilde{T}_x$ that minimizes $\sum_{[x', x''] \in T_x} r(x', x'')$ as a *least resistance tree.*

Now, the set of stochastically stable allocations $SRC(R)$ can be characterized as follows.

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[15] Every recurrent class corresponds to exactly one invariant distribution, and the set of all invariant distributions of a Markov process is the convex hull of the invariant distribution for all its recurrent classes (for details see Appendix B)

Lemma 2. ***Young (1993, Theorem 4 (ii))***
For any dynamic recontracting process $(X, M(R))$ *that fulfills Assumptions 1, 2, and 3, it holds that* $x \in X$ *is stochastically stable if and only if* $\gamma(x) \leq \gamma(y)$ *for each* $y \in X$.

Interpretation The set of recurrent classes of a dynamic recontracting process exhibits a certain kind of (internal) coalitional stability because no recurrent class is blocked by any other recurrent class –*i.e.,* mistakes by agents are needed to let the recontracting process propagate from one recurrent class to another.[16] In the case of singleton recurrent classes this stability requirement coincides with the (internal) stability property of the core. Stochastic stability also requires (internal) *stability* –$SRC(R) \subseteq RC(R)$– but in addition also considers the (relative) *accessibility* of recurrent classes –*i.e.,* the number of mistakes agents needed to let the process *reach* a recurrent class from all other allocations, as depicted in the stochastic potential.

An allocation is stochastically stable if and only if it *minimizes* the number of mistakes needed to reach the allocation through the dynamic recontracting processes –*i.e.,* if the allocation x minimizes $\gamma(x)$. This additional aspect suggests stochastic stability as a selection device for recurrent classes, and thereby core allocations.

Note that allocations in the same recurrent class have the same stochastic potential. To see this consider two states x and y in the same recurrent class. Now take any least resistance x-tree $\widetilde{T}_x$. Obviously, there exists a path from y to x solely consisting of edges with zero resistance, and such a path has to be part of $\widetilde{T}_x$ (otherwise $\widetilde{T}_x$ would not be resistance minimizing). Likewise, there also exists a path from x to y solely consisting of edges with zero resistance. But then we can construct a y-tree $\widetilde{T}_y$ with $\sum_{[x',x'']\in\widetilde{T}_x} r(x',x'') = \sum_{[x',x'']\in\widetilde{T}_y} r(x',x'')$ which implies that $\gamma(x) \geq \gamma(y)$. Graphically one can obtain the y-tree $\widetilde{T}_y$ by first drawing the zero resistance path from x to y and then reattaching the missing vertices by using only branches of the original x-tree $\widetilde{T}_x$ — thus $\widetilde{T}_y$ is obtained from $\widetilde{T}_x$ by "tree surgery". As the argument to obtain $\gamma(x) \geq \gamma(y)$ is symmetric with respect to x and y, it follows that $\gamma(x) = \gamma(y)$.

This "tree-surgery argument" can also be applied to make the following point. Consider two states x and y that are members of two different recurrent classes and suppose for simplicity that there is a least resistance x-tree with a direct edge $[y, x]$. This tree can be transformed into a y-tree by reversing that edge. As x and y are members of two different recurrent classes it follows that $r(x, y) > 0$ and $r(y, x) > 0$. If, however, $r(y, x) > r(x, y)$ we have just constructed a y-tree with a tree-resistance $\sum_{[x',x'']\in T_y} r(x',x'') < \sum_{[x',x'']\in T_x} r(x',x'')$. But this implies $\gamma(y) < \gamma(x)$ and, by Lemma 2, $x \notin SRC(R)$. Hence, if the resistance minimizing link (or, more generally, path) from y to x has a higher resistance than a link (or path) from x to y, the recurrent class that includes x cannot be stochastically stable. Note that this observation, which is also the underlying idea behind Ellison's 2000 radius/coradius formalism, only provides a sufficient condition for stochastic stability. However, it illustrates that stochastic stability is driven by the trade-off between *stability* of a recurrent class and the respective *accessibility* from other recurrent classes of the process.

Finally, note that the stochastic potential is only determined by the resistance of a tree in the set of least resistance trees. Consider, for instance, an economy $R \in \mathcal{R}$ with $RC(R) =$

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[16] Here the internal non-blocking conditions means that no allocation in a recurrent class can be blocked by an allocation from another recurrent class.

$\{\{x\},\{y\}\}$. To establish stochastic stability of $\{x\}$ we only have to find *one* x-tree with tree-resistance $\gamma(y)$. The *number* of least resistance trees for either $\{x\}$ or $\{y\}$ does not influence the result. This already suggests that the stochastic potential does not capture all aspects of accessibility and stability of allocations. We return to this observation in Subsection 5.2.

Computation To compute $SRC(R)$, Lemma 2 offers the following procedure. (i) Determine the set of recurrent classes $RC(R)$, (ii) select a state x for every recurrent class and construct an x-tree that minimizes $\sum_{[x',x'']\in T} r(x',x'')$. (iii) Compute $\gamma(x)$. Because of the identity of stochastic potentials within a given recurrent class, we simply refer to $\gamma(x)$ as the stochastic potential of the recurrent class that incudes x. The set of stochastically stable recurrent classes is the set of all recurrent classes with minimal stochastic potential.

To illustrate our methodology, we apply it to the examples discussed in Section 4. We start with the trivial case of a single recurrent class.

Example 1 (continued). ***Empty Core and Non-Singleton SRC(R)***
Recall that the only recurrent class of the unperturbed dynamics for Example 1 is $\{x^2,x^3,x^4\}$. Hence by Lemma 1, Lemma 2, and the observation that allocations in the same recurrent class have the same stochastic potential, we obtain that $SRC(R)=\big\{\{x^2,x^3,x^4\}\big\}=RC(R)$. $\diamond$

We continue with an example that illustrates the computational techniques to elicit the stochastic potential.

Example 2 (continued). ***SRC(R) = RC(R)*** $\supsetneq$ ***Core(R)***
Recall that the recurrent classes of the unperturbed dynamics for Example 2 are $\{x^2,x^3,x^4\}$ and $Core(R)=\{x^5\}$. The only other individually rational allocation is x^1. Moreover, all states in $\{x^2,x^3,x^4\}$ have the same stochastic potential. Hence, we can restrict ourselves, for instance, to the construction of a resistance minimizing x^2-tree. In any least resistance tree, allocations of the blocking cycle have to be connected with mistake-free edges. Note furthermore that x^2 can be reached from x^1 (and x^3, x^4) without mistake. Hence, we are left with x^5. Observe that every state in $\{x^2,x^3,x^4\}$ can be reached from x^5 with one mistake ($\{2,3\}$ can recontract to obtain x^2 with a mistake by agent 2, $\{1,2\}$ can recontract to obtain x^3 with a mistake by agent 1, and $\{1,3\}$ can recontract to obtain x^4 with a mistake by agent 3) and there can be no other (direct or indirect) path through recontracting with fewer mistakes. Therefore $\widetilde{T}_{x^2}=\{[x^1,x^2],[x^2,x^3],[x^3,x^4],[x^5,x^2]\}$ is a resistance minimizing x^2-tree and $\gamma(x^2)=1$.

As for $\gamma(x^5)$ observe that the grand coalition can directly recontract at any of the allocations in $\{x^2,x^3,x^4\}$ to x^5 with only one mistakes (starting from x^2 agent 3 makes a mistake, starting from x^3 agent 2 makes a mistake, and starting from x^4 agent 1 makes a mistake). As $\{x^2,x^3,x^4\}$ is a recurrent class, this is also the minimal edge (or path) resistance. Moreover, x^5 can be reached from x^1 without mistakes. Therefore $\widetilde{T}_{x^5}=\{[x^2,x^3],[x^3,x^4],[x^4,x^5],[x^1,x^5]\}$ is a resistance minimizing x^5-tree and $\gamma(x^5)=1$. Thus, $\gamma(x^2)=\gamma(x^5)$ and $SRC(R)=\big\{\{x^5\},\{x^2,x^3,x^4\}\big\}=RC(R)$. $\diamond$

The following example demonstrates a general drawback of the requirement of stochastic stability in dynamic recontracting processes a la Serrano and Volij (2004).

Example 3 (continued). ***SRC(R) = RC(R) = Core(R) and*** $|$***Core(R)***$|>1$
Recall that the recurrent classes of the unperturbed dynamics for Example 3 are the core allocations $\{x^2\}$, $\{x^3\}$, and $\{x^4\}$. Note that whenever the process is in one of these allocations,

each single agent can recontract with him- or herself to obtain x^1 by making one mistake. Furthermore, any of the allocations in $\{x^2, x^3, x^4\}$ $\{1,2,3\}$-blocks x^1. Hence, recontracting between any two recurrent classes can be achieved with one mistake (which is also the minimal for transitions between recurrent classes). Therefore, $\widetilde{T}_{x^2} = \{[x^1, x^2], [x^3, x^1], [x^4, x^1]\}$ is a resistance minimizing x^2-tree, $\widetilde{T}_{x^3} = \{[x^1, x^3], [x^2, x^1], [x^4, x^1]\}$ is a resistance minimizing x^3-tree, and $\widetilde{T}_{x^4} = \{[x^1, x^4], [x^2, x^1], [x^3, x^1]\}$ is a resistance minimizing x^4-tree and $\gamma(x^2) = \gamma(x^3) = \gamma(x^4) = 2$ such that $SRC(R) = \big\{\{x^2\}, \{x^3\}, \{x^4\}\big\}$. $\diamond$

The last example nicely illustrates a weakness of stochastic stability as a selection device for recurrent classes. Observe that a direct transition from x^2 to x^3 needs two mistakes while all other direct transitions between core allocations only ask for one mistake. This indicates that x^2 is more difficult to exit for the recontracting process than other core allocations, and x^3 is more difficult to access. This suggests x^2 as a more accessible and more stable allocation, or simply a better prediction for the final allocation. But due to the indirect paths from one recurrent class via x^1 to another recurrent class, all core allocations have a stochastic potential of 2 and the conjectured superiority of x^2 can not be established. We formalize this observation as our first main result.

Proposition 2. ***Stochastic Stability and Recurrent Classes***
For a dynamic recontracting process $(X, M(R))$ that fulfills Assumptions 1-3 it holds that
(i) $\gamma(x) = |RC(R)| - 1$ for all $x \in RC(R)$ and $\gamma(x) = 0$ otherwise.
(ii) $RC(R) = SRC(R)$ for every $R \in \mathcal{R}$.

Proof. We first prove the following auxiliary result.

Claim 1. *For every pair of allocations $x, x' \in IR(R)$ there is a directed sequence of edges from x to x' with at most one mistake.*

Observe that the claim is trivially fulfilled if $x = x'$. Hence, assume that $x \neq x'$ from now on. To prove the claim we have to distinguish three cases. First, suppose that $x = e$. Then individual rationality of x' implies that $r(x, x') = 0$. Second suppose that $x' = e$. As $x \neq e$ there has to be a pair of agents $i, j \in N$ that have to trade to achieve allocation x (i.e. there is a commodity l with $x_j(l) = i$). When the singleton coalition $\{i\}$ forms, agent i can claim back his endowment of commodity l without the consent of the other agents. This requires at most one mistake (by agent i). As $x_{N\backslash\{i\}}$ is no longer feasible (recall $x_j(l) = i$), all agents in $N\backslash\{i\}$ receive their initial endowment. Hence, we have established an edge from x to $x' = e$ with a resistance of at most one (such that $r(x, x') \leq 1$). Finally suppose that neither x nor x' equals the initial endowment e. Then the previous findings imply that the sequence of edges $\{[x, e], [e, x']\}$ constitutes a path of resistance of at most one, which concludes the prove of the claim.

We are now ready to prove the proposition. Given the claim above, it is now easy to construct least-resistance trees.

Fix an economy R and denote the respective (finite) set of recurrent classes by $RC(R) = \{\{RC_1\}, ..., \{RC_m\}\}$. It follows directly from the definition of a recurrent class that $\gamma(x) \geq (m-1)$ for any allocation $x \in X$. Next, we prove that $\gamma(x) \leq (m-1)$ for all $x \in RC(R)$ which implies $\gamma(x) = m-1$ for all $x \in RC(R)$ (part (ii)) and implies – together with Lemma 2 – part(i).

Pick an allocation $x \in RC(R)$ and construct an $x-$tree as follows. Consider first the recurrent class RC_k with $x \in RC_k$. By the definition of a recurrent class $r(x', x) = 0$ for any $x' \in RC_k$, *i.e.,* there exists a set of edges that constitute the restriction of an x-tree to RC_k and has zero resistance. Suppose all $x' \in RC_k$ are connected through that set of edges. Then pick another recurrent class $RC_k \neq RC_l$ and consider an allocation $y \in RC_l$. Again, the definition of a recurrent class implies that $r(y', y) = 0$ for any $y' \in RC_l$ such that there exists the restriction of a y-tree to RC_l which has zero resistance. Suppose all $y' \in RC_l$ are connected through that tree. Repeat this procedure for all remaining recurrent classes. With this we have constructed graph of resistance zero that connects all allocations *within* a given recurrent class but does not establish edges between recurrent classes. Claim 1, however, implies that $r(y, x) \leq 1$ for any $x, y \in X$. Hence, for every recurrent class $RC_m \in RC(R)$ there is an allocation $z \in RC_m$ and a path from z to x which has a resistance of at most one. If we establish such a path between the respective allocations in any recurrent class and x, we have constructed an $x - tree$ with a resistance of at most $(m - 1)$. Hence, $\gamma(x) = m - 1$ (part(i)). But then Lemma 2 implies that $SRC(R) = RC(R)$. □

This result offers two insights. First, the requirement of stochastic stability does not work as a selection device for recurrent classes. In addition to coalitional stability, the concept of stochastic stability also incorporates a notion of coalitional accessibility of recurrent classes (*i.e.,* it compares the number of mistakes needed to reach an allocation in a certain recurrent class from all other recurrent classes) as captured in the stochastic potential. However, like we emphasized before, this notion of accessibility is rather limited as it only values the *existence* of least resistance trees, but not their number and overall probability (see *e.g.,* the different stability and accessibility features of allocations x^2 and x^3 in Example 3). This limitation drives the lack of selective power of stochastic stability as documented in proposition 2.

Second, the prominent role of paths via x^1 in Example 3 and the proof of Proposition 2 points at a modeling problem of the type of recontracting processes we investigate. Serrano and Volij (2004) argue (see footnote 6) that the recontracting process is robust to alternative specifications. In particular it is robust to the requirement that any agent in $N \setminus S$ whose bundle remain feasible keep it instead of being automatically sent back to their endowments. We see here that this robustness may not extend. Notice however that when preferences are strict, Proposition 2 fails only if $n > 3$. Thus the lack of selection in our examples remains even if we modify Assumption 3. Notice also that if $n > 3$ but RC(R) contains no cycle, Proposition 2 remains also unaffected by a change in assumption 3. Moreover, our aim is to extend Serrano and Volij (2004) –including their assumption on the new allocation of the complement coalition– to a multiple-type housing market. To this end, Proposition 2(ii) indicates that the selective power of stochastic stability as established by Serrano and Volij (2004) for non-singleton indifference sets does *not* carry over to multiple-type housing markets with strict preferences unlike one could have conjectured.

The following example (a modified version of Example 2) shows that stochastic stability indeed serves as a selection device once non-singleton indifference sets are introduced.

Example 6. ***Non-Singleton Indifference Sets:*** $SRC(R) \subset RC(R)$
Reconsider Example 2 with preferences modified as follows

$$(3,1) \succ_1 (2,2) \succ_1 (1,2) \sim_1 (1,1) \succ_1 \textit{anything},$$

$$(2,1) \succ_2 (3,3) \succ_2 (3,2) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(2,3) \succ_3 (1,1) \succ_3 (1,3) \succ_3 (3,3) \succ_3 \textit{anything}.$$

Recall that $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(1,1),(3,2),(2,3)\},$$
$$x^3 = \{(1,2),(2,1),(3,3)\}, \quad x^4 = \{(3,1),(2,2),(1,3)\},$$
$$x^5 = \{(2,2),(3,3),(1,1)\}.$$

Next, we determine the set of recurrent classes. When moving from an allocation y to an allocation z, if agents involved in the transition are all left indifferent, this should be counted as mistakes since weak blocking requires that at least one agent be made strictly better off. Nonetheless, mistakes that leave agents indifferent may put less weight to $r(y,z)$ than mistakes in which an agent gets worse off.[17] *E.g.,* if the process is in allocation x^3 and the singleton coalition $\{1\}$ becomes active, the unperturbed process still does not permit a transition to x^1 – even though agent 1 is indifferent. Hence, $RC(R) = \{x^2, x^3, x^4, x^5\}$. However, if a lower weight is put on this mistake by agent 1 than *e.g.,* on the mistake by the same agent to ask for his endowment when the process is at allocation x^5 (which leaves him with the bundle $(1,1)$ instead of the strictly preferred bundle $(2,2)$) we end up with a modified analysis of stochastic stability.
One of the least resistance tree for x^2 is $\widetilde{T}_{x^2} = \{[x^3,x^4],[x^4,x^2],[x^5,x^1],[x^1,x^2]\}$. Only the transition from x^5 to the endowment involves a mistake that makes agent 1 worse-off. On the other hand, $\widetilde{T}_{x^5} = \{[x^4,x^2],[x^2,x^3],[x^3,x^1],[x^1,x^5]\}$ is the unique least-resistance tree for allocation x^5. The only mistake is indifference-based. When going from x^3 to x^1, agent 1 claims his endowment back and the complement $N\backslash 1$ is sent back to its respective endowment. But agent 1 is indifferent between the bundles $(1,2)$ and $(1,1)$. In consequence $\gamma(x^5) < \gamma(x^2)$. The only stochastically stable allocation is x^5 because it is easier to access x^5 from the cycle than to access the cycle from x^5. ◇

This example demonstrates how the introduction of a single indifference leads to a breakdown of Proposition 2. Hence, small manipulations of preference profiles lead to major changes in the set of solutions.

This section has shown that stochastic stability does not serve as a proper selection device for recurrent classes in our setting. Either, stochastic stability does not shrink the set of recurrent classes or the respective selection is highly sensitive to small manipulations of the underlying preference relations.

In our view, the consequences of Proposition 2 clearly motivates the search for a more robust solution concept that can deliver sharper predictions on allocations that will be visited by the system in the long-run. We take one further step in the analysis of stability and accessibility of allocations. In the following subsection we will analyze the limit invariant distribution which captures not only the number of mistakes needed to switch between recurrent classes but also encounters details of stability and accessibility of an allocation, such as the number of coalitions that can agree or decide to improve upon it.

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[17] One way to implement this would be to follow Serrano and Volij (2004) and let $r(y,z) = |S|/|N|$ when all agents involved in S in the transition from y to z are left indifferent. Otherwise, if at least one agent in S is made worse-off in the transition from y to z, then $r(y,z)$ is simply the number of mistakes in which agents were made worse-off. Formally, this requires an analysis of state-dependent mistakes as spelled out in Serrano and Volij (2004)

5.2 The Limit Invariant Distribution

So far we offered a method to identify the set of stochastically stable allocations $SRC(R)$, *i.e.*, the support of the limit invariant distribution $\mu^*(R) = \lim_{\epsilon\to 0}\mu_\epsilon(R)$. But the fact that an allocation $x \in X$ is in the support of the limit invariant distribution (*i.e.*, $\mu^*(x) > 0$) does of course not indicate anything about the actual probability that the system will be in allocation $x \in X$ in the long run. This is captured by the value of $\mu^*(x)$. The next proposition presents a method to assess the limit invariant distribution that makes explicit use of the polynomial structure of $M^\epsilon(x', x'') = \sum_{i=0..n} m_i(x', x'')\epsilon^i$. For further reference we define a least resistance transition $\widetilde{m}(x', x'')$ as the component $m_i(x', x'') > 0$ that minimizes i (*i.e.*, for any two allocations x' and x'', $\widetilde{m}(x', x'')$ depicts the respective transition probability with the smallest possible number of mistakes).

Proposition 3. *For each perturbed dynamic recontracting process* $(X, M^\epsilon(R))$ *that fulfills Assumptions 1-4, it holds that* $\mu^*(x) = \frac{p(x)}{\sum_{y\in X} p(y)}$ *with,*

$$p(x) = \sum\nolimits_{\{T\in\mathcal{T}_x | \Pi_{[x',x'']\in T} r(x',x'')=\gamma(x)\}}[\Pi_{[x',x'']\in T}\widetilde{m}(x', x'')].$$

Proof. (Freidlin and Wentzell, 2004) have shown that the (unique) invariant distribution of an irreducible Markov chain (X, M) is given by $\mu(x) = \frac{q(x)}{\sum_{x\in X} q(x)}$ with $q(x) = \sum_{T\in\mathcal{T}_x} \Pi_{[x',x'']\in T} M(x', x'')$. If the Markov process is a perturbed dynamic recontracting process $(X, M^\epsilon(R))$ that fulfills Assumptions 1-4 such that $M^\epsilon(x', x'') = \sum_{i=0..n} m_i(x', x'')\epsilon^i$, $q(x)$ is a polynomial of degree $n(|X| - 1)$ in the mistake probability ϵ (there can be only one mistake per agent per edge, and there are not more than $(|X| - 1)$ edges in a spanning tree in X), *i.e.*, we can rewrite $q(x) \equiv \sum_{i=0..n(|X|-1)} q_i(x)\epsilon^i$.
By Proposition 2(ii), $\lim_{\epsilon\to 0} q(x) > 0$ if and only if $x \in RC(R)$. Moreover, $q_i(x) = 0$ for all $i < \gamma(x)$ and $q_{\gamma(x)}(x) > 0$ (by definition of the stochastic potential there are no spanning trees with less than $\gamma(x)$ mistakes). Finally, observe that $q_{\gamma(x)} = \sum_{\{T\in\mathcal{T}_x | \Pi_{[x',x'']\in T} r(x',x'')=\gamma(x)\}} \Pi_{[x',x'']\in T}\widetilde{m}(x', x'')$ (a contribution to $q(x)$ of lowest order in ϵ is the sum over least resistance spanning trees *i.e.*, trees that only contain least resistance edges).
Consider a multiple-type housing market with one recurrent class. Then, $\lim_{\epsilon\to 0} q(x) = q_0(x) > 0$ for all $x \in RC(R)$ and $\mu^* = \lim_{\epsilon\to 0}\mu^\epsilon(x) = \frac{q_0(x)}{\sum_{y\in RC(R)} q_0(y)}$ with $q_0(x) = \sum_{\{T\in\mathcal{T}_x | \Pi_{[x',x'']\in T} r(x',x'')=0\}} \Pi_{[x',x'']\in T} m_0(x', x'')$ which equals $p(x)$ for a unique recurrent class (or $\gamma(x) = 0$).
Suppose from now on that $|RC(R)| \geq 2$. Recall from Proposition 2(i) that $\gamma(x) = (|RC| - 1)$). From $q_{\gamma(x)} > 0$ for all $x \in RC(R)$ it follows that $\lim_{\epsilon\to 0} \frac{\partial^j q(x)}{\partial\epsilon^j} = 0$ for all $j = 0..(\gamma(x) - 1)$ and $\lim_{\epsilon\to 0} \frac{\partial^{\gamma(x)} q(x)}{\partial\epsilon^{\gamma(x)}} = \frac{1}{\gamma(x)!} q_{\gamma(x)}(x)$ (which is strictly positive if and only if $x \in RC(R)$). Then, iterated application of de l'Hospitals rule implies that

$$\mu^*(x) = \lim_{\epsilon\to 0}\mu^\epsilon(x) = \lim_{\epsilon\to 0}\frac{q(x)}{\sum_{y\in X} q(y)} = \frac{\lim_{\epsilon\to 0}\frac{\partial^{\gamma(x)} q(x)}{\partial\epsilon^{\gamma(x)}}}{\lim_{\epsilon\to 0}\sum_{y\in X}\frac{\partial^{\gamma(x)} q(y)}{\partial\epsilon^{\gamma(x)}}} = \frac{q_{\gamma(x)}(x)}{\sum_{y\in RC(R)} q_{\gamma(x)}(x)}.$$

□

Proposition 3 offers the following recipe to elicit the limit invariant distribution $\mu^*(x)$ for an allocation $x \in RC(R)$. (i) Construct *all* least-resistance x-trees (*i.e.*, determine $\{T \in \mathcal{T}_x \mid$

$\Pi_{[x',x'']\in T} r(x',x'') = \gamma(x)\}$. By Proposition 2(i) this amounts to a construction of all trees of resistance $|RC(R)|-1$. (ii) Compute the product of transition probabilities for all edges in a given least-resistance x-tree (*i.e.,* $\Pi_{[x',x'']\in T}\widetilde{m}(x',x'')$) and (iii) sum over all least resistance x-trees.

We illustrate the method by revisiting some of the previously introduced examples.[18] Our findings are the second main result of the paper. We start with Example 4 to show that the limit invariant distribution serves as a selection device among elements of the core (or Walrasian allocations).

Example 4 (continued). ***Multiple Walrasian Allocations***
Consider again the multiple-type housing market given by the following preference relation

- $(1,2) \succ_1 (3,3) \succ_1 (2,3) \succ_1 (1,1) \succ_1$ *anything*,
- $(1,3) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (2,2) \succ_2$ *anything*,
- $(2,1) \succ_3 (3,1) \succ_3 (3,3) \succ_3$ *anything.*

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4\}$ with,

- $x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(2,3),(1,2),(3,1)\}$,
- $x^3 = \{(3,3),(1,2),(2,1)\}, \quad x^4 = \{(1,2),(3,3),(2,1)\}$.

Recall that in this case

$$RC(R) = SRC(R) = Core(R) = W(R) = \{\{x^3\},\{x^4\}\}.$$

Proposition 2(ii) furthermore indicates that $\gamma(x^3) = \gamma(x^4) = 1$ (*i.e.,* every least resistance tree includes only one mistake). But at least one mistake is needed for every edge that leads away from one of the core allocations. The only edges leading away from x^3 with not more than one mistake are given by the grand coalition agreeing upon x^4 (with a mistake by agent 2) or all three singleton coalitions that by mistake block via x^1. In the same way, x^3 can only be left towards x^4 (with a blocking by the grand coalition and a mistake by agent 1) or towards x^1 with mistakes by the respective singleton coalitions. The only edges with zero mistakes –that have to be exploited to form a spanning tree in the space of allocations– are from x^1 to any other individually rational allocation and from x^2 to x^3 (all by the grand coalition). Figure 1 depicts the three different types of x^3-least-resistance trees. Two more trees can be constructed by attaching x^1 to x^2 and x^4, respectively, instead of attaching it to x^3 in the left tree. We compare this with the set of x^4 least-resistance-trees. As depicted in Figure 2, the tree on the left again symbolizes three different trees through the possible ways to attach x^1). Because x^3 can $\{1,2,3\}$-block x^2 with no mistake, but x^4 cannot, there is no x^4-analogue to the middle tree in Figure 1.
According to Proposition 3

$$\frac{\mu^*(x^3)}{\widetilde{m}(x^2,x^3)} = \widetilde{m}^{\epsilon}(x^4,x^3)(\widetilde{m}(x^1,x^2)+\widetilde{m}(x^1,x^3)+\widetilde{m}(x^1,x^4))+\widetilde{m}(x^4,x^1)(\widetilde{m}(x^1,x^3)+\widetilde{m}(x^1,x^2)),$$

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[18] For each of the example, we recapitulate the complete set of preferences and of allocations to help the reader follow the reasoning construction of the limit invariant distribution.

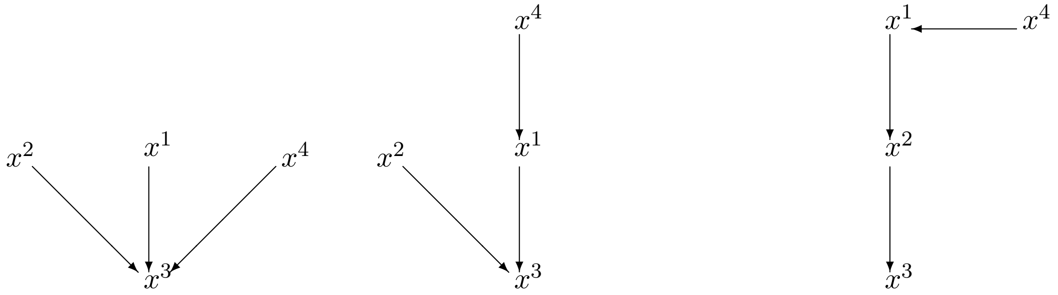

Figure 1: Least Resistance x^3-trees in Example 4

Figure 2: Least Resistance x^4-trees in Example 4

$$\frac{\mu^*(x^4)}{\widetilde{m}(x^2,x^3)} = \widetilde{m}(x^3,x^4)(\widetilde{m}(x^1,x^2)+\widetilde{m}(x^1,x^3)+\widetilde{m}(x^1,x^4))+\widetilde{m}(x^3,x^1)\widetilde{m}(x^1,x^4).$$

Now recall that the probability to make a mistake does not depend on the identity of the agent (see Assumption 4) and suppose that the probability for a certain coalition to form does not depend on the current allocation. Then, $\widetilde{m}(x^3,x^4)=\widetilde{m}(x^4,x^3)$ and $\widetilde{m}(x^3,x^1)=\widetilde{m}(x^4,x^1)$. Finally, suppose that $\widetilde{m}(x^1,x^3)=\widetilde{m}(x^1,x^4)$ (*i.e.,* the grand coalition agrees upon x^3 and x^4 if the process is in x^1 with the same probability). Then, $\mu^*(x^3)>\mu^*(x^4)$ such that the system is more likely to be found in allocation x^3 then in x^4. Observe that this result is driven by the fact that x^3 can be agreed upon by the grand coalition if the system is in allocation x^2 while this is not feasible for allocation x^4. This difference in mistake-free edges –or *accessibility*– leads to a larger set of least resistance x^3-trees and thereby enhances the respective component in the limit invariant distribution. ◇

This example shows that the various core (or Walrasian) allocations of a multiple-type housing market are not necessarily equally good predictors for the final allocation. Accordingly, one may wonder wether a Walrasian/core allocation is at least a better prediction than non-Walrasian/non-core allocations. To elaborate on this point we reconsider Example 3.

Example 3 (continued). ***SRC(R) = RC(R) = Core(R) $\supsetneq$ W(R)***
Recall that preferences are given by

$$(1,2)\succ_1(3,3)\succ_1(2,3)\succ_1(1,1)\succ_1 \textit{anything},$$

$$(1,3) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(3,1) \succ_3 (2,1) \succ_3 (3,3) \succ_3 \textit{anything}.$$

such that the set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(2,3),(1,2),(3,1)\},$$

$$x^3 = \{(1,2),(3,3),(2,1)\}, \quad x^4 = \{(3,3),(1,2),(2,1)\}.$$

The core consists of the allocations $\{x^2, x^3, x^4\}$ and the unique Walrasian allocation is $\{x^3\}$ such that $RC(R) = SRC(R) = Core(R) \supsetneq W(R)$.
Observe that a direct edge from x^2 to x^3 takes two mistakes while all other transitions from one core allocation to another need only one mistake. This suggests that x^2 is less likely to be left than x^3 and x^4 (*i.e.,* is more stable) while x^3 is less likely to be reached by the recontracting process than x^2 and x^4 (*i.e.,* is less accessible). In Appendix A we prove that this intuition is true and indeed $\mu^*(x^2) > \mu^*(x^4) > \mu^*(x^3)$ if the probability that a certain coalition forms does not depend on the allocation and a given coalition agrees upon each improvement with the same probability. Hence, x^2 is the most accessible and stable allocation while the x^3, the (unique) Walrasian allocation, is the worst prediction for the long-run behavior of the process. ◇

The next example shows that the limit invariant distribution does not only allow for a distinction between different singleton recurrent classes but does also allow for an analysis of the probability that a recontracting process ends up in a cycle of individually rational allocations.

Example 2 (continued). ***SRC(R) = RC(R) $\supsetneq$ Core(R)***

$$(3,1) \succ_1 (2,2) \succ_1 (1,2) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(2,1) \succ_2 (3,3) \succ_2 (3,2) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(2,3) \succ_3 (1,1) \succ_3 (1,3) \succ_3 (3,3) \succ_3 \textit{anything}.$$

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with,

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(1,1),(3,2),(2,3)\},$$

$$x^3 = \{(1,2),(2,1),(3,3)\}, \quad x^4 = \{(3,1),(2,2),(1,3)\},$$

$$x^5 = \{(2,2),(3,3),(1,1)\}.$$

We have that $Core(R) = W(R) = \{x^5\}$ and $RC(R) = SRC(R) = \{\{x^2, x^3, x^4\}, \{x^5\}$.
Recall that in this market every individual allocation but x^1 is member of a recurrent class (either the cycle consisting of $(\{x^2, x^3, x^4\})$ or the unique core allocation x^5). Hence, $RC(R) = SRC(R) = \{\{x^2, x^3 x^4\}, \{x^5\}\}$. Proposition 2 (ii) indicates that $\gamma(x^2) = \gamma(x^3) = \gamma(x^4) = \gamma(x^5) = 1$. This implies that every least resistance tree can only contain one mistake, which is also necessary to connect –directly or indirectly– the cycle and the core-allocation. As a consequence, x^2, x^3, and x^4 have to be connected with the mistake-free and unique recontracting-cycle in any least resistance tree.

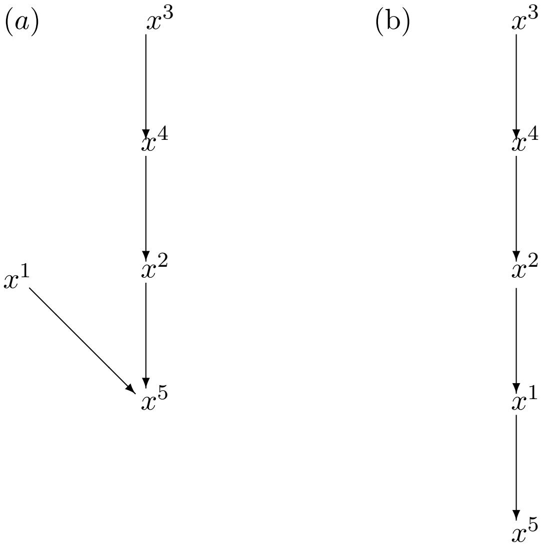

Figure 3: Least Resistance x^5-trees

Any x^5-tree needs a sequence of edges from an element of the cycle to x^5 with one mistake. Figure 3 depicts examples of the two distinct types of least resistance x^5-trees. Either, an element of a cycle is directly connected with x^5 (*i.e.,* the grand coalition forms and one agent makes a mistake –see tree (a)) or the cycle-element is connected with x^1 (a singleton coalition forms and the respective agents makes one mistake) and the grand coalition agrees upon x^5 afterwards (see tree (b)). The different permutations of cycle elements and the feasible connections between x^1 and the rest of the tree lead to 15 trees in total (12 of type (a) and 3 of type (b)) that are listed in the Appendix.

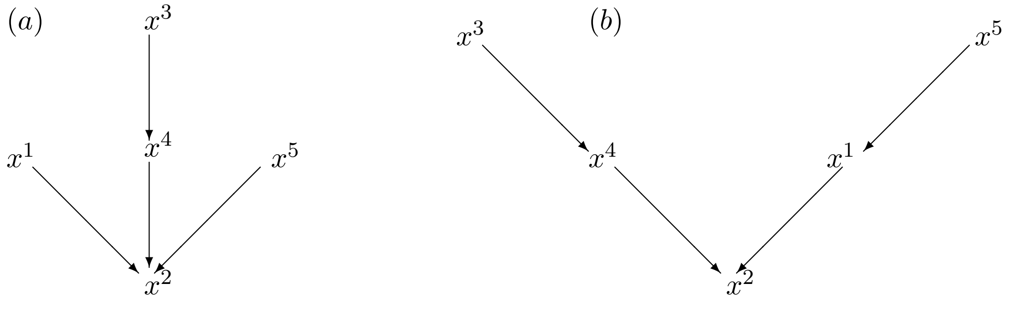

Figure 4: Least Resistance x^2-trees

Likewise any least resistance tree of a cycle-element (x^2, say) has to include a (sequence of) edge(s) from x^5 to an element of the cycle. Figure 4 depicts examples of the two distinct types of least resistance x^2-trees. Either, x^5 is directly connected with an element of the cycle (*i.e.,* a coalition of two agents forms and one of these agents makes a mistake –see tree (a)) or x^5 is connected with x^1 (a singleton coalition forms, the respective agent makes one mistake, and the grand coalition or $\{2, 3\}$ agree upon x^2 afterwards) which is depicted in tree (b). The different permutations of cycle elements and the feasible connections between x^1 and the rest of the tree lead to 15 trees in total (12 of type (a) and 3 of type (b)) that are listed in the Appendix.

In contrast to Example 4 (see above) the *number* of least resistance trees for x^2 and x^5 is **identical**. However, different edges turn out to be bottle-necks. In the following, we present an informal discussion of the relevant effects - a complete treatment is relegated to Appendix A. To start out, observe that x^2 and x^5-trees differ in three aspects. First, any least resistance x^5-tree of type (a) in Figure 3 contains a blocking of a cycle allocation by the grand coalition to reach x^5, while x^2-trees of type (a) in Figure 4 need a blocking of x^5 by a particular coalition of two agents to reach a cycle-allocation. If the probability that the grand coalition is chosen equals the probability that any coalition with two agents is chosen, this difference does not matter. Second, any least resistance x^5-tree of type (b) in Figure 3 contains a blocking of a cycle allocation by a singleton coalition to reach x^1 (by mistake), while x^2-trees of type (b) in Figure 4 need a blocking of x^5 by a singleton coalition to reach an element of the cycle. Now recall that in every cycle allocation one of the agents is already at his initial endowment, hence if he alone is given the opportunity to recontract the system can not switch to x^1. Hence, $\widetilde{m}(x^5, x^1) > \widetilde{m}(x^2, x^1)$ if the probability that a singleton coalition forms does not depend on the allocation. This leads to a larger contribution from this type of trees to $\mu^*(x^2)$ than to $\mu^*(x^5)$. Third, x^5-trees of type (b) in Figure 3 also contain a blocking of x^1 by x^5, while x^2-trees of type (b) in Figure 4 need a blocking of x^1 by x^2. Note that for the former allocation to be agreed upon the grand coalition has to form while the latter allocation can also be achieved by coalitions of two agents. Hence, $\widetilde{m}(x^1, x^2) > \widetilde{m}(x^1, x^5)$. In Appendix A we prove that this intuition holds and $\mu^*(x^2) > \mu^*(x^5)$ if $\widetilde{m}(x, x^1) < \widetilde{m}(x^5, x^1)$ for every $x \in \{x^2, x^3, x^4\}$, $\widetilde{m}(x^1, x^2) > \widetilde{m}(x^1, x^5)$, and coalitions of size three form with the same probability as coalitions of size two. Hence, the probability that the system is found in *one* cycle allocation is higher than the respective probability to find it in the core.[19] Note, however, that this result is not driven by the lack of mistake-free edges that lead to the core allocation x^5 (as in the previous examples), but by details of the process that determine the opportunities to reach and exit the initial endowment. ◇

The above-mentioned examples highlight the following features of the limit invariant distribution. First, the probability that a dynamic recontracting process is in a certain recurrent class does not depend on its topology (singleton or cycle). Neither is it necessarily enhanced by the fact that a certain allocation is Walrasian. Rather, the limit invariant distribution depends on two issues. (i) The number and relative transition probabilities of resistance minimizing paths from one recurrent class to another. Consider as an example the importance of the edges to and from the initial endowment in Example 2. (ii) The accessibility of a recurrent class from individually rational allocations that are not member of a recurrent class. An example is allocation x^2 in Example 4.

6 Concluding Remarks

6.1 Solution Concepts for Dynamic Recontracting Processes

Recurrent Classes In order to develop a non-empty solution set for any multi-type housing market, we have modeled economic interaction as a dynamic recontracting (Markov) process.

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[19] In Appendix A we show that $\mu^*(x^2) = \mu^*(x^3) = \mu^*(x^4)$ under these conditions due to symmetry.

We characterized this process for a given multiple-type housing market R by its set of recurrent classes $RC(R)$, the set of stochastically stable states $SRC(R)$, and the limit invariant distribution $\mu^*(R)$. The set of recurrent classes of the dynamic recontracting process is non-empty and contains all sets of allocations that cannot be blocked by any allocation outside the respective set. Obviously, every element of the core will form a singleton recurrent class. Moreover, we have shown the existence of non-singleton recurrent classes (in settings with an empty core as well as vis-a-vis a non-empty core).

Stochastically Stable States In the attempt to select among the recurrent classes of the dynamic recontracting process, we applied the concept of stochastic stability. Agents sign contracts that make them worse off with a small probability (referred to as a mistake). If this perturbation of the dynamic recontracting process becomes sufficiently small, the process will – in the long-run – converge to a collection of recurrent classes (the set of stochastically stable recurrent classes) that does not necessarily include every recurrent class. Consider, for instance, a market R with two singleton recurrent classes. One can be left (towards the other recurrent class) only if at least two agents make a mistake, while the other class can be left after with just one mistake. Then, the process will converge to the first recurrent class with probability one – this recurrent class is the only stochastically stable recurrent class of R. Hence, while recurrent classes are characterized by the (core-like) feature of internal stability (no allocation in a recurrent class can be blocked by recontracting among the members of a coalition upon an allocation outside the recurrent class), stochastically stable states are also characterized by their accessibility from other recurrent classes. We formalize the interplay of stability and accessibility through the stochastic potential that indicates the minimal number of mistakes needed to reach a recurrent class from all other recurrent classes. However, we showed that every recurrent class is stochastically stable as long as preferences are strict (and the process is modelled with weak blocking).

Limit Invariant Distributions To offer sharper predictions for the final allocations of multiple-type housing markets, we introduced a method to directly access the limit invariant distribution. By construction, the limit invariant distribution $\mu^*(R)$ of a dynamic recontracting process is the unique probability distribution over allocations that the process will converge to in the long-run. Hence, it indeed offers a sharp prediction and it is always applicable. Compared to the set of (stochastically stable) recurrent classes, these are clearly valuable features. Moreover, considering the complete set of least resistance trees fully incorporates a very intuitive notion of stability and accessibility as we illustrated in several examples. However, these features come at the cost of the computation of *all* least resistance trees (compared to the construction of one tree for a stochastic stability analysis). But as a least resistance tree only consists of least resistance edges, this is a simple combinatorial exercise as was also demonstrated in the examples.

Our paper aims at an illustration of the method to compute the limit invariant distribution. The particular features of multiple-type housing markets proved useful to demonstrate its applicability and intuitive appeal. However, it should be obvious that results similar to Proposition 3 are also feasible and desirable for other Markov processes that suffer from a multiplicity of stochastically stable recurrent classes (as e.g. evolutionary equilibrium selection in (asymmetric) non-cooperative games or models of social and economic network generation).

6.2 Relation to Other Solution Concepts

Dynamic Recontracting and the Core The concepts proposed in this paper distinguish themselves from the core in two respects. First, both $RC(R)$ and $SRC(R)$ are non-empty due to the finiteness of the Markov process. In particular, cycles as in Example 1 emerge naturally as recurrent classes (and stochastically stable states). Moreover, core allocations are the only singleton recurrent classes of the process. It therefore seems natural to invoke the requirement of stochastic stability to select among recurrent classes in the very same way as the literature on evolutionary equilibrium selection in non-cooperative game theory used stochastic stability to select among Nash equilibria. However, this kind of selection is only applicable if indifference in preferences are possible or if strong blocking is used. Nonetheless, its major weakness is the sole consideration of the stochastic potential (*i.e.,* the smallest number of mistakes needed to reach an allocation from all other allocations) while not taking into account the respective number of paths that lead to an allocation with the smallest number of mistakes. A direct analysis of the limit invariant distribution as conducted in Section 5.2 proceeds one step further as it requires internal stability and accessibility through *all* least resistance trees. As the components of the limit invariant distribution depict the probability that the system converges to a certain allocation, it thereby ultimately solves the problem of multi-valuedness of the set of recurrent classes (and the core). Our examples illustrate the intuitive appeal of such a core-selection procedure that trades-off stability and accessibility and thereby enriches the static concept of the core with the dynamic aspects of recontracting processes.

Dynamic Recontracting and Walrasian Allocations In contrast to the set of (stochastically stable) recurrent classes, the set of Walrasian allocations can be empty, even in the presence of a non-empty core. Given the message of Proposition 2, Walrasian allocations are always stochastically stable –if preferences are strict. An analysis of the limit invariant distributions, however, suggests that (i) Walrasian allocations may be the worst prediction for the final allocation of a dynamic recontracting process and (ii) Different Walrasian allocations of the same market do not share the same properties in terms of coalitional accessibility. Some are easier to reach and exit than others and thereby have different weight in the long-run probability distribution over allocations. This underlines that there may be frictions in the market and the trading process that are indeed not captured by the Walrasian equilibrium concept.

6.3 Weak and Strong Blocking

To keep the exposition as simple as possible we focused on weak coalitional blocking. Strong blocking has been investigated for the one commodity case in Serrano and Volij (2004). From a modelling perspective, the key difference turns out to be the special state dependent probability of mistakes which is necessary for strong blocking but is not needed in our setting. In particular, Serrano and Volij (2004) assume that contracts which leave an agent indifferent are signed with probability ϵ while contracts which make an agent (strictly) worse of are signed with probability ϵ^λ with $\lambda > 1$. We demonstrated that the introduction of non-singleton indifference sets or strong blocking reestablishes the selective power of stochastic stability that is absent otherwise.

For expositional ease, we restricted ourselves to strict preferences and weak blocking in the presentation of the computational method to elicit the limit invariant distribution. In contrast to the concept of stochastic stability that crucially relies on non-singleton indifference sets and strong blocking, it should be obvious from Section 5.2 that this method offers enough flexibility to be applicable regardless of the structure of the indifference-sets and the blocking notion.

6.4 Is this the end of the story?

Our analysis has shown that attempts to study trading institutions may lead away from the predictions of the core or Walrasian allocations. We have seen some instances where a cycle of allocations may be the best predictor of the final allocation of our dynamic trading process. Indeed, this conclusion is in part driven by the myopia of agents. Recall that at each date agents agree to retrade if it is beneficial to do so. In accepting such trades, agents do not envision the possible blockings along the retrading path. Nevertheless, it is clear from our analysis that there are aspects of *stability* and in particular of *accessibility* of allocations that are not captured by the core. Some core allocations turn out to be harder to reach than others while non-core allocations may emerge naturally through sequence of trades. This hints that in studying cooperative solution concepts, dynamic aspects cannot be ignored.

Finally, several possible extensions of this work can be considered. First, myopic agents could be substituted by farsighted agents. We conjecture that such a replacement would eliminate cycles of allocations. Second, an analysis of dynamic processes for general NTU games could be a fruitful path of research.

Appendix

A Examples

Example 1: ***An Empty Core***
Agents' preferences equal

$$(3,1) \succ_1 (1,2) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(2,1) \succ_2 (3,2) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(2,3) \succ_3 (1,3) \succ_3 (3,3) \succ_3 \textit{anything},$$

and $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(1,1),(3,2),(2,3)\},$$

$$x^3 = \{(1,2),(2,1),(3,3)\}, \quad x^4 = \{(3,1),(2,2),(1,3)\}.$$

Hence, $Core(R) = W(R) = \emptyset$. Furthermore, $RC(R) = SRC(R) = \{x^2, x^3, x^4\}$. ◇

Example 2: ***The Unique Walrasian Allocation Equals the Core Allocation***
Agents' preferences equal

$$(3,1) \succ_1 (2,2) \succ_1 (1,2) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(2,1) \succ_2 (3,3) \succ_2 (3,2) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(2,3) \succ_3 (1,1) \succ_3 (1,3) \succ_3 (3,3) \succ_3 \textit{anything},$$

and $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(1,1),(3,2),(2,3)\},$$

$$x^3 = \{(1,2),(2,1),(3,3)\}, \quad x^4 = \{(3,1),(2,2),(1,3)\},$$

$$x^5 = \{(2,2),(3,3),(1,1)\}.$$

Clearly, x^2 $\{2,3\}$-blocks x^1, x^3 $\{1,2\}$-blocks x^2, x^4 $\{1,3\}$-blocks x^3, and x^2 $\{2,3\}$-blocks x^4. Since x^5 cannot be blocked by any coalition, $Core(R) = \{x^5\}$. Next, we prove that allocation x^5 is Walrasian. It is easy to check that the following (in)equalities hold, *e.g.*, for price system $p \equiv (p_1, p_2)$ such that $p_1 = (0, 1, \frac{1}{2})$ and $p_2 = (1, 0, \frac{1}{2})$.

$$(1)\ p_1(1) + p_2(1) = p_1(2) + p_2(2), \quad (2)\ p_1(2) + p_2(2) = p_1(3) + p_2(3),$$

$$(3)\ p_1(3) + p_2(3) = p_1(1) + p_2(1), \quad (4)\ p_1(3) + p_2(1) > p_1(1) + p_2(1),$$

$$(5)\ p_1(2) + p_2(1) > p_1(2) + p_2(2), \quad (6)\ p_1(2) + p_2(3) > p_1(3) + p_2(3).$$

Hence, $W(R) = \{x^5\}$. Furthermore, $RC(R) = SRC(R) = \big\{\{x^2, x^3, x^4\}, \{x^5\}\big\}$.

In the following we indicate the set of least resistance trees for each of the recurrent classes. As it follows from Proposition 2(i) that $\gamma(x^5) = \gamma(x^k) = 1$ for $k = 2, 3, 4$ it suffices to consider those transitions that need at most one mistake. In in the following table we indicate the respective (set of) coalition(s) that facilitates a transition from an initial state (row) to a final state (column) with zero mistakes (we omit the diagonal elements as they are irrelevant for cycle-free graphs). For expositional ease we introduce the following abbreviations.

$$N \equiv \{1,2,3\}$$

$$C_1 \equiv \{\{1\},\{2\},\{3\}\}$$

$$C_2 \equiv \{\{1,2\},\{1,3\},\{2,3\}\}$$

	x^1	x^2	x^3	x^4	x^5
x^1	x	$\{N, \{2,3\}\}$	$\{N, \{1,2\}\}$	$\{N, \{1,3\}\}$	N
x^2	$\emptyset$	x	$\{1,2\}$	$\emptyset$	$\emptyset$
x^3	$\emptyset$	$\emptyset$	x	$\{1,3\}$	$\emptyset$
x^4	$\emptyset$	$\{2,3\}$	$\emptyset$	x	$\emptyset$
x^5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	x

The next table lists the respective coalitions that facilitate a transition if a certain member makes a mistake.[20]

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[20] In this example this agent is uniquely determined.

	x^1	x^2	x^3	x^4	x^5
x^1	x	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x^2	$\{C_1, \{1,2\}, \{1,3\}\}$	x	N	$\{1,3\}$	N
x^3	$\{C_1, \{1,3\}, \{2,3\}\}$	$\{2,3\}$	x	N	N
x^4	$\{C_1, \{1,2\}, \{2,3\}\}$	N	$\{1,2\}$	x	N
x^5	C_1	$\{2,3\}$	$\{1,2\}$	$\{1,3\}$	x

With the transition opportunities as depicted in the previous tables, we can now list all least resistance trees for the (unique) core allocation x^5 and a cycle allocation (x^2, say). We start with the set of x^5-trees. If every least resistance tree includes exactly one mistake (*i.e.,* $\gamma(x^5) = 1$), this mistake has to be made on a (cycle-free) sequence of edges from an element of the cycle to x^5. From any allocation in the cycle there are exactly two such paths. Either, the grand coalition forms and the agent who is worse off at allocation x^5 (note that this is exactly one agent for every allocation in the cycle) agrees upon x^5 by mistake, or a singleton coalition forms and the respective agent ask for his initial endowment even though he is then worse off (note that in every cycle allocation there are two such singleton allocations, *e.g.,* at x^2 only agent 2 and 3 can recontract on their initial endowment and thereby induce a transition to x^1). Agent 1, in contrast, is already at his initial endowment). These two paths induce the two types of least resistance trees depicted in Figure 3. Next, observe that the allocations of the cycle have to be connected via the respective (unique) mistake-free edges. As for type (a) note that there are three different allocations from which the cycle can be left to x^5. Moreover, there can be a (mistake-free) edge from x^1 to any other of the four allocations. This leaves as with the following 12 trees of type (a).

$$\{[x^3,x^4],[x^4,x^2],[x^2,x^5],[x^1,x^2]\};\{[x^3,x^4],[x^4,x^2],[x^2,x^5],[x^1,x^3]\};$$

$$\{[x^3,x^4],[x^4,x^2],[x^2,x^5],[x^1,x^4]\};\{[x^3,x^4],[x^4,x^2],[x^2,x^5],[x^1,x^5]\};$$

$$\{[x^4,x^2],[x^2,x^3],[x^3,x^5],[x^1,x^2]\};\{[x^4,x^2],[x^2,x^3],[x^3,x^5],[x^1,x^3]\};$$

$$\{[x^4,x^2],[x^2,x^3],[x^3,x^5],[x^1,x^4]\};\{[x^4,x^2],[x^2,x^3],[x^3,x^5],[x^1,x^5]\};$$

$$\{[x^2,x^3],[x^3,x^4],[x^4,x^5],[x^1,x^2]\};\{[x^2,x^3],[x^3,x^4],[x^4,x^5],[x^1,x^3]\};$$

$$\{[x^2,x^3],[x^3,x^4],[x^4,x^5],[x^1,x^4]\};\{[x^2,x^3],[x^3,x^4],[x^4,x^5],[x^1,x^5]\}.$$

Likewise there are three different allocations in the cycle from which x^1 can be reached (and subsequently left towards x^5). Hence, we have to add the following three trees.

$$\{[x^3,x^4],[x^4,x^2],[x^2,x^1],[x^1,x^5]\};$$

$$\{[x^2,x^3],[x^3,x^4],[x^4,x^1],[x^1,x^5]\};$$

$$\{[x^4,x^2],[x^2,x^3],[x^3,x^1],[x^1,x^5]\}.$$

In a similar way, we can construct the set of x^2-trees. If every least resistance tree includes exactly one mistake (*i.e.,* $\gamma(x^2) = 1$), this mistake has to be made on a (cycle-free) sequence of edges from x^5 to an element of the cycle. To any allocation in the cycle there are two such paths. Either, a particular coalition of size two forms and the agent who is worse off at the cycle-allocation agrees upon the cycle allocation by mistake. Note that there is exactly one coalition of size two that can actually contract upon each cycle allocation (the agent who is left at his endowment can not be a member of such a coalition). Or one of the three singleton coalition forms and the respective agent asks for his initial endowment even though he is then worse off, afterwards the grand coalition or $\{2,3\}$ agree upon x^2. These two types of paths induce the two types of least resistance trees depicted in Figure 4 (again the allocations of the cycle have to be connected via the respective (unique) mistake-free edges). As for type (a) note that there are three different allocations from which the cycle can be accessed from x^5. Moreover, there can be a (mistake-free) edge from x^1 to any other of the four allocation. This leaves as with the following 12 x^2-trees of type (a).

$$\{[x^3,x^4],[x^4,x^2],[x^5,x^2],[x^1,x^2]\};\{[x^3,x^4],[x^4,x^2],[x^5,x^2],[x^1,x^3]\};$$

$$\{[x^3,x^4],[x^4,x^2],[x^5,x^2],[x^1,x^4]\};\{[x^3,x^4],[x^4,x^2],[x^5,x^2],[x^1,x^5]\};$$

$$\{[x^4,x^2],[x^2,x^3],[x^5,x^3],[x^1,x^2]\};\{[x^4,x^2],[x^2,x^3],[x^5,x^3],[x^1,x^3]\};$$

$$\{[x^4,x^2],[x^2,x^3],[x^5,x^3],[x^1,x^4]\};\{[x^4,x^2],[x^2,x^3],[x^5,x^3],[x^1,x^5]\};$$

$$\{[x^2,x^3],[x^3,x^4],[x^5,x^4],[x^1,x^2]\};\{[x^2,x^3],[x^3,x^4],[x^5,x^4],[x^1,x^3]\};$$

$$\{[x^2,x^3],[x^3,x^4],[x^5,x^4],[x^1,x^4]\};\{[x^2,x^3],[x^3,x^4],[x^5,x^4],[x^1,x^5]\}.$$

Likewise there are three different allocations in the cycle that can be reached from x^1 (after x^5 has been left towards x^1). Hence, we have to add the following three trees.

$$\{[x^3,x^4],[x^4,x^2],[x^1,x^2],[x^5,x^1]\};$$

$$\{[x^2,x^3],[x^3,x^4],[x^1,x^4],[x^5,x^1]\};$$

$$\{[x^4,x^2],[x^2,x^3],[x^1,x^3],[x^5,x^1]\}.$$

This is sufficient information to apply the formula for μ^* in Proposition 3. For expositional ease let us make the following assumptions. First, recall from Assumption 4 that the probability for an agent to commit a mistake does not depend on his identity, the coalition, or the allocation. Moreover, suppose the probability that a certain coalition forms does not depend on the identity of the agents and that a coalition chooses each improving allocation with the same probability. Then,

$$\begin{aligned}
\widetilde{m}(x^2,x^3) &= \widetilde{m}(x^3,x^4) = \widetilde{m}(x^4,x^2),\\
\widetilde{m}(x^1,x^2) &= \widetilde{m}(x^1,x^3) = \widetilde{m}(x^1,x^4),\\
\widetilde{m}(x^2,x^1) &= \widetilde{m}(x^3,x^1) = \widetilde{m}(x^4,x^1),\\
\widetilde{m}(x^2,x^5) &= \widetilde{m}(x^3,x^5) = \widetilde{m}(x^4,x^5),\\
\widetilde{m}(x^5,x^2) &= \widetilde{m}(x^5,x^3) = \widetilde{m}(x^5,x^4).
\end{aligned}$$

With this we get

$$\frac{\mu^*(x^5)}{\mu^*(x^2)} = \frac{9\widetilde{m}(x^2,x^5)\widetilde{m}(x^1,x^2) + 3\widetilde{m}(x^1,x^5)\widetilde{m}(x^2,x^5) + 3\widetilde{m}(x^1,x^5)\widetilde{m}(x^2,x^1)}{9\widetilde{m}(x^5,x^2)\widetilde{m}(x^1,x^2) + 3\widetilde{m}(x^1,x^5)\widetilde{m}(x^5,x^2) + 3\widetilde{m}(x^5,x^1)\widetilde{m}(x^1,x^2)}.$$

Hence, the ratio of $\mu^*(x^5)$ and $\mu^*(x^2)$ is determined by the relation $\widetilde{m}(x^2,x^5)/\widetilde{m}(x^5,x^2)$, $\widetilde{m}(x^1,x^5)/\widetilde{m}(x^1,x^2)$, and $\widetilde{m}(x^2,x^1)/\widetilde{m}(x^5,x^1)$. As for $\widetilde{m}(x^2,x^5)/\widetilde{m}(x^5,x^2)$ recall from the second table (see above) that a least resistance edge from x^2 to x^5 needs the grand coalition to form and agent 3 making a mistake, while the least resistance edge from x^5 to x^2 needs the coalition $\{2,3\}$ to form and agent 2 making a mistake. Suppose coalitions of size two and size three form with the same probability, than $\widetilde{m}(x^2,x^5) = \widetilde{m}(x^5,x^2)$. As for $\widetilde{m}(x^1,x^5)/\widetilde{m}(x^1,x^2)$ recall from the first table (see above) that x^2 can be recontracted upon at x^1 by the grand coalition *and* $\{2,3\}$, while the grand coalition is needed to agree upon x^5. If all improving allocations are chosen from a coalition with the same probability, we therefore get $\widetilde{m}(x^1,x^2) > \widetilde{m}(x^1,x^5)$. But these conditions also ensure that $\widetilde{m}(x^2,x^1) < \widetilde{m}(x^5,x^1)$ which implies that $\mu^*(x^2) > \mu^*(x^5)$. Finally the symmetry of the preference relation with respect to a permutation of allocations x^2, x^3, and x^4 implies that $\mu^*(x^2) = \mu^*(x^3) = \mu^*(x^4)$ whenever coalition formation does not depend on allocations and the identity of the agents. ◇

Example 3: *Multiple Core Allocations and Unique Walrasian Allocation*
Agents' preferences equal

$$(1,2) \succ_1 (3,3) \succ_1 (2,3) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(1,3) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(3,1) \succ_3 (2,1) \succ_3 (3,3) \succ_3 \textit{anything},$$

and $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(2,3),(1,2),(3,1)\},$$

$$x^3 = \{(1,2),(3,3),(2,1)\}, \quad x^4 = \{(3,3),(1,2),(2,1)\}.$$

Clearly, x^2 $\{1,2,3\}$-blocks x^1. Since x^2, x^3, and x^4 cannot be blocked by any coalition, $Core(R) = \{x^2, x^3, x^4\}$. Next, we check if any of the core allocations is Walrasian.

Allocation x^2 is not Walrasian. If it was, then the following (in)equalities would hold:

$$(1)\ p_1(1)+p_2(1) = p_1(2)+p_2(3), \quad (2)\ p_1(2)+p_2(2) = p_1(1)+p_2(2),$$

$$(3)\ p_1(3)+p_2(3) = p_1(3)+p_2(1), \quad (4)\ p_1(3)+p_2(3) > p_1(1)+p_2(1),$$

$$(5)\ p_1(1)+p_2(2) > p_1(1)+p_2(1), \quad (6)\ p_1(1)+p_2(3) > p_1(2)+p_2(2).$$

By (2) and (6), we obtain $p_1(1)+p_2(3) > p_1(1)+p_2(2)$. Thus, $p_2(3) > p_2(2)$. By (5), $p_2(2) > p_2(1)$ and by (3), $p_2(1) = p_2(3)$. Hence, $p_2(3) > p_2(2) > p_2(1) = p_2(3)$; a contradiction.

Allocation x^3 is Walrasian. It is easy to check that the following (in)equalities hold, *e.g.*, for price system $p \equiv (p_1, p_2)$ such that $p_1 = (2, \frac{1}{2}, 1)$ and $p_2 = (1, 1, \frac{1}{2})$:

$$(1)\ p_1(1)+p_2(1) = p_1(1)+p_2(2), \quad (2)\ p_1(2)+p_2(2) = p_1(3)+p_2(3),$$

$$(3)\ p_1(3)+p_2(3) = p_1(2)+p_2(1), \quad (4)\ p_1(1)+p_2(2) > p_1(2)+p_2(2),$$

$$(5)\ p_1(1)+p_2(3) > p_1(2)+p_2(2), \quad (6)\ p_1(3)+p_2(1) > p_1(3)+p_2(3).$$

Allocation x^4 is not Walrasian. If it was, then the following (in)equalities would hold:

$$(1)\ p_1(1)+p_2(1) = p_1(3)+p_2(3), \quad (2)\ p_1(2)+p_2(2) = p_1(1)+p_2(2),$$

(3) $p_1(3) + p_2(3) = p_1(2) + p_2(1)$, (4) $p_1(1) + p_2(2) > p_1(1) + p_2(1)$,

(5) $p_1(1) + p_2(3) > p_1(2) + p_2(2)$, (6) $p_1(3) + p_2(1) > p_1(3) + p_2(3)$.

By (2) and (5), we obtain $p_1(1)+p_2(3) > p_1(1)+p_2(2)$. Thus, $p_2(3) > p_2(2)$. By (4), $p_2(2) > p_2(1)$ and by (6), $p_2(1) > p_2(3)$. Hence, $p_2(3) > p_2(2) > p_2(1) > p_2(3)$; a contradiction. Hence, $W(R) = \{x^3\}$. Furthermore, $RC(R) = SRC(R) = Core(R) = \{\{x^2\}, \{x^3\}, \{x^4\}\}$.

To compute the limit invariant distribution, we now indicate the set of least resistance trees for each of the recurrent classes. As it follows from Proposition 2(i) that $\gamma(x^k) = 2$ for $k = 2, 3, 4$ it suffices to consider those edges that need at most one mistake (otherwise no spanning tree between three recurrent classes can be formed). In in the following table we indicate the respective (set of) coalition(s) that facilitates a transition from an initial state (row) to a final state (column) with zero mistakes (we omit the diagonal elements as they are irrelevant for cycle-free graphs).

	x^1	x^2	x^3	x^4
x^1	x	N	N	N
x^2	$\emptyset$	x	$\emptyset$	$\emptyset$
x^3	$\emptyset$	$\emptyset$	x	$\emptyset$
x^4	$\emptyset$	$\emptyset$	$\emptyset$	x

The next table lists the respective coalitions that facilitate a transition if exactly one member makes a mistake.[21]

	x^1	x^2	x^3	x^4
x^1	x	$\emptyset$	$\emptyset$	$\emptyset$
x^2	C_1	x	$\emptyset$	N
x^3	C_1	N	x	N
x^4	C_1	N	N	x

The crucial symmetry breaking in the transition matrix is due to the fact that it takes two mistakes for a direct edge from x^2 to x^3 while all other direct edges between core allocations need only one mistake (and x^1 can be accessed from *any* core allocation by one mistake of the respective singleton coalition). Hence, the process is less likely to exit x^2 than to exit x^3 or x^4 and also less likely to enter x^3 than to enter x^2 or x^4. Hence, one might suspect that $\mu^*(x^2) > \mu^*(x^4) > \mu^*(x^3)$.

To prove this claim, we list the complete set of least resistance trees for each core-allocation. There are the following 16 x^2-trees.[22]

$\{[x^3, x^2], [x^4, x^3], [x^1, x^2]\}$; $\{[x^3, x^2], [x^4, x^3], [x^1, x^3]\}$; $\{[x^3, x^2], [x^4, x^3], [x^1, x^4]\}$;

$\{[x^4, x^2], [x^3, x^4], [x^1, x^2]\}$; $\{[x^4, x^2], [x^3, x^4], [x^1, x^3]\}$; $\{[x^4, x^2], [x^3, x^4], [x^1, x^4]\}$;

$\{[x^3, x^2], [x^4, x^2], [x^1, x^2]\}$; $\{[x^3, x^2], [x^4, x^2], [x^1, x^3]\}$; $\{[x^3, x^2], [x^4, x^2], [x^1, x^4]\}$;

$\{[x^1, x^2], [x^4, x^3], [x^3, x^1]\}$;

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[21] The agent who makes a mistake is uniquely determined in this example.

[22] Trees listed in one row only differ through a different edge from the initial endowment.

$\{[x^1,x^3],[x^3,x^2],[x^4,x^1]\}$;

$\{[x^1,x^2],[x^3,x^4],[x^4,x^1]\}$;

$\{[x^1,x^4],[x^4,x^2],[x^3,x^1]\}$;

$\{[x^3,x^2],[x^1,x^2],[x^4,x^1]\}$;

$\{[x^4,x^2],[x^1,x^2],[x^3,x^1]\}$;

$\{[x^1,x^2],[x^4,x^1],[x^3,x^1]\}$;

We continue with the following 8 x^3-trees.

$\{[x^4,x^3],[x^2,x^4],[x^1,x^2]\}$; $\{[x^4,x^3],[x^2,x^4],[x^1,x^3]\}$; $\{[x^4,x^3],[x^2,x^4],[x^1,x^4]\}$;

$\{[x^4,x^2],[x^2,x^1],[x^1,x^3]\}$

$\{[x^2,x^4],[x^4,x^1],[x^1,x^3]\}$;

$\{[x^2,x^1],[x^1,x^4],[x^4,x^3],\}$;

$\{[x^4,x^3],[x^2,x^1],[x^1,x^3],\}$;

$\{[x^2,x^1],[x^4,x^1],[x^1,x^3]\}$;

Finally, there are 12 x^4-trees.

$\{[x^3,x^2],[x^2,x^4],[x^1,x^4]\}$; $\{[x^3,x^2],[x^2,x^4],[x^1,x^2]\}$; $\{[x^3,x^2],[x^2,x^4],[x^1,x^3]\}$;

$\{[x^2,x^4],[x^3,x^4],[x^1,x^2]\}$; $\{[x^2,x^4],[x^3,x^4],[x^1,x^3]\}$; $\{[x^2,x^4],[x^3,x^4],[x^1,x^4]\}$;

$\{[x^2,x^1],[x^1,x^3],[x^3,x^4]\}$;

$\{[x^1,x^4],[x^3,x^2],[x^2,x^1]\}$;

$\{[x^1,x^2],[x^2,x^4],[x^3,x^1]\}$;

$\{[x^3,x^4],[x^1,x^4],[x^2,x^1]\}$;

$\{[x^2,x^4],[x^1,x^4],[x^3,x^1]\}$;

$\{[x^1,x^4],[x^2,x^1],[x^3,x^1]\}$;

To simplify the computation of $\mu^*(x)$ recall that the probability of a mistake does not depend on the allocation or the identity of the agent. Suppose further that the probability for a certain coalition to become active does not depend on the allocation. Then, $\widetilde{m}(x^2,x^1) = \widetilde{m}(x^3,x^1) = \widetilde{m}(x^4,x^1)$. Moreover, assume that the grand coalition agrees upon recontracting on each core-allocation with the same probability if the process is at x^1 (*i.e.,* $\widetilde{m}(x^1,x^2) = \widetilde{m}(x^1,x^3) = \widetilde{m}(x^1,x^4)$). Finally, suppose that all transitions between core allocations that only need one mistake have the same probability(*i.e.,* $\widetilde{m}(x^2,x^4) = \widetilde{m}(x^3,x^2) = \widetilde{m}(x^3,x^4) = \widetilde{m}(x^2,x^4) = \widetilde{m}(x^4,x^2) = \widetilde{m}(x^4,x^3)$). This is, for instance, the case if the grand coalition agrees upon each improvement with equal probability. The formula in Proposition 3 then leads to the following expressions for the respective components of the limit invariant distribution.

$$\mu^*(x^2) = 9\widetilde{m}(x^1,x^2)(\widetilde{m}(x^2,x^4))^2 + 6\widetilde{m}(x^2,x^1)\widetilde{m}(x^1,x^2)\widetilde{m}(x^2,x^4) + \widetilde{m}(x^1,x^2)(\widetilde{m}(x^2,x^1))^2$$

$$\mu^*(x^3) = 3\widetilde{m}(x^1,x^2)(\widetilde{m}(x^2,x^4))^2 + 4\widetilde{m}(x^2,x^1)\widetilde{m}(x^1,x^2)\widetilde{m}(x^2,x^4) + \widetilde{m}(x^1,x^2)(\widetilde{m}(x^2,x^1))^2$$

$$\mu^*(x^4) = 6\widetilde{m}(x^1,x^2)(\widetilde{m}(x^2,x^4))^2 + 5\widetilde{m}(x^2,x^1)\widetilde{m}(x^1,x^2)\widetilde{m}(x^2,x^4) + \widetilde{m}(x^1,x^2)(M(x^2,x^1))^2$$

which indicates that $\mu^*(x^2) > \mu^*(x^4) > \mu^*(x^3)$ confirming our initial intuition motivated by the different accessibility and stability of the various core allocations. Note in particular that the unique Walrasian allocation x^3 is the worst predictor for the long-run behavior of the process. $\diamond$

Example 4: *Multiple Walrasian Allocations*
Agent's preferences equal

$(1,2) \succ_1 (3,3) \succ_1 (2,3) \succ_1 (1,1) \succ_1$ *anything*,

$(1,3) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (2,2) \succ_2$ *anything*

$(2,1) \succ_3 (3,1) \succ_3 (3,3) \succ_3$ *anything*,

and $IR(R) = \{x^1, x^2, x^3, x^4\}$ with,

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(2,3),(1,2),(3,1)\},$$

$$x^3 = \{(3,3),(1,2),(2,1)\}, \quad x^4 = \{(1,2),(3,3),(2,1)\}.$$

Clearly, x^2 $\{1,2,3\}$-blocks x^1 and x^3 $\{1,2,3\}$-blocks x^2. Since x^3 and x^4 cannot be blocked by any coalition, $Core(R) = \{x^3, x^4\}$. Next, we check if any of the core allocations is Walrasian.

Allocation x^3 is Walrasian. It is easy to check that the following (in)equalities hold, *e.g.*, for price system $p \equiv (p_1, p_2)$ such that $p_1 = (2,2,0)$ and $p_2 = (1,2,3)$:

(1) $p_1(1) + p_2(1) = p_1(3) + p_2(3)$, (2) $p_1(2) + p_2(2) = p_1(1) + p_2(2)$,

(3) $p_1(3) + p_2(3) = p_1(2) + p_2(1)$, (4) $p_1(1) + p_2(2) > p_1(1) + p_2(1)$,

(5) $p_1(1) + p_2(3) > p_1(2) + p_2(2)$.

Allocation x^4 is Walrasian. It is easy to check that the following (in)equalities hold, *e.g.*, for price system $p \equiv (p_1, p_2)$ such that $p_1 = (2,1,2)$ and $p_2 = (1,1,1)$:

(1) $p_1(1) + p_2(1) = p_1(3) + p_2(3)$, (2) $p_1(2) + p_2(2) = p_1(1) + p_2(2)$,

(3) $p_1(3) + p_2(3) = p_1(2) + p_2(1)$, (4) $p_1(1) + p_2(2) > p_1(2) + p_2(2)$,

(5) $p_1(1) + p_2(3) > p_1(2) + p_2(2)$.

Hence, $W(R) = \{x^3, x^4\}$.

By Proposition 1 (iii) only individually rational allocations are candidates to be elements of a recurrent class. By Proposition 1 (ii), $\{x^3\}$ and $\{x^4\}$ are recurrent classes. Since x^3 $\{1,2,3\}$-blocks x^2 and x^4 $\{1,2,3\}$-blocks x^1, $\{x^3\}$ and $\{x^4\}$ are the only recurrent classes. Therefore, $RC(R) = SRC(R) = Core(R) = \big\{\{x^3\}, \{x^4\}\big\}$. ◇

Example 5: *No Walrasian Allocations and a Multi-Valued Core*
Agents' preferences equal

$$(1,2) \succ_1 (3,3) \succ_1 (2,3) \succ_1 (1,1) \succ_1 \textit{anything},$$

$$(3,2) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (2,2) \succ_2 \textit{anything},$$

$$(1,3) \succ_3 (3,1) \succ_3 (2,3) \succ_3 (2,1) \succ_3 (3,3) \succ_3 \textit{anything},$$

and $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with

$$x^1 = \{(1,1),(2,2),(3,3)\}, \quad x^2 = \{(1,1),(3,2),(2,3)\},$$

$$x^3 = \{(2,3),(1,2),(3,1)\}, \quad x^4 = \{(3,3),(1,2),(2,1)\},$$

$$x^5 = \{(1,2),(3,3),(2,1)\}.$$

Clearly, x^2 $\{2,3\}$-blocks x^1, x^2 $\{2,3\}$-blocks x^4, and x^2 $\{2,3\}$-blocks x^5. Since x^2 and x^3 cannot be blocked by any coalition, $Core(R) = \{x^2, x^3\}$. Next, we check if any of the core allocations is Walrasian.

Allocation x^2 is not Walrasian. If it was, then the following (in)equalities would hold:

(1) $p_1(2) + p_2(2) = p_1(3) + p_2(2)$, (2) $p_1(3) + p_2(3) = p_1(2) + p_2(3)$,

(3) $p_1(2) + p_2(3) > p_1(1) + p_2(1)$, (4) $p_1(3) + p_2(3) > p_1(1) + p_2(1)$,

(5) $p_1(1) + p_2(2) > p_1(1) + p_2(1)$, (6) $p_1(3) + p_2(1) > p_1(3) + p_2(3)$,

(7) $p_1(1) + p_2(3) > p_1(3) + p_2(3)$.

By (5), $p_2(2) > p_2(1)$ and by (6), $p_2(1) > p_2(3)$. Hence, (8) $p_2(2) > p_2(3)$.
Next, by (1), $p_1(2) = p_1(3)$ and therefore, $p_1(3) + p_2(3) = p_1(2) + p_2$. By (7), $p_1(1) > p_1(3)$ and therefore, $p_1(1) + p_2(1) > p_1(3) + p_2(1)$. Using the previous (in)equalities, (3) implies $p_1(3) + p_2(3) > p_1(3) + p_2(1)$. Hence, $p_2(3) > p_2(1)$; a contradiction to (8).

Allocation x^3 is not Walrasian. If it was, then the following (in)equalities would hold:

(1) $p_1(1) + p_2(1) = p_1(2) + p_2(3)$, (2) $p_1(2) + p_2(2) = p_1(1) + p_2(2)$,

(3) $p_1(3) + p_2(3) = p_1(3) + p_2(1)$, (4) $p_1(3) + p_2(3) > p_1(1) + p_2(1)$,

(5) $p_1(1) + p_2(2) > p_1(1) + p_2(1)$, (6) $p_1(3) + p_2(2) > p_1(2) + p_2(2)$,

(7) $p_1(1) + p_2(3) > p_1(3) + p_2(3)$.

By (7), $p_1(1) > p_1(3)$ and by (6), $p_1(3) > p_1(2)$. By (2), $p_1(2) = p_1(1)$. Hence, $p_1(1) > p_1(3) > p_1(2) = p_1(1)$; a contradiction.
Hence, $W(R) = \emptyset$.

By Proposition 1 (iii) only individually rational allocations are candidates to be elements of a recurrent class. According to Proposition 1(ii) $\{x^2\}$ and $\{x^3\}$ are (singleton) recurrent classes. Since x^2 $\{1,2,3\}$-block x^1 and x^2 $\{2,3\}$-blocks x^4 and x^5, $\{x^2\}$ and $\{x^3\}$ are the only recurrent classes. Therefore, $RC(R) = SRC(R) = Core(R) = \{x^2, x^3\}$. ◇

B A Markov-Process Dictionary

- A *Markov process* (or *Markov chain*) is a collection (X, M) of a discrete (state) space X and a mapping $M : X \times X \to [0, 1]$ where $M(x, x')$ depicts the probability that the state will be in $x' \in X$ in period $t + 1$ whenever it was in $x \in X$ in period t. Clearly, $\sum_{x' \in X} M(x, x') = 1$. In this contribution we restrict ourselves to finite, homogeneous Markov processes where X is a finite set (the set of allocations) and transition probabilities induced by the recontracting process (and captured in M) do not depend on time. A *path* of such a Markov process is a mapping from a countable time set T (usually $\mathbb{Z}^+$) to the state space that depicts the evolution of the process $x(t) : T \to X$.

- A *recurrent class* (or *absorbing set* or *limit set*) $A \subseteq X$ is a minimal set of allocations that once entered throughout the process is never abandoned, *i.e.*, for all $x \in A$ and $x' \notin A$, $M(x, x') = 0$. We denote the set of recurrent classes by RC.

- A *singleton recurrent class* or an *absorbing state* is a state $a \in X$ with $M(a, a) = 1$ (and as a consequence, for all $x \in X \setminus \{a\}$, $M(a, x) = 0$). States that do not belong to any recurrent class are called *transient*. Note that any finite Markov process has at least one .

- A recurrent class is *aperiodic* whenever it does not contain any deterministic and non-trivial cycle, *i.e.*, there is no sequence of at least two states $\{x_1, x_2, ..., x_n\}$ such that for all $i = 1, .., n - 1$, $M(x_i, x_{i+1}) = M(x_n, x_1) = 1$. Note that a sufficient condition for the aperiodicity of a recurrent class A is that there is an $x \in A$ such that $M(x, x) > 0$, *i.e.*, that the Markov process exhibits sufficient inertia. Any singleton recurrent class is obviously aperiodic. A dynamic recontracting process defined by Assumptions 1, 2, and 3 is aperiodic.

- Every Markov process induces (a set of) *invariant distributions* $\mu : X \to [0, 1]$ with $\sum_{x \in X} \mu(x) = 1$ and $\mu \cdot M = \mu$. Every recurrent class $A \subseteq X$ corresponds to exactly one invariant distribution with support A, *i.e.*, $\sum_{x \in A} \mu(x) = 1$. The set of all invariant distributions of a Markov process is the convex hull of the invariant distributions of all its recurrent classes. The support of an invariant distribution μ is therefore a (non-empty) set of recurrent classes.

- By the *ergodic theorem* an invariant distribution which is induced by a recurrent class $A \subseteq X$ describes the time-average behavior of the system once it reached A. That is, $\mu(x)$ depicts the fraction of time that the system spends in state $x \in A$ if the initial probability distribution over states has support A.

- A Markov process is *ergodic* if it has a unique recurrent class. Note that the invariant distribution of an ergodic Markov process is unique and thus depicts the time-average behavior independent of the initial probability distribution over states.

- A Markov process is called *irreducible* if it is ergodic and the unique recurrent class coincides with the state space X.

- By the *fundamental theorem of Markov processes* an invariant distribution which is induced by an aperiodic recurrent class $A \subseteq X$ describes the probability that the system will be in state x if it reached a state in A and propagated forever, *i.e.*, $\mu(x) = \lim_{T \to \infty} \nu \cdot$

P^T for every $x \in A$ and all probability distributions over states $\nu : X \to [0,1]$ whose support is contained in A.

- A *perturbed Markov process* (X, M^ϵ) is a Markov process such that all transition probabilities $M^\epsilon(x, x')$ are continuous in ϵ, and for all $x, x' \in X$, $\lim_{\epsilon \to 0} M^\epsilon(x, x') = M(x, x')$. More specifically, if $M^\epsilon(x, x') > 0$ for $\epsilon > 0$ this implies that there is an $r \geq 0$ such that $\infty > \lim_{\epsilon \to 0} \epsilon^{-r} M^\epsilon(x, x') > 0$. Hence, we restrict ourselves to *regular* perturbations and M^ϵ is polynomial in ϵ (*i.e.,* we can write $M^\epsilon(x, x') = \sum_{i=0..\hat{i}} m_i(x, x')\epsilon^i$ where $\hat{i}$ is a finite number and $m(x, x') > 0$ for at least some i).

- The *limit invariant distribution* μ^* of a Markov process (X, M) is the invariant distribution μ^ϵ of a perturbed process (X, M^ϵ) in the limit of $\epsilon \to 0$. Note that any perturbed Markov process is irreducible, hence its invariant distribution is unique. Moreover, $\lim_{\epsilon \to 0} \mu^\epsilon \equiv \mu^*$ exists and is an invariant distribution of (X, M) (*e.g.*, Young, 1993, Theorem 4 (i)). Hence, the support of the limit invariant distribution (denoted by *SRC*) is a set of recurrent classes of the unperturbed process $SRC \equiv \{A \in RC | \mu^*(x) > 0 \forall x \in A\} \subseteq RC$.

- A state in the support of μ^* is called a *stochastically stable state.*

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