Externalities, Potential, Value and Consistency

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Abstract

We provide new characterization results for the value of games in partition function form. In particular, we use the potential of a game to define the value. We also provide a characterization of the class of values which satisfies one form of reduced game consistency.

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1 Introduction

In a variety of economic and social contexts, the activities of one group of agents affect payoffs of other groups. Consider, for instance, the issue of political alliances between different groups of countries. The benefit to each group will typically depend on the strength of the alliance between opposing groups of countries. Similarly, the benefits to one group of agents from activities aimed at controlling pollution depend upon whether other agents are also engaged in similar pollution abatement exercises.

In a framework where such *externalities* across coalitions or groups are absent, Shapley (1953) provided what has become the focal method for distributing the surplus generated by cooperation amongst groups of agents. Shapley obtained a remarkable uniqueness result by showing that there is only *one* solution - the *Shapley value* - which satisfies some seemingly mild axioms. The Shapley value essentially gives each player the average of his marginal contributions to different coalitions

There have been other axiomatic characterizations of the Shapley value. For instance, Young [18] uses a monotonicity principle which states that if a game changes so that some player's contribution to all coalitions increases or stays the same then the player's reward should not decrease. Young shows that the Shapley value is the only *efficient* and *symmetric* solution that is monotonic in this sense.

Hart and Mas-Colell [5] provide two derivations of the Shapley value. First, they use the analytical tool of a *potential function* to formalise the notion of rewarding players according to marginal productivity. The potential function has the property that the sum of the players' marginal products (according to the potential) adds up to the worth of the grand coalition. Moreover, the Shapley value happens to coincide with the vector of marginal products. Thus, this provides another very interesting interpretation of the value. Hart and Mas-Colell also define an internal consistency property of solution concepts and show that the Shapley value is the unique solution satisfying this consistency property and the so-called *standard property on two-person games.*[1]

Given the widespread presence of externalities, it is important to study the distributional issue in environments with externalities. Games in *partition function form*,[2] in which the "worth" of any coalition depends on how players outside the coalition are organised, provides an appropriate framework within which one can describe solution concepts for games with externalities. Not surprisingly, this has received some recent attention. For instance, Macho-Stadler et al. [6] provide characterizations of extensions of the Shapley value to partition function games, using axioms which are designed to capture the intuitive content of Shapley's original axioms.[3] In contrast, de Clippel and Serrano [4] follow the approach of Young[17], and also provide alternative characterizations of value concepts for partition function games.

In this paper, we follow the methodology of Hart and Mas-Colell[5]. The poten-

[1] This property states that on two-person games, the gains from cooperation be split equally between the two players.

[2] This is due to Thrall and Lucas [17].

[3] Bolger[2] and Myerson[8] are earlier contributions along the same lines.

tial approach requires that “subgames” be well-defined for each player set. This is a trivial issue for characteristic function games since a subgame is simply the “projection” of the original game to the appropriate player set. However, there is no such unambiguous answer for games in partition function form since the worth of each coalition depends on how the complementary coalition is partitioned.

We adopt the following procedure. First, we define *restriction operators*, which are functions that map each partition function game on any set of players N to its subgames with player sets $N \setminus \{i\}$. We impose minimal requirements or axioms which these restriction operators should satisfy. Then Hart and Mas-Colell's approach allows us to identify a large class of extensions of the Shapley value from games without externalities to games with externalities.

Our basic requirement is that the order in which players are removed from a game should be irrelevant for the restricted game. In other words, first removing player 1 and then player 2 or first removing player 2 and then player 1 should result in the same game restricted to the player set without 1 and 2. We call this requirement *path-independence*. Our first main result shows that any path-independent restriction operator defines uniquely a potential for games in partition function form. This unique potential coincides with that of a particular characteristic function game *without externalities*. A natural step is to define the value of the partition function game to be the Shapley value of this game in characteristic function form. For any path-independent restriction operator r, we call this value the r-Shapley value.

Following Hart and Mas-Colell [5], we also define an analogous internal consistency property. We then show that a large number of solutions, including the ones derived by us through the potential approach, satisfy this definition of internal consistency and the standard property on two-person games.

Path independence by itself does not have much bite in the sense that large numbers of very different restriction operators satisfy this property. We impose additional axioms to select from this large class. Each operator in our class gives rise to a different value through the potential. We show that all our values satisfy the basic properties of the Shapley value, suitably extended to the more general framework of games with externalities.

The plan of this paper is the following. Section 2 describes the general framework and some notation which is used throughout the paper. Section 3 introduces restriction operators as well as our characterization results on restriction operators. Section 4 demonstrates that the values derived through the potential approach satisfy appropriate modifications of the original Shapley axioms. Section 5 contains our results on consistency. The Appendix contains the proofs of all our results.

2 Framework and Notation

Let $\mathbb{N}$ denote the universal set of players. The universal set $\mathbb{N}$ may be finite or infinite. Any set of players is a finite subset N of $\mathbb{N}$. A coalition S is a non-empty subset of N. Let 2^N denote the set of all non-empty subsets of N. A partition of N is a set $\pi = \{T_1, \ldots, T_k\}$ such that (i) for all $i, j \in \{1, \ldots, k\}$, $T_i \cap T_j = \emptyset$, and (ii)

$\cup_{i=1}^{k} T_i = N$. Let Π_N denote the set of all partitions of N, and Π_S denote the set of all partitions of any coalition $S \subset N$. For any coalition S and any partition π, let $S \cap \pi = \{S \cap T : T \in \pi\}$. For any coalition S, S^c denotes the set $N \setminus S$. For any $S \subseteq N$, $\pi^t(S) = \{\{i\} | i \in S\}$. That is, $\pi^t(S)$ is the "trivial" partition of S consisting of the singleton members of S.

Given player set N, let (N, v) denote a game in *partition function form*. That is, v specifies a real number for every coalition S and for every partition of S^c. We represent this as $v(S; \pi(S^c))$, and call this the *worth* of coalition S when S^c is partitioned according to $\pi(S^c)$. We will call $(S; \pi(S^c))$ an *embedded coalition*. For any player set N, we will simply write $v(N)$ instead of $v(N; \emptyset)$.

The game (N, v) is *with externalities* if the worth of at least one coalition depends on the partition of the other players, i.e. $v(S; \pi(S^c)) \neq v(S; \pi'(S^c))$ for at least one coalition S and some $\pi(S^c), \pi'(S^c) \in \Pi_{S^c}$.

A game (N, v) is without externalities if the worth of any coalition S is independent of how the complementary coalition S^c is partitioned. That is, a game without externalities is the "traditional" TU game in characteristic form with $v : 2^N \to \mathbb{R}$. We will typically use w, w' etc., to denote games without externalities and v, v' to denote games with externalities.

Let $\mathbb{V}$ denote the class of partition function games while $\mathbb{W}$ denotes the set of games in characteristic form.

A *solution concept* or *value* is a mapping φ which associates with every game (N, v) in $\mathbb{V}$ a vector in $\mathbb{R}^{|N|}$ satisfying (i) $\sum_{i \in N} \varphi_i(N, v) = v(N)$ and (ii) for every null game the value assigns zero to every player, i.e. for any (N, v) such that $v(S; \pi(S^c)) = 0$ for any coalition $S \subseteq N$ and any partition $\pi(S^c)$ of S^c, $\varphi_i(N, v) = 0$ for all $i \in N$.[4] A value determines the payoffs of the individual players in any game.

3 The Potential Approach

The traditional approach in economics of paying individuals according to their marginal productivity has no straightforward analogue in cooperative game theory because the sum of the players' marginal contributions to the grand coalition is typically not a feasible payoff vector. Hart and Mas-Colell (1989) develop the *potential function* as a new analytical tool which helps in formalising the notion of rewarding players according to their marginal contributions. In particular, they define the potential as a real-valued function P on the set of all TU games (without externalities) such that the marginal contributions of all players according to P add up to the worth of the grand coalition, and moreover the resulting payoff vector coincides with the Shapley value of the TU game.

More formally, they define a function $P : \mathbb{W} \to \mathbb{R}$ which assigns a real number $P(N, w)$ to every TU game (N, w), and define the marginal contribution of player i

[4]Property (ii) is extremely weak and is only needed in the proof of Corollary 4. All other results hold without this requirement.

to be

$$D^i P(N,w) = P(N,w) - P(N \setminus \{i\}, w)$$

Note that $(N \setminus \{i\}, w)$ is the projection of (N,w) on $N \backslash \{i\}$. The function P is a potential if $P(\emptyset, 0) = 0$ and

$$\sum_{i \in N} D^i P(N,w) = w(N)$$

for every TU game (N,w). Hart and Mas-Colell (1989) show that there is a unique potential function and that for every game (N,w), the payoff vector $(D^i P(N,w))_{i \in N}$ coincides with the Shapley value of the game, i.e.,

$$D^i P(N,w) = Sh_i(N,w) = \sum_{S \subseteq N. s.t.\ i \in S} \frac{(s-1)!(n-s)!}{n!} [w(S) - w(S \setminus \{i\})]$$

where s and n denote the cardinalities of the sets S and N respectively.

Our principal goal in this paper is to use the potential approach to derive a value for games in partition function form. Notice that this approach requires us to specify *subgames* $(N \setminus \{i\}, v)$ for each game (N, v). This is perfectly straightforward for characteristic function games since $(N \setminus \{i\}, w)$ is simply the restriction of (N,w) to $N \setminus \{i\}$. Unfortunately, there is no unambiguous way of deriving subgames for games in partition function form.

Consider, for instance, $N = \{1,2,3\}$, and suppose $v(N) = a$, $v(\{i,j\};\{k\}) = b$, and

$$v(\{i\};\{j,k\}) = c, v(\{i\}; \pi^t(\{j,k\})) = d$$

Then, what is $(\{2,3\}, v^{-1})$ where v^{-1} denotes the corresponding partition function for the player set $\{2,3\}$. Since there is only one possible partition of $\{1,2,3\}$ in which $\{2,3\}$ is a member, it is natural to define $v^{-1}(\{2,3\}) \equiv v(\{2,3\};\{1\})$.[5] The problem appears when one tries to specify $v^{-1}(\{2\};\{3\})$ from knowledge of v on the player set $\{1,2,3\}$. Should we take a simple or weighted average of $v(\{2\};\{1,3\})$ and $v(\{2\};\pi^t(\{1,3\}))$? Or take the maximum (or minimum) value amongst these?

3.1 Restriction Operators

We follow the following procedure. Define a *restriction operator* to be a mapping r from $\mathbb{V}$ to $\mathbb{V}$ which specifies for each game $(N,v) \in \mathbb{V}$ a "subgame" game $(N \backslash \{i\}, v^{-i,r})$ for each $i \in N$. Notice that once a particular restriction operator r has been specified, then it is straightforward to define an r-potential function. We will subsequently use r-potential functions to derive corresponding values following the Hart and Mas-Colell procedure.

In order to define a restriction operator, we need some notation.

Let $\pi(S) = (S_1, S_2 \ldots, S_K)$ be a partition of some set S. Then, for any $i \notin S$, $\pi^{+i}(S)$ is the *set* of partitions of $S \cup \{i\}$ where i either joins one of the coalitions S_k of $\pi(S)$, the other coalitions remaining unchanged, or it is the partition

[5] In general, one can specify $v^{-i}(N \setminus \{i\}) = v(N \setminus \{i\}; \{i\})$.

$(S_1, \ldots, S_K, \{i\})$. For instance, if $S = \{1,2,3\}$ and $\pi(S) = \{\{1,2\},\{3\}\}$, then $\pi^{+4}(S) = \{\{\{1,2,4\},\{3\}\};\{\{1,2\},\{3,4\}\};\{\{1,2\},\{3\},\{4\}\}\}$.

Given any (N, v) in $\mathbb{V}$, the general form of a restriction operator is specified below:

$$v^{-i,r}(S; \pi((S \cup \{i\})^c)) = r^N_{i,S,\pi((S\cup\{i\})^c)}\left((v(S;\pi))_{\pi \in \pi^{+i}((S\cup\{i\})^c)}\right)$$

So, the worth of any coalition S corresponding to $(S; \pi((S \cup \{i\})^c))$ in the subgame when player i is absent is some function of the worths $(S; \pi)$ where π is some element of $\pi^{+i}((S\cup\{i\})^c)$. For instance, if $N = \{1,2,3,4\}$, then $v^{-4,r}(\{1\};\{2,3\}\}) = r^N_{4,\{1\},\{2,3\}}(v(\{1\};\{2,3,4\}), v(\{1\};\{2,3\},\{4\}))$. In the subgame the worth of any coalition S for a fixed partition of the other players depends only on the worths of S where player i joins one of the existing members of the fixed partition or remains alone. For any coalition S and any fixed partition in the subgame, our specification of a restriction operator supposes that coalition S and this fixed partition is maintained in the original game.[6]

Of course, this specification hardly imposes any restriction on the $r^N_{i,S,\pi((S\cup\{i\})^c)}$ functions, and hence on the restriction operators. For instance, the specification allows $r^N_{i,S,\pi((S\cup\{i\})^c)}$ to depend on i, the coalition S, the partition of its complement in the subgame, and the original player set. In order to make notation simpler, we will typically drop the superscript N and the subscripts S and $\pi((S\cup\{i\})^c)$ whenever no confusion can result from this and write r_i instead of $r^N_{i,S,\pi((S\cup\{i\})^c)}$. Similarly, we will drop the superscripts N and r and write v^{-i} instead of $v^{-i,r}$. In this paper we use an axiomatic approach to restriction operators. Each axiom is meant to be a "reasonable" property of a restriction operator. As we proceed, we will impose more and more axioms on restriction operators and finally single out a certain class of restriction operators via axioms which are parallel to two axioms used by Shapley in his characterization of the Shapley value.

The most basic axiom is that of *Path Independence.* For any $i, j \in N$, let $v^{-ij} = (v^{-i})^{-j}$.

Definition 1 *A restriction operator r satisfies Path Independence if for all $(N, v) \in \mathbb{V}$, for all $i, j \in N$, $v^{-ij} = v^{-ji}$.*

If Path Independence is not satisfied, then the subgame on the player set $N\setminus\{i,j\}$ is not well-defined. So, Path Independence is almost a necessary condition to use the potential approach since the latter requires well-defined subgames. In particular, then the subgame v^{-S}, where some coalition $S \subset N$ leaves the game v, is well-defined: under Path Independence the players belonging to S are sequentially removed from v in any arbitrary order.

[6]Although our specification of a restriction operator is natural, our first two main results (Theorem 1 and Theorem 2 below) remain even true for more general restriction operators. For instance, we may use instead $v^{-i,r}(S;\pi((S\cup\{i\})^c)) = r^N_{i,S,\pi((S\cup\{i\})^c)}((v(S;\pi))_{\pi\in\Pi_{S^c}})$, i.e., we may not require any interdependence between the partition of the other players in the subgame and in the original game.

Given any well-defined restriction operator r, it is now easy to define the r-potential function of any game (N, v) in $\mathbb{V}$. More formally, we define a function $P^r : \mathbb{V} \to \mathbb{R}$ which assigns a real number $P^r(N, v)$ to every game (N, v). The marginal contribution of player i is

$$D^i P^r(N, v) = P^r(N, v) - P^r(N \setminus \{i\}, v^{-i})$$

The function P^r is an r-potential if $P^r(\emptyset, 0) = 0$ and

$$\sum_{i \in N} D^i P^r(N, v) = v(N)$$

for all games (N, v) in $\mathbb{V}$.

Path independence by itself has an important implication for the subsequent analysis. In particular, it allows us to relate the r-potential of any game with externalities to the potential of a game *without* externalities. Put differently, we can derive a characteristic function game from any game in partition function form, and then define the value of the partition function form game to be the Shapley value of the associated characteristic function game. Of course, this has also been the approach followed by other recent contributions.[7] The novelty of our approach is that we use the potential function to derive the associated characteristic function game, and so the latter will depend on the specific restriction operator r used to define the r-potential.

Given any restriction operator r, and game (N, v), define the characteristic function $w_v^r : 2^N \to \mathbb{R}$ as follows:

$$w_v^r(N) = v(N), \text{ and for all } S \subset N, w_v^r(S) = v^{-S^c,r}(S)$$

Theorem 1 *Let r be a restriction operator satisfying Path Independence. Then for all $(N, v) \in \mathbb{V}$, we have*

(i) $P^r(N, v) = P(N, w_v^r)$.

(ii) $D_i P^r(N, v) = D_i P(N, w_v^r) = Sh_i(N, w_v^r)$.

The proof of this and subsequent theorems is in the Appendix.

3.2 The r-Shapley Value

Let r be any restriction operator satisfying Path Independence. Following Hart and Mas-Colell (1989), Theorem 1 allows us to use the r-potential to define a value as Sh^r, where

$$Sh_i^r(N, v) \equiv D_i P^r(N, v) = P^r(N, v) - P^r(N \setminus \{i\}, v^{-i,r}) \text{ for all } i \in N$$

Alternatively,

$$Sh^r(N, v) = Sh(N, w_v^r).$$

We will call Sh^r the *r-Shapley value.*

[7] See, for instance [4] and [6].

3.3 Examples

The class of restriction operators satisfying Path Independence is very big. One example is where a priori a partition of the universal set is given and any game is reduced by taking the worth of a coalition when the other players are organized according to the fixed partition. In other words, any game is reduced by "projecting it onto the coordinate" of this partition. More formally, let $\mathbb{P}$ be a partition of $\mathbb{N}$. Given any (N, v) in $\mathbb{V}$, let for all $i \in N$,

$$v^{-i,r_{\mathbb{P}}}(S; \pi((S \cup \{i\})^c)) = v(S; S^c \cap \mathbb{P})$$

if $(S \cup \{i\})^c \cap \mathbb{P} = \pi((S \cup \{i\})^c)$; and $v^{-i,r_{\mathbb{P}}}(S; \pi((S \cup \{i\})^c)) = 0$ otherwise. We will call $r_{\mathbb{P}}$ the $\mathbb{P}$-coordinate (restriction) operator. It is easy to check that this restriction operator satisfies Path Independence.

The next restriction operator captures optimistic expectations of a fixed coalition and a fixed partition when a player leaves a game. It supposes that the leaving player joins a member of the fixed partition such that the worth of the fixed coalition is maximized.

The max (restriction) operator is specified below. For all $(N, v) \in \mathbb{V}$ and all $i \in N$, let

$$v^{-i,max}(S; \pi((S \cup \{i\})^c)) = \max_{\pi \in \pi^{+i}((S \cup \{i\})^c)} v(S; \pi).$$

Analogously the min operator is defined via replacing max by min in the above equality. The min operator captures pessimistic expectations of a coalition about the organization of the other players once a player leaves. Both the max operator and the min operator satisfy Path Independence.

Another operator is the following. It simply restricts any game by assigning to any coalition and any partition the worth of this coalition and this partition and the leaving player staying alone. The trivial operator is specified below. For all $(N, v) \in \mathbb{V}$ and all $i \in N$, let

$$v^{-i,triv}(S; \pi((S \cup \{i\})^c)) = v(S; \pi((S \cup \{i\})^c) \cup \{i\}).$$

Clearly, the trivial operator satisfies Path Independence. Furthermore, it is easy to verify that for all (N, v) and all $S \subset N$ we have

$$w_v^{triv}(S) = v(S; \pi^t(S^c)).$$

The *triv*-Shapley value has played an important role in de Clippel and Serrano [4].

4 Consistency

Hart and Mas-Colell (1989) provide an axiomatic derivation of the Shapley value in characteristic function games through the use of an internal consistency condition which has come to be called a "reduced game" consistency property.[8] The intuitive

[8] Different versions of the reduced game property have proved very useful in the characterization of a variety of cooperative solution concepts. See for instance [1], [3], [7], [9], [10], and [14].

content of this concept of consistency is the following. Suppose φ is a solution concept, and that a group of players can be "bought off" by paying them according to φ. These players do not actually leave the game, but can be persuaded to cooperate with any coalition provided they are paid according to φ. This then precipitates a reduced game on the complementary player set S^c, and φ is said to satisfy (reduced game) consistency if it prescribes the same payoffs to players in S^c in both the reduced game as well as the original game for the grand coalition.

There are different definitions of reduced games, each corresponding to a different interpretation of what it means for the coalition S to be paid according to the solution concept φ. Below we define the natural extension of the reduced game formulated by Hart and Mas-Colell to partition function form games. As one might imagine, each restriction operator gives rise to one such reduced game and to a corresponding notion of consistency.

It is worth pointing out here that a "reduced" game is different from a subgame. This difference is more transparent when the original game is without externalities. As we have pointed out earlier, the subgame on player set S is simply the projection of the original game to S and subsets of S. That is, there is no ambiguity about the subgame. However, there are different versions of a reduced game even in this case.

Definition 2 *Fix a restriction operator r satisfying Path Independence. Let φ be a value and $(N, v) \in \mathbb{V}$. For any $S \subset N$, the reduced game $(S, v_S^{\varphi,r})$ is defined as follows. For all $R \subseteq S$ and all $\pi(S \setminus R) \in \Pi_{S\setminus R}$,*

$$v_S^{\varphi,r}(R; \pi(S \setminus R)) = v(R \cup S^c; \pi(S \setminus R)) - \sum_{k \in S^c} \varphi_k(R \cup S^c, v^{-(S\setminus R),r})$$

So, the Hart and Mas-Colell reduced game on the player set S specifies that if players in S^c join forces with some $R \subseteq S$, then they are paid what they would obtain (according to φ) in the subgame restricted to the player set $S^c \cup R$. Of course, this subgame depends on the specific restriction operator.[9]

This reduced game leads to the following definition of consistency.

Definition 3 *A value φ is r-consistent iff for all $(N, v) \in \mathbb{V}$, all $S \subset N$ and all $i \in S$, we have*

$$\varphi_i(N, v) = \varphi_i(S, v_S^{\varphi,r})$$

Hart and Mas-Colell (1989) showed that on the class of games $\mathbb{W}$, the Shapley value is the only solution satisfying and the "standard property" which requires that on two-person games, the solution splits the gains from cooperation equally between the two players. More formally,

Definition 4 *A value φ satisfies the standard property on two-person games if for all $(\{i, j\}, v) \in \mathbb{V}$, $\varphi_i(\{i, j\}, v) = v(\{i\}) + \frac{1}{2}[v(\{i, j\}) - v(\{i\}) - v(\{j\})]$.*[10]

[9]Notice that the Hart and Mas-Colell reduced game for TU games is obtained from Definition 2 by dropping $\pi(S\setminus R)$ from it.

[10]Abusing notation, for two player games we write $v(\{i\})$ instead of $v(\{i\}; \{j\})$.

In this section, we prove an analogous result for partition function games. For this uniqueness result we need an additional axiom on restriction operators.

Definition 5 *A restriction operator r satisfies Translation Invariance if for all player sets N, all $i \in N$, all $S \subseteq N\backslash\{i\}$, and all $\pi((S \cup \{i\})^c)$, and any real numbers $x_1, \ldots, x_k$ and c,*

$$r^N_{i,S,\pi((S\cup\{i\})^c)}(x_1 - c, \ldots, x_k - c) = r^N_{i,S,\pi((S\cup\{i\})^c)}(x_1, \ldots, x_k) - c.$$

Translation Invariance simply says that if the original worths are translated by some constant, then in the restricted game the worth should be also translated by the same constant.

We are now ready for the main theorem of this section.

Theorem 2 *Let r be a restriction operator satisfying Path Independence and Translation Invariance. Then, a value φ satisfies r-consistency and the standard property on two-person games if and only if $\varphi \equiv Sh^r$.*

The following corollaries follow from Theorem 2 because the classes of operators described in the corollaries satisfy both Path Independence and Translation Invariance.

Corollary 1 *A value φ satisfies* max*-consistency and the standard property on two-person games if and only if $\varphi \equiv Sh^{\max}$.*

Corollary 2 *A value φ satisfies triv-consistency and the standard property on two-person games if and only if $\varphi \equiv Sh^{triv}$.*

Although the $\mathbb{P}$-coordinate operator does not satisfy Translation Invariance, we obtain from Theorem 2 and its proof a similar corollary for it.

Corollary 3 *Let $\mathbb{P}$ be a partition of the universal set $\mathbb{N}$. A value φ satisfies $r_{\mathbb{P}}$-consistency and the standard property on two-person games if and only if $\varphi \equiv Sh^{r_{\mathbb{P}}}$.*

Next we establish that the average approach characterized by [6] belongs to the class of restriction operators satisfying both Path Independence and Regularity. In other words, their characterized average approach is captured by the r-potential. The average approach associates with any partition function form game a TU game by taking for any coalition S a weighted average of its worths for all partitions of the other players. Then the value of any game is simply the Shapley value of its associated TU game. One difference is that in [6] the player set remains fixed throughout whereas here the player set may vary.

For simplicity, let $\mathbb{N}$ be finite. A weight system is a vector $\alpha = (\alpha(S; \pi(S^c)))_{S \subseteq \mathbb{N}, \pi(S^c) \in \Pi_{S^c}}$ such that for any coalition S we have $\alpha(S; \pi(S^c)) \geq 0$ for all $\pi(S^c) \in \Pi_{S^c}$ and

$\sum_{\pi(S^c)\in\Pi_{S^c}} \alpha(S;\pi(S^c)) = 1$. Then any game $(\mathbb{N}, v)$ is associated with a TU game $\tilde{v}$ simply by

$$\tilde{v}(S) = \sum_{\pi(S^c)\in\Pi_{S^c}} \alpha(S;\pi(S^c))v(S;\pi(S^c))$$

for any coalition S. The principal result of [6] characterizes the values which can be constructed through the average approach where the weight system α satisfies (3) in their Theorem 1. Repeated use of (3) in their Theorem 1 yields

$$\alpha(S;\pi(S^c)) = \sum_{i\in S} \sum_{\pi(\mathbb{N}\backslash\{i\}):S^c\cap\pi(\mathbb{N}\backslash\{i\})=\pi(S^c)} \alpha(i;\pi(\mathbb{N}\backslash\{i\})). \tag{1}$$

In other words, all weights are determined by the weights on $\alpha(i;\pi(\mathbb{N}\backslash\{i\}))$.

When trying to associate with the average approach a restriction operator, one might be tempted by simply taking the weighted average of the coordinate operators $r_{\mathbb{P}}$ (where $\mathbb{P}$ is a partition of $\mathbb{N}$). This may not work because we may have $\alpha(S;S^c) \neq \alpha(S^c;S)$ for a coalition S. Then it is not clear which weight to put on the operator $r_{\{S,S^c\}}$. In some sense we need to incorporate the two entries of any embedded coalition, namely the coalition and the partition of its complement. Then for identical partitions the weights could differ depending on the first entry of the embedded coalition. Notice that (1) facilitates this since all weights are determined by the weights put on embedded coalitions of the form $(i;\pi(\mathbb{N}\backslash\{i\}))$.

In order to connect their characterized average approach with restriction operators, we need to take the weighted average of somewhat modified coordinate restriction operators. Indeed, it will turn out that (1) is crucial for this operation. Let $\mathbb{P}_i$ denote a partition of $\mathbb{N}$ such that $\{i\} \in \mathbb{P}_i$ and $\Pi^i_{\mathbb{N}}$ denote the set of all such partitions. For any partition $\mathbb{P}_i$ we put weight $\alpha(i;\mathbb{P}_i\backslash\{i\})$ on the following modified $\mathbb{P}_i$-coordinate restriction operator. This operator takes the $\mathbb{P}_i$-coordinate of the original game for the restricted game only if i belongs to the coalition S and the other players' partition in the restricted game is consistent with the partition $\mathbb{P}_i$. For any game (N, v), any $j \in N$ and any coalition $S \subseteq N\backslash\{j\}$, let

$$v^{-j,r_{\mathbb{P}_i}}(S;\pi((S\cup\{j\})^c)) = v(S;S^c\cap\mathbb{P}_i)$$

if $i \in S$ and $\pi((S\cup\{j\})^c) = (S\cup\{j\})^c\cap\mathbb{P}_i$; and $v^{-j,r_{\mathbb{P}_i}}(S;\pi((S\cup\{i\})^c)) = 0$ otherwise. Now $r_{\mathbb{P}_i}$ checks whether i belongs to S and whether $\pi((S\cup\{j\})^c)$ is consistent with $\mathbb{P}_i$. If this is the case, then $\alpha(i;\mathbb{P}_i\backslash\{i\})r_{\mathbb{P}_i}$ puts weight $\alpha(i;\mathbb{P}_i\backslash\{i\})$ on the number $v(S;S^c\cap\mathbb{P}_i)$.[11] It is easy to see that $r_{\mathbb{P}_i}$ satisfies Path Independence.

For any weight system α satisfying (1), in our setting the α-average restriction operator r_α is simply

$$r_\alpha = \sum_{i\in\mathbb{N}} \sum_{\mathbb{P}_i\in\Pi^i_{\mathbb{N}}} \alpha(i;\mathbb{P}_i\backslash\{i\})r_{\mathbb{P}_i}.$$

[11] The restriction operator $r_{\mathbb{P}_i}$ combines the ideas of the i-unanimity TU games and the $\mathbb{P}_i$-coordinate operators.

Notice that r_α satisfies Path Independence because this property is preserved under linear combinations. Now for any game $(\mathbb{N}, v)$ we have for any coalition S,

$$\begin{aligned}
w_v^{r_\alpha}(S) &= \sum_{i\in\mathbb{N}} \sum_{\mathbb{P}_i\in\Pi^i_{\mathbb{N}}} \alpha(i;\mathbb{P}_i\backslash\{i\}) w_v^{r_{\mathbb{P}_i}}(S) \\
&= \sum_{i\in S} \sum_{\mathbb{P}_i\in\Pi^i_{\mathbb{N}}} \alpha(i;\mathbb{P}_i\backslash\{i\}) v(S; S^c\cap\mathbb{P}_i) \\
&= \sum_{\mathbb{P}\in\Pi_{\mathbb{N}}} \sum_{i\in S:\{i\}\in\mathbb{P}} \alpha(i;\mathbb{P}\backslash\{i\}) v(S; S^c\cap\mathbb{P}) \\
&= \sum_{\pi(S^c)\in\Pi_{S^c}} \sum_{i\in S} \sum_{\mathbb{P}_i\in\Pi^i_{\mathbb{N}}:S^c\cap\mathbb{P}_i=\pi(S^c)} \alpha(i;\mathbb{P}_i\backslash\{i\}) v(S;\pi(S^c)) \\
&= \sum_{\pi(S^c)\in\Pi_{S^c}} \alpha(S;\pi(S^c)) v(S;\pi(S^c)) \\
&= \tilde{v}(S),
\end{aligned}$$

where the first equality is the definition of r_α, the second follows from the definition of $r_{\mathbb{P}_i}$, the third interchanges the two summations, the fourth follows from the fact that for any partition of $\mathbb{N}$ only the partition induced on S^c matters, the fifth from α satisfying (1), and the sixth is the definition of the average approach. Therefore, for any game $(\mathbb{N}, v)$ we have

$$Sh^{r_\alpha}(\mathbb{N}, v) = Sh(\mathbb{N}, w_v^{r_\alpha}) = Sh(\mathbb{N}, \tilde{v}) \tag{2}$$

and the r_α-Shapley value associates with the game $(\mathbb{N}, v)$ the same vector as the average approach characterized by [6].

Although in [6] the player set is a fixed finite set $\mathbb{N}$, it is easy to adopt the construction above to infinite universal sets of players.[12] Now we obtain the following corollary from Theorem 2 and its proof and from (2).

Corollary 4 *Let $\mathbb{N}$ be finite and α be a weight system satisfying (1). A value φ satisfies r_α-consistency and the standard property on two-person games if and only if $\varphi \equiv Sh^{r_\alpha}$. In particular, for any game $(\mathbb{N}, v)$ we have $Sh^{r_\alpha}(\mathbb{N}, v) = Sh(\mathbb{N}, \tilde{v})$.*

5 Restriction Operators à la Shapley

We now define some other axioms on restriction operators. All these axioms are the counterparts for restriction operators of the corresponding axioms of values on TU games. First, we define a dummy player. In games without externalities, a dummy player is one whose marginal contribution is zero to all coalitions. By analogy, a dummy player must be one whose marginal contribution is zero to all embedded coalitions. However, this does not uniquely define the concept of a dummy player. One can still define two notions of a dummy player.

[12]Details are available from the authors upon request.

Definition 6 *Let (N, v) be a game in $\mathbb{V}$.*

(i) Player $i \in N$ is a dummy player of type 1, if for all $S \subseteq N$ containing i, and for all partitions $\pi(S^c)$, $v(S;\pi(S^c)) - v(S \setminus \{i\};\pi) = 0$ for all $\pi \in \pi^{+i}(S^c)$.

(ii) Player $i \in N$ is a dummy player of type 2, if for all $S \subseteq N$ containing i, and for all partitions $\pi(S^c)$, $v(S;\pi(S^c)) - v(S \setminus \{i\};\pi(S^c) \cup \{i\}) = 0$.

The difference between the two definitions hinges on what player i is supposed to do *after* she leaves coalition S. Player i is said to be a dummy player of type 1 if no assumption is made about what coalition she joins after leaving S—her marginal contribution to S is zero irrespective of which coalition she joins. A player is a type 2 dummy player if her marginal contribution to any embedded coalition is zero when she remains alone after leaving S. Clearly, a type 1 dummy player is a type 2 dummy player, though the converse is not true.

Definition 7 *A restriction operator r satisfies the Weak Dummy Axiom if for all $(N, v) \in \mathbb{V}$, if player p is a dummy player of type 1 in (N, v), then player p is a dummy player of type 1 in $(N \setminus \{i\}, v^{-i})$ for all $i \neq p$.*

Definition 8 *A restriction operator r satisfies the Strong Dummy Axiom when the following are true for all $(N, v) \in \mathbb{V}$:*

(i) If player p is a dummy player of type 1 in (N, v), then player p is a dummy player of type 1 in $(N \setminus \{i\}, v^{-i})$ for all $i \neq p$.

(ii) If player p is a dummy player of type 2 then player p is a dummy player of type 2 in $(N \setminus \{i\}, v^{-i})$ for all $i \neq p$.

The Weak Dummy Axiom does not impose any condition on type 2 dummy players. Hence, the Strong Dummy Axiom implies the Weak Dummy Axiom. We also remind the reader that these are axioms on the restriction operator and *not* on the value. So, the dummy axioms defined above are related but distinct from the dummy axioms on solution concepts which are, for instance, used in the characterization of the Shapley value.

Definition 9 *A restriction operator r satisfies Scale Invariance if for all $(N, v), (N, v') \in \mathbb{V}$, for all a, b, if $v = a + bv'$,*[13] *then for all $i \in N$, $v^{-i} = a + bv'^{-i}$.*

This axiom incorporates the notion that if two games on the player set are transforms of one another, then the subgames should also be similarly related. Note that Scale Invariance implies Translation Invariance.

[13] Abusing notation, $v = a + bv'$ means $v(S;\pi(S^c)) = a + bv'(S;\pi(S^c))$ for all $S \subseteq N$ and all $\pi(S^c) \in \Pi_{S^c}$.

Definition 10 *A restriction operator r satisfies Non-negativity if for all* $(N, v) \in \mathbb{V}$, *if* $v \geq \mathbf{0}$[14], *then there exists* $i \in N$ *such that for some* $S \subseteq N \setminus \{i\}$ *and* $\pi \in \Pi_{(S \cup \{i\})^c}$ *we have* $v^{-i}(S; \pi) \geq 0$.

This axiom is extremely weak. All it requires is that if the partition function is non-negative, then there must be *some* player i such that in the subgame on player set $N \setminus \{i\}$, *some* embedded coalition has non-negative worth. No restriction is placed on worths of other embedded coalitions in the subgame or on worths of embedded coalitions in other subgames.

We first illustrate through examples why some of the possible restriction operators do not satisfy one or more of the axioms defined above.

Consider, for instance, the simple average (restriction) operator $\bar{r}$, where for all $(N, v) \in \mathbb{V}$, for all $i \in N$,

$$v^{-i,\bar{r}}(S; \pi((S \cup \{i\})^c)) = \frac{1}{t} \sum_{\pi \in \pi^{+i}((S \cup \{i\})^c)} v(S; \pi)$$

where $t = |\pi^{+i}((S \cup \{i\})^c)|$. This operator does not satisfy the Weak Dummy axiom: let $N = \{1, 2, 3, 4\}$ and 4 be a dummy player of type 1 in (N, v). Now,

$$v^{-3,\bar{r}}(\{1\}; \{2\}, \{4\}) = 1/3\,[v(\{1\}; \{2, 3\}, \{4\}) + v(\{1\}; \{2\}, \{3, 4\}) + v(\{1\}; \{2\}, \{4\}, \{3\})]$$

and

$$v^{-3,\bar{r}}(\{1, 4\}; \{2\}) = 1/2\,[v(\{1, 4\}; \{2, 3\}) + v(\{1, 4\}; \{2\}, \{3\})]$$

If 4 is a dummy player of type 1 in $v^{-3,\bar{r}}$, then we must have

$$v^{-3,\bar{r}}(\{1\}; \{2\}, \{4\}) = v^{-3,\bar{r}}(\{1, 4\}; \{2\})$$

Because 4 is a dummy of type 1 in v, $v(\{1, 4\}; \{2, 3\}) = v(\{1\}; \{2, 3\}, \{4\})$ and $v(\{1, 4\}; \{2\}, \{3\}) = v(\{1\}; \{2\}, \{3, 4\}) = v(\{1\}; \{2\}, \{4\}, \{3\})$. But, then the last equality will hold only if $v(\{1, 4\}; \{2\}, \{3\}) = v(\{1, 4\}; \{2, 3\})$. The fact that 4 is a dummy player in v does not imply the last equality.

It is easy to see that $\bar{r}$ satisfies both Path Independence and Scale Invariance. Recall the definition of the max operator: for all $(N, v) \in \mathbb{V}$ and all $i \in N$,

$$v^{-i,max}(S; \pi((S \cup \{i\})^c)) = \max_{\pi \in \pi^{+i}((S \cup \{i\})^c)} v(S; \pi)$$

Clearly the max operator fails to satisfy Scale Invariance for $b < 0$. On the other hand, the max operator satisfies Path Independence and the Weak Dummy Axiom.

These examples show that the axioms on restriction operators defined earlier have some "bite". Indeed, the next two theorems provide characterizations of the class of restriction operators which satisfy the two versions of the dummy axiom along with the other axioms.

[14]We use the convention that $x \geq \mathbf{0}$ means $x_i \geq 0$ for all i with strict inequality for some i.

Theorem 3 *Let r be a restriction operator satisfying the Weak Dummy Axiom, Scale Invariance and Path Independence. Then, there exists θ, such that for all $(N, v) \in \mathbb{V}$ and all $i \in N$,*

$$v^{-i,r}(S; \pi((S \cup \{i\})^c)) = \theta \sum_{\pi \in \pi^{+i}((S \cup \{i\})^c)} v(S; \pi) + (1 - t\theta) v(S; \pi((S \cup \{i\})^c) \cup \{i\})$$

where $t = |\pi^{+i}((S \cup \{i\})^c)|$.[15] *Moreover, if the restriction operator satisfies Non-negativity, then $\frac{1}{t-1} \geq \theta \geq 0$.*

The Weak Dummy Axiom and Scale Invariance are the counterparts for restriction operators of Shapley's Dummy Axiom and Linearity Axiom for values on TU games. Recall that Path Independence is almost a necessary condition for restriction operators to be well-defined. Furthermore, observe that Theorem 3 does not impose any requirement for treating symmetric players symmetrically. This just follows from the axioms used in Theorem 3. We will call any restriction operator satisfying the Weak Dummy Axiom, Scale Invariance and Path Independence a *restriction operator à la Shapley.*

By Theorem 3, restriction operators à la Shapley are a linear combination of the simple average operator and the trivial operator. In particular, the weights are such that the simple average operator is offset by the trivial operator if a player is a type 1 dummy player and the Weak Dummy Axiom is satisfied. In the next section we discuss in detail the properties of these r-Shapley values.

Theorem 3 used the Weak Dummy Axiom. The Strong dummy axiom implies the following.

Theorem 4 *Let r be a restriction operator satisfying the Strong Dummy Axiom, Scale Invariance and Path Independence. Then $r = triv$.*

The corollary below follows from Theorem 2 because Scale Invariance implies Translation Invariance.

Corollary 5 *Let r be a restriction operator satisfying Path Independence, Scale Invariance, Non-negativity and the Weak Dummy Axiom. Then, a value φ satisfies r-consistency and the standard property on two-person games if and only if $\varphi \equiv Sh^r$.*

6 Properties of Shapley Values à la Shapley

The r-Shapley value is "à la Shapley" if r satisfies Shapley's counterparts of the (Weak) Dummy Axiom and Linearity for restriction operators and Path Independence (for its well-definedness). Throughout this section let r be a restriction operator à la Shapley.

We now define the natural extensions of Shapley's classic axioms for games in partition function form.

[15] Note that $\pi((S \cup \{i\})^c) \cup \{i\})$ is an element of $\pi^{+i}((S \cup \{i\})^c)$. So, the total weight on $v(S; \pi(S \cup \{i\})^c)) \cup \{i\})$ is $(1 - (t-1)\theta)$.

Definition 11 *Let φ be a solution on $\mathbb{V}$. Then, φ satisfies*

(i) Linearity: if

(a) For all (N, v), $(N, v') \in \mathbb{V}$, $\varphi(N, v + v') = \varphi(N, v) + \varphi(N, v')$; and

(b) For any scalar $\lambda \in \mathbb{R}$ and any game $(N, v) \in \mathbb{V}$, $\varphi(N, \lambda v) = \lambda\varphi(N, v)$.

(ii) Symmetry: if for any permutation of σ of N, $\varphi(N, \sigma v) = \sigma\varphi(N, v)$.

(iii) Efficiency: if for all $(N, v) \in \mathbb{V}$, $\sum_{i\in N} \varphi_i(N, v) = v(N)$.

(iv) Weak Dummy Property: if for all $(N, v) \in \mathbb{V}$, $\varphi_i(N, v) = 0$ if i is a type 1 dummy player.

(v) Strong Dummy Property: if for all $(N, v) \in \mathbb{V}$, $\varphi_i(N, v) = 0$ if i is a type 2 dummy player.[16]

We now show that for any restriction operator r à la Shapley, the corresponding r-Shapley value satisfies the first four properties described above. Moreover, if in addition r satisfies the stronger dummy axiom, then Sh^r satisfies the Strong Dummy property as well.

Consider, first the property of Linearity. Take any two games (N, v) and (N, v'). Then, it is easy to check that

$$w^r_{v+v'} = w^r_v + w^r_{v'}$$

and for any scalar λ,

$$w^r_{\lambda v} = \lambda w^r_v.$$

Moreover, since the Shapley value on $\mathbb{W}$ satisfies Linearity, it follows that

$$Sh^r(N, v + v') = Sh(N, w^r_{v+v'}) = Sh(N, w^r_v) + Sh(N, w^r_{v'}) = Sh^r(N, v) + Sh^r(N, v')$$

and

$$Sh^r(N, \lambda v) = Sh(N, w^r_{\lambda v}) = Sh(N, \lambda w^r_v) = \lambda Sh(N, w^r_v) = \lambda Sh^r(N, v).$$

These are sufficient to show that Sh^r satisfies Linearity on $\mathbb{V}$.

Now, consider any permutation σ of N. Then,

$$w^r_{\sigma v} = \sigma w^r_v$$

Since the Shapley value satisfies Symmetry on $\mathbb{W}$, this last equality shows that the property of Symmetry can be extended to $\mathbb{V}$. It is also obvious that Sh^r satisfies Efficiency.

For the Weak Dummy Property, choose any $(N, v) \in \mathbb{V}$ and suppose player i is a dummy player of type 1 in v. Choose any S containing i. Now, from repeated

[16] Notice that these Dummy properties are *restrictions* on the solution concept and are distinct from the Dummy axioms which were imposed on the restriction operators.

application of the Weak Dummy Axiom on r, i is a type 1 dummy player in $v^{-S^c,r}$. Hence,

$$v^{-S^c,r}(S) = v^{-S^c,r}(S \setminus \{i\}; \{i\})$$

Also,

$$w_v^r(S) = v^{-S^c,r}(S) = v^{-S^c,r}(S \setminus \{i\}; \{i\}) = v^{-(S \cup \{i\})^c,r}(S \setminus \{i\}) = w_v^r(S \setminus \{i\})$$

This shows that i is a dummy player in w_v^r and by the Weak Dummy Property of the Shapley value on TU games, $Sh_i^r(N, v) = 0$.

A similar analysis applies to the Strong Dummy Property and the Strong Dummy Axiom.

Remark 1 Although above we focused only on restriction operators à la Shapley, it is clear that for any arbitrary restriction operator r, the r-Shapley value satisfies any property in Definition 11 if the associated characteristic function games satisfy the "operational content" of the property. This is simply due to the fact that the Shapley value on TU games satisfies all the properties in Definition 11 on the set of TU games. For instance, Sh^r satisfies Symmetry if $w_{\sigma v}^r = \sigma w_v^r$.

7 Conclusion

8 References

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9 Appendix

In the Appendix we often use the following (simplified) notation. Whenever there is no scope for confusion, we will write a set $S = \{i, j, k\}$ as ijk, etc. For any set S and $i \in S^c$, we write $S + i$ to denote the set $S \cup \{i\}$, and $S - i$ to denote the set $S - \{i\}$. Similarly, $S + ij$ denotes the set $S \cup \{i, j\}$. We will also write a partition $\{\{ij\}, \{kl\}\}$ as $\{ij, kl\}$, etc. That is elements in a partition will be separated by a "comma".

Proof of Theorem 1:

We only need to prove (i) since (ii) is an immediate consequence of (i) and Theorem 1 of Hart and Mas-Colell (1989). We prove (i) by induction on $|N|$.

If $|N| = 1$, say $N = \{i\}$, then by definition of w^r_v, $w^r_v(i) = v(i)$. Hence, (i) is true for one person games since by definition of w^r_v, $P(i, w^r_v) = w^r_v(i) = v(i)$. Also, $P^r(i, v) = v(i)$ from the definition of a potential.

Let N be a player set and suppose by induction that (i) is true for all player sets containing fewer than $|N|$ players. Let $i \in N$. We first show that for all $S \subseteq N - i$

$$w^r_v(S) = w^r_{v^{-i,r}}(S). \tag{3}$$

where of course $v^{-i,r}$ is the partition function induced by r on player set $N - i$. Note that from Path Independence,

$$v^{-S^c} = (v^{-i})^{-(S^c - i)} \tag{4}$$

where we have dropped reference to the restriction operator for notational simplicity. Hence, for all $S \subseteq N - i$, we have

$$w^r_v(S) = v^{-S^c}(S) = (v^{-i})^{-(S^c-i)}(S) = w^r_{v^{-i,r}}(S)$$

where the first and the third equality follow from the definition of w, and the second equality follows from (4). Hence, (3) is true.

Let $w^r_v|_{N-i}$ denote the subgame of w^r_v on the player set $N - i$. Now we obtain

$$\begin{aligned}
P^r(N, v) &= \frac{v(N)}{|N|} + \sum_{i \in N} P^r(N - i, v^{-i,r}) \\
&= \frac{w^r_v(N)}{|N|} + \sum_{i \in N} P(N - i, w^r_{v^{-i,r}}) \\
&= \frac{w^r_v(N)}{|N|} + \sum_{i \in N} P(N - i, w^r_v|_{N-i})) \\
&= P(N, w^r_v)
\end{aligned}$$

where the first equality follows from the definition of an r-potential, the second from our induction hypothesis that (i) is true for all player sets containing fewer than $|N|$ players, the third from (3), and the fourth from the definition of the characteristic function potential. $\square$

Before proving Theorem 2, we introduce a property which is of instrumental importance for this proof. As we show, this property is satisfied by all of the examples given in the main text.

Definition 12 *A restriction operator r is Regular if for all solutions φ, for all $(N, v) \in \mathbb{V}$, and $S \subset N$, the reduced game $v^{\varphi,r}_S$ satisfies the following for all $R \subseteq S$*

$$(v^{\varphi,r}_S)^{-S\setminus R,r}(R) = v^{-S\setminus R,r}(R \cup S^c) - \sum_{k \in S^c} \varphi_k(R \cup S^c, v^{-S\setminus R,r})$$

Lemma 1 *Let r be a restriction operator satisfying Path Independence. If r satisfies Translation Invariance, then r satisfies Regularity.*

Proof. Let φ be a solution, $(N,v) \in \mathbb{V}$, and $S \subset N$. Then for the reduced game $v_S^{\varphi,r}$ we have for any $R \subseteq S$,

$$\begin{aligned}
(v_S^{\varphi,r})^{-S\setminus R,r}(R) &= r_{S\setminus R}\left(\left(v_S^{\varphi,r}(R;\pi(S\setminus R))\right)_{\pi(S\setminus R)\in\Pi_{S\setminus R}}\right) \\
&= r_{S\setminus R}\left(\left(v(R\cup S^c;\pi(S\setminus R)) - \sum_{k\in S^c}\varphi_k(R\cup S^c, v^{-(S\setminus R),r})\right)_{\pi(S\setminus R)\in\Pi_{S\setminus R}}\right) \\
&= r_{S\setminus R}\left((v(R\cup S^c;\pi(S\setminus R))_{\pi(S\setminus R)\in\Pi_{S\setminus R}}\right) - \sum_{k\in S^c}\varphi_k(R\cup S^c, v^{-(S\setminus R),r}) \\
&= v^{-S\setminus R,r}(R\cup S^c) - \sum_{k\in S^c}\varphi_k(R\cup S^c, v^{-S\setminus R,r}),
\end{aligned}$$

where the first and the fourth equality use the definition of the restriction operator (where we dropped the superscripts for notational simplicity and write r_{ij} instead of $r_j(r_i)$ etc.), the second uses the definition of the reduced game $v_S^{\varphi,r}$, and the third follows from repeated application of Translation Invariance of r. Hence, r satisfies Regularity. ■

Proof of Theorem 2:

Throughout the proof, fix some restriction operator r satisfying Path Independence and Translation Invariance. By Lemma 1, r satisfies Regularity. Since r is fixed, we omit all reference to r in the rest of the proof.

We first show that Sh satisfies consistency. Choose any $(N,v) \in \mathbb{V}$, and $S \subset N$. We need to show that for all $i \in S$,

$$Sh_i(N,v) = Sh_i(S, v_S^{Sh}) \text{ where } v_S^{Sh} \text{ is the reduced game on } S.$$

For all $R \subseteq S$, and $\pi \in \Pi_{S\setminus R}$,

$$v_S^{Sh}(R;\pi) = v(R\cup S^c;\pi) - \sum_{k\in S^c} Sh_k(R\cup S^c, v^{-S\setminus R}) \tag{5}$$

where $v^{-S\setminus R}$ is the subgame (of v) on the player set $S^c \cup R$.

Let w, w_S and $w^{-S\setminus R}$ be the characteristic function games associated with v, v_S^{Sh} and $v^{-S\setminus R}$ respectively. Let $\left(v_S^{Sh}\right)^{-(S\setminus R)}$ denote the subgame of v_S^{Sh} on the player set R. Since the restriction operator satisfies Regularity,

$$\left(v_S^{Sh}\right)^{-(S\setminus R)}(R) = v^{-(S\setminus R)}(R\cup S^c) - \sum_{k\in S^c} Sh_k\left(R\cup S^c, v^{-(S\setminus R)}\right)$$

Hence,

$$\begin{aligned}
w_S(R) &= \left(v_S^{Sh}\right)^{-(S\setminus R)}(R) \\
&= v^{-(S\setminus R)}(R\cup S^c) - \sum_{k\in S^c} Sh_k\left(R\cup S^c, v^{-(S\setminus R)}\right) \\
&= w(R\cup S^c) - \sum_{k\in S^c} Sh_k\left(R\cup S^c, w^{-(S\setminus R)}\right) \\
&= w(R\cup S^c) - \sum_{k\in S^c} Sh_k\left(R\cup S^c, w|_{R\cup S^c}\right) \\
&= w_S^{Sh}(R).
\end{aligned}$$

Therefore for all $i \in S$,

$$Sh_i\left(S, v_S^{Sh}\right) = Sh_i\left(S, w_S\right) = Sh_i\left(S, w_S^{Sh}\right) = Sh_i\left(N, w\right) = Sh_i\left(N, v\right),$$

where the third equality follows from consistency of the Shapley value on TU games. That is, Sh satisfies consistency.

It is obvious that Sh satisfies the standard game property. We now prove the converse by showing that there can be only one solution satisfying the standard property on two-person games and r- consistency for any given r.

First, if φ is a solution satisfying these two properties, then φ must be efficient. The proof of this is very similar to that in Hart and Mas-Colell [5].

So, if $n = 1$, then $\varphi_i(i, v) = v(i)$ and so there must be a unique solution. Similarly, the standard property on two-person games ensures that there is a unique solution on all two-person games. Suppose now that there is a unique solution satsifying consistency and the standard property on two-person games on all games $(N, v) \in \mathbb{V}$ when $|N| < m$.

Suppose now that $|N| = m$ and both φ and ψ are two different solutions satisfying these two properties. Since φ and ψ are different solutions, and both are efficient, there must exist $(N, v) \in \mathbb{V}$ and $i, j \in N$ such that $\varphi_i(N, v) > \psi_i(N, v)$, while $\varphi_j(N, v) < \psi_j(N, v)$. Define $S = \{i, j\}$. Now,

$$v_S^{\varphi}(i) = v(N-j; j) - \sum_{k\neq i,j} \varphi_k(N-j, v^{-j})$$

$$v_S^{\psi}(i) = v(N-j; j) - \sum_{k\neq i,j} \psi_k(N-j, v^{-j})$$

From the induction hypothesis, φ and ψ coincide on $(N-j, v^{-j})$. Hence,

$$\sum_{k\neq i,j} \psi_k(N-j, v^{-j}) = \sum_{k\neq i,j} \varphi_k(N-j, v^{-j})$$

This implies that

$$v_S^{\varphi}(i) = v_S^{\psi}(i)$$

Similarly,

$$v_S^{\varphi}(j) = v_S^{\psi}(j)$$

Then, from consistency,

$$\varphi_i(S, v_S^{\varphi}) = \varphi_i(N, v) > \psi_i(N, v) = \psi_i(S, v_S^{\psi})$$

So, $\varphi_j(S, v_S^{\varphi}) \geq \psi_j(S, v_S^{\psi})$. This contradicts the assumption that $\varphi_j(N, v) < \psi_j(N, v)$. $\square$

Proof of Corollary 1:

We show that the max operator satisfies Path Independence and Translation Invariance. Again, we omit any explicit reference to the operator in order to simplify the notation.

Fix any game (N, v).

To check that the max operator satisfies Path Independence, choose any $i, j \in N$, and $S \subset N - ij$. Let $\pi \equiv \{T_1, \dots, T_K\}$ be any partition of $(S + ij)^c$. Let Π' denote the set of partitions of S^c satisfying

$$\Pi' = \{\pi' \in \Pi_{S^c} | T' \in \pi' \to T_k \subset T' \cup \{i, j\} \text{ for some } k = 1, \dots K\}$$

That is, Π' is the set of partitions of S^c where each of i and j can join any of the elements of π, or remain single or form the set $\{i, j\}$. Then,

$$v^{-ij}(S; \pi) = \max_{\pi' \in \Pi'} \left[v(S; \pi')\right] = v^{-ji}(S; \pi)$$

This shows that the max operator satisfies Path Independence.

To check that max satisfies Translation Invariance, let $x_1, \dots, x_k$ and c be any real numbers. Then,

$$\max_{l=1,\dots,k} [x_l - c] = \max_{l=1,\dots,k} [x_l] - c.$$

This shows that max satisfies Translation Invariance. $\square$

Proof of Corollary 2:

We show that the trivial operator satisfies Path Independence and Translation Invariance. Again, we omit any explicit reference to the operator in order to simplify the notation.

Fix any game (N, v). To check that the trivial operator satisfies Path Independence, choose any $i, j \in N$, and $S \subset N - ij$. Let π be any partition of $(S + ij)^c$. Then,

$$v^{-ij}(S; \pi) = v(S; \pi \cup \{i\} \cup \{j\}) = v^{-ji}(S; \pi)$$

This shows that the trivial operator satisfies Path Independence. It is straightforward that the trivial operator satisfies Translation Invariance. $\square$

Proof of Corollary 3:

Let $\mathbb{P}$ be a partition of the universal set $\mathbb{N}$. By Theorem 2 and its proof it suffices to show that $r_{\mathbb{P}}$ satisfies both Path Independence and Regularity. Again, we omit any explicit reference to the operator in order to simplify the notation.

Fix any game (N, v). To check that the $\mathbb{P}$-coordinate operator satisfies Path Independence, choose any $i, j \in N$, and $S \subset N - ij$. Let π be any partition of $(S + ij)^c$. Now if $(S + ij)^c \cap \mathbb{P} \neq \pi$, then $v^{-ij}(S; \pi) = 0 = v^{-ji}(S; \pi)$; and if $(S + ij)^c \cap \mathbb{P} = \pi$, then $v^{-ij}(S; \pi) = v(S; S^c \cap \mathbb{P}) = v^{-ji}(S; \pi)$. This shows that the $\mathbb{P}$-coordinate operator satisfies Path Independence.

In order to check Regularity, let φ be a value, $(N, v) \in \mathbb{V}$ and $S \subset N$. For all $R \subseteq S$, we have by definition of $r_{\mathbb{P}}$

$$\begin{aligned}(v_S^{\varphi})^{-S\backslash R}(R) &= v_S^{\varphi}(R; (S\backslash R) \cap \mathbb{P}) \\ &= v(R \cup S^c; (S\backslash R) \cap \mathbb{P}) - \sum_{k \in S^c} \varphi_k(R \cup S^c, v^{-S\backslash R}) \\ &= v^{-S\backslash R}(R \cup S^c) - \sum_{k \in S^c} \varphi_k(R \cup S^c, v^{-S\backslash R}),\end{aligned}$$

which is the desired conclusion. □

Proof of Corollary 4:

Let $\mathbb{N}$ be finite and α be a weight system satisfying (1). In view of Theorem 2 and its proof it suffices to show that r_α satisfies both Path Independence and Regularity. Because these properties are preserved under linear combinations of restriction operators satisfying those properties, it is enough to show for any $i \in \mathbb{N}$ and any $\mathbb{P}_i \in \Pi_{\mathbb{N}}^i$ that $r_{\mathbb{P}_i}$ satisfies both Path Independence and Regularity. Again, we omit any explicit reference to the operator $r_{\mathbb{P}_i}$ to simplify the notation.

Path Independence of $r_{\mathbb{P}_i}$ can be checked similarly as in the proof of Corollary 3.

In order to check Regularity, let φ be a value, $(N, v) \in \mathbb{V}$ and $S \subset N$. For all $R \subseteq S$, we have by definition of $r_{\mathbb{P}_i}$: (i) if $i \in R$, then

$$\begin{aligned}(v_S^{\varphi})^{-S\backslash R}(R) &= v_S^{\varphi}(R; (S\backslash R) \cap \mathbb{P}_i) \\ &= v(R \cup S^c; (S\backslash R) \cap \mathbb{P}_i) - \sum_{k \in S^c} \varphi_k(R \cup S^c, v^{-S\backslash R}) \\ &= v^{-S\backslash R}(R \cup S^c) - \sum_{k \in S^c} \varphi_k(R \cup S^c, v^{-S\backslash R});\end{aligned}$$

(ii) if $i \notin R \cup S^c$, then $(v_S^{\varphi})^{-S\backslash R}(R) = 0$, $v^{-S\backslash R}(R \cup S^c) = 0$, and for all $k \in S^c$, $\varphi_k(R \cup S^c, v^{-S\backslash R}) = 0$ by $v^{-S\backslash R}(T; \pi((R \cup S^c)\backslash T)) = 0$ for all $T \subseteq R \cup S^c$ and all partitions $\pi((R \cup S^c)\backslash T)$ (i.e. $(R \cup S^c, v^{-S\backslash R})$ is a null game and the value assigns zero to any player for this game); and (iii) if $i \in S^c$: NOT CLEAR. Hence, both in (i) and (ii), $r_{\mathbb{P}_i}$ satisfies Regularity. □

The following lemma is an important tool for the proof of Theorem 3. It simply says that for any subgame, the function associating the worth of any coalition S and a partition of its complement is identical with the one where from any member of the partition all players except one are removed and all removed players join the coalition. In other words, these functions are identical if in the subgame the partition contains the same number of elements (and the same players left the original game). The proof uses the Weak Dummy Axiom.

Lemma 2 *Let r be a restriction operator satisfying Path Independence and the Weak Dummy Axiom. Choose any $S \subseteq (N-i)$ and any partition $\pi((S+i)^c) = \{S_1, \ldots, S_K\}$. For any choice of $j_k \in S_k, k = 1, \ldots K$, define $N' = \cup_{k=1}^K (S_k - j_k)$. Let*

$$\begin{aligned} v^{-i}(S; \pi((S+i)^c)) &= f(\{x_{ik}\}_{k=1}^K, x_0) \\ v^{-i}(S \cup N'; \pi^t(\{j_1, j_2, \ldots, j_K\})) &= g(\{z_{ik}\}_{k=1}^K, z_0) \end{aligned}$$

where for each $k = 1, \ldots K$,

(i) $x_{ik} = v(S; \{S_1, \ldots, S_{k-1}, S_k + i, S_{k+1}, \ldots, S_K\})$.

(ii) $z_{ik} = v(S \cup N'; \{\{j_1\}, \ldots, \{j_{k-1}\}, \{j_k, i\}, j_{k+1}, \ldots, \{j_K\}\})$.

(iii) $x_0 = v(S; \{S_1, S_2 \ldots, S_K, \{i\}\})$.

(iv) $z_0 = v(S \cup N'; \pi^t(\{j_1, \ldots, j_K, i\}))$.

Then $f = g$.

Proof. Let us choose all the members of N' to be dummy players of type 1 in v. By the Weak Dummy Axiom, members of N' remain dummy players of type 1 in v^{-i}. Hence, $v^{-i}(S; \pi((S+i)^c)) = v^{-i}(S \cup N'; \pi^t(\{j_1, \ldots, j_K, i\}))$. Moreover, for all $k = 1, \ldots K$,

$$\begin{aligned} z_{ik} &= v(S \cup N'; \{\{j_1\}, \ldots, \{j_{k-1}\}, \{j_k, i\}, \{j_{k+1}\}, \ldots, \{j_K\}\}). \\ &= v(S; \{S_1, \ldots, S_{k-1}, S_k + i, S_{k+1}, \ldots, S_K\}) \\ &= x_{ik} \end{aligned}$$

and $z_0 = v(S \cup N'; \pi^t(\{j_1, \ldots, j_K, i\})) = v(S; \{S_1, S_2 \ldots, S_K, \{i\}\}) = x_0$.

Note that this choice of dummy players does not impose any restrictions on the vector $(\{z_{ik}\}_{k=1}^K, z_0)$ and hence on $(\{x_{ik}\}_{k=1}^K, x_0)$. Therefore, $g(\{z_{ik}\}_{k=1}^K, z_0) = g(\{x_{ik}\}_{k=1}^K, x_0) = f(\{x_{ik}\}_{k=1}^K, x_0)$ for any $(\{x_{ik}\}_{k=1}^K, x_0)$.

Hence, $f = g$. ∎

Proof of Theorem 3:

Choose any $i \in N$ and $S \subseteq N - i$. We prove the theorem by induction on the number of elements in $\pi^{+i}(S^c)$.

Suppose $|\pi^{+i}(S^c)| = 1$. Then $v^{-i}(S) = r_i(v(S; \{i\}))$. By Scale Invariance, $r_i(v(S; i)) = v(S; i)$. Hence the induction hypothesis is satisfied.

Now, suppose $|\pi^{+i}(S^c)| = 2$. Let us first show that the induction hypothesis holds for elements of the type $(S; j)$. From our specification,

$$v^{-i}(S; j) = r_i(v(S; ji), v(S; j, i))$$

By Scale Invariance,

$$r_i(v(S; ji), v(S; j, i)) = \theta_i v(S; ji) + (1 - \theta_i) v(S; j, i)$$

We have already proved that $v^{-ij}(S) = v^{-i}(S;j)$. Hence,

$$v^{-ij}(S) = \theta_i v(S;ji) + (1-\theta_i)v(S;j,i)$$

By similar arguments,

$$v^{-ji}(S) = \theta_j v(S;ji) + (1-\theta_j)v(S;j,i)$$

However, Path Independence implies $v^{-ij}(S) = v^{-ji}(S)$. Therefore $\theta_i = \theta_j = \theta$.

We can use Lemma 2 to extend our analysis for all partitions with $|\pi^{+i}(S^c)| = 2$.

So, the theorem is true when $\pi^{+i}(S^c)$ has no more than two elements. Suppose the theorem is true for all partitions when $\pi^{+i}(S^c)$ has K or less elements for some K. We want to show that the theorem remains true when $\pi^{+i}(S^c)$ has $K+1$ elements.

Choose N such that $|N| \geq K+2$. Choose $i \in N$ and $S \subset N - i$ such that $\pi((S+i)^c) = \pi^t((S+i)^c)$ has exactly K elements. Hence, $\pi^{+i}((S+i)^c)$ has $K+1$ elements.

Our first aim is to calculate $v^{-i}(S;\pi^t((S+i)^c))$, with the help of the induction hypothesis.

From our specification,

$$v^{-i}(S;\pi^t((S+i)^c)) = r_i\left(\{v(S;ik,\pi^t((S+ik)^c))\}_{k\notin(S+i)}, v(S;\pi^t(S^c))\right)$$

Denoting $v(S;ik,\pi^t((S+ik)^c)) = x_{ik}$ and $v(S;\pi^t(S^c)) = x_0$, we can rewrite the previous equation as

$$v^{-i}(S;\pi^t((S+i)^c)) = r_i(\{x_{ik}\}_{k\notin S+i}, x_0)$$

Similarly, for any $m \neq i$, we can write $v^{-m}(S;\pi^t((S+m)^c))$ as

$$v^{-m}(S;\pi^t((S+m)^c)) = r_m(\{x_{mk}\}_{k\notin S+m}, x_0)$$

where $x_{mk} = v(S;mk,\pi^t((S+mk)^c))$, and $x_{im} = x_{mi}$.

We can use the induction hypothesis to calculate $v^{-im}(S;\pi^t((S+im)^c))$. Indeed,

$$\begin{aligned}
&v^{-im}(S;\pi^t((S+im)^c)) \\
&\quad= \theta \sum_{k\notin(S+im)} v^{-i}(S;mk,\pi^t((S+imk)^c)) + (1-K\theta)v^{-i}(S;\pi^t((S+i)^c)) \\
&\quad= \theta \sum_{k\notin(S+im)} [\theta \sum_{\ell\notin(S+imk)} v(S;mk,i\ell,\pi^t((S+imk\ell)^c)) + \theta v(S;mki,\pi^t((S+imk)^c)) \\
&\qquad +(1-K\theta)v(S;mk,\pi^t((S+mk)^c))] + (1-K\theta)v^{-i}(S;\pi^t((S+i)^c)) \qquad (6)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&v^{-mi}(S;\pi^t((S+im)^c)) \\
&\quad= \theta \sum_{k\notin(S+im)} [\theta \sum_{\ell\notin(S+imk)} v(S;ik,m\ell,\pi^t((S+imk\ell)^c) + \theta v(S;mki,\pi^t((S+imk)^c)) \\
&\qquad +(1-K\theta)v(S;ik,\pi^t((S+ik)^c))] \\
&\qquad +(1-K\theta)v^{-m}(S;\pi^t((S+m)^c)) \qquad (7)
\end{aligned}$$

From Path Independence,

$$v^{-im}(S;\pi^t((S+im)^c)) = v^{-mi}(S;\pi^t((S+i,m)^c)) \tag{8}$$

Also,

$$\begin{aligned}\sum_{k\notin(S+im)}\sum_{\ell\notin(S+imk)} v(S;mk,i\ell,\pi^t((S+imk\ell)^c)) &= \sum_{\ell\notin(S+im)}\sum_{k\notin(S+im\ell)} v(S;mk,i\ell,\pi^t((S+imk\ell)^c))\\ &= \sum_{k\notin(S+im)}\sum_{\ell\notin(S+imk)} v(S;ik,m\ell,\pi^t((S+imk\ell)^c))\end{aligned}$$

Thus,

$$\begin{aligned}&\left[v^{-i}(S;\pi^t((S+i)^c)) - v^{-m}(S;\pi^t((S+m)^c))\right]\\ =\ &\theta\sum_{k\notin(S+im)}\left[v(S;ik,\pi^t((S+ik)^c)) - v(S;mk,\pi^t((S+mk)^c))\right]\\ =\ &\theta\sum_{k\notin(S+im)}(x_{ik}-x_{mk})\end{aligned} \tag{9}$$

By choosing $x_{ik} = x_{mk} \equiv x_k$ for all $k \notin (S+im)$, we obtain

$$r_i(\{x_{ik}\}_{k\notin(S+i)}, x_0) = r_m(\{x_{mk}\}_{k\notin(S+m)}, x_0) \equiv r(\{x_k\}_{k\notin(S+i)}, x_0) \tag{10}$$

So, from (9),

$$r(\{x_{ik}\}_{k\notin(S+i)}, x_0) = \theta\sum_{k\notin(S+im)}(x_{ik}-x_{mk}) + r(\{x_{mk}\}_{k\notin(S+m)}, x_0) \tag{11}$$

Let $p \notin (S+im)$ be a dummy player of type 1 in v. Then, p is also a dummy player in v^{-i}, and so

$$v^{-i}(S;\pi^t((S+i)^c)) = v^{-i}(S+p;\pi^t((S+ip)^c))$$

From the induction hypothesis,

$$v^{-i}(S+p;\pi^t((S+ip)^c)) = \theta\sum_{k\notin(S+ip)} v(S+p;ki,\pi^t((S+ipk)^c)) + (1-K\theta)v(S+p;\pi^t((S+p)^c))$$

Since p is a dummy player, $v(S+p;ki,\pi^t((S+ikp)^c)) = v(S;ki,\pi^t((S+ik)^c))$ and $v(S+p;\pi^t((S+p)^c)) = v(S;\pi^t(S^c))$. Also, $v(S;pi,\pi^t((S+p)^c)) = v(S;\pi^t(S^c))$. So,

$$\begin{aligned}v^{-i}(S;\pi^t((S+i)^c)) &= v^{-i}(S+p;\pi^t((S+i,p)^c))\\ &= \theta\sum_{k\notin(S+ip)} v(S;ki,\pi^t((S+ik)^c)) + (1-K\theta)v(S;\pi^t(S^c))\\ &= \theta\sum_{k\notin(S+i)} v(S;ki,\pi^t((S+ik)^c)) + (1-(K+1)\theta)v(S;\pi^t(S))\end{aligned}$$

That is,

$$r(\{x_{ik}\}_{k\notin(S+i)},x_0)=\theta\sum_{k\notin(S+i)}x_{ik}+(1-(K+1)\theta)x_0 \tag{12}$$

Notice that this equality is proved under the assumption that $x_{ip}=x_0$, and so we do not as yet have a general expression for r.

Finally, from (11),

$$\begin{aligned}
r(\{x_{mk}\}_{k\notin(S+m)},x_0) &= \theta\sum_{k\notin(S+i)}x_{ik}+(1-(K+1)\theta)x_0-\theta\sum_{k\notin(S+im)}(x_{ik}-x_{mk})\\
&= x_{mi}+\theta\sum_{k\notin(S+im)}x_{mk}+(1-(K+1)\theta)x_0\\
&= \theta\sum_{k\notin(S+m)}x_{mk}+(1-(K+1)\theta)x_0
\end{aligned}$$

Hence,

$$v^{-i}(S;\pi^t((S+i)^c))=\theta\sum_{k\notin(S+i)}v(S;ik,\pi^t((S+i,k)^c))+(1-(K+1)\theta)v(S;\pi^t(S))$$

We can use Lemma 2 to extend our analysis for all partitions with $|\pi^{+i}(S)|=K+1$. This completes the induction step.

In order to complete the proof of the theorem, we have to demonstrate the implication of Non-negativity. Consider a game $(N,v)\in\mathbb{V}$ such that for all $S\subseteq N$,

(i) $v(S,\pi)=0$ if there is some $i\in S^c$ such that $\{i\}\in\pi$.

(ii) $v(S,\pi)>0$ otherwise.

Now, suppose $\theta<0$. Choose any $S\subset N$ and any $i\notin S$. Then,

$$v^{-i}(S,\pi((S\cup\{i\})^c))=\theta\sum_{\pi\in\pi^{+i}((S\cup\{i\})^c)}v(S;\pi)<0$$

where the first inequality follows from the fact that $v(S;\pi)=0$ if $\{i\}\in\pi$.

But, this violates Non-negativity, and so $\theta\geq 0$.

This concludes the proof of Theorem 3. □

Proof of Theorem 4:

Suppose the restriction operator satisfies Strong Dummy Axiom, Scale Invariance, Monotonicity and Path Independence. Then, for all $i\in N$, for all v,

$$v^{-i}(S;\pi^t((S+i)^c))=v(S;\pi^t((S+i)^c)\cup\{i\})$$

That is, the strong dummy axiom implies that $\theta=0$. This can be checked as follows. Let $N=\{1,2,3,4\}$. By Theorem 3,

$$\begin{aligned}
v^{-4}(\{1,2\},\{3\}) &= \theta v(\{1,2\},\{3,4\})+(1-\theta)v(\{1,2\},\{3\},\{4\})\\
v^{-4}(\{1\},\{2\},\{3\}) &= \theta v(\{1\},\{2\},\{3,4\})+\theta v(\{1\},\{2,4\},\{3\})+(1-2\theta)v(\{1\},\{2\},\{3\},\{4\})
\end{aligned}$$

Suppose 2 is a dummy player of type 2 in v. By strong dummy axiom player 2 must be a type 2 dummy player in v^{-4}. Thus, $v^{-4}(\{1,2\},\{3\}) = v^{-4}(\{1\},\{2\},\{3\})$. This is only possible if $\theta = 0$, because the assumption of type 2 dummy player does not impose any restriction on $v(\{1\},\{2,4\},\{3\})$. Hence the result. It is easy to check that this restriction operator will satisfy all the axioms. □

Proof of Corollary 5:

Suppose r satisfies Path Independence, Scale Invariance, Non-negativity and the Weak Dummy Axiom. In view of Theorem 2 and its proof, it is sufficient to show that r satisfies Regularity.

Let w_S be the TU game associated with $v_S^{Sh^r}$. Thus, $w_S(R) = \left(v_S^{Sh^r}\right)^{-(S\setminus R)}(R)$. By repeated use of Theorem 3,

$$\begin{aligned}
\left(v_S^{Sh^r}\right)^{-(S\setminus R)}(R) &= \sum_{\pi\in\Pi_{S\setminus R}} \alpha_\pi v_S^{Sh^r}(R;\pi) \\
&= \sum_{\pi\in\Pi_{S\setminus R}} \alpha_\pi \left[v(R\cup S^c;\pi) - \sum_{k\in S^c} Sh_k^r\left(v^{-(S\setminus R)}\right)\right] \\
&= \sum_{\pi\in\Pi_{S\setminus R}} \alpha_\pi v(R\cup S^c;\pi) - \left[\sum_{k\in S^c} Sh_k^r\left(v^{-(S\setminus R)}\right)\right]\left[\sum_{\pi\in\Pi_(S\setminus R)} \alpha_\pi\right] \\
&= v^{-(S\setminus R)}(R\cup S^c) - \sum_{k\in S^c} Sh_k^r\left(v^{-(S\setminus R)}\right)
\end{aligned}$$

One can check that $\alpha_\pi = \theta^{(|S\setminus R|-|\pi|)}\Pi_{k=0}^{|P|-1}(1-k\theta)$ and $\sum_{\pi\in\Pi_{S\setminus R}}\alpha_\pi = 1$. of course, the actual form of α_π is not important. What is important is that the restriction operator is linear, and that α_π does not depend upon T in $v^{-(S\setminus R)}(T)$. □