

# Network of Commons\*

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## Abstract

A tragedy of the commons appears when the users of a common resource have incentives to exploit it more than the socially efficient level. We analyze the situation when the tragedy of the commons is embedded in a network of users and sources. Users play a game of extractions, where they decide how much resource to draw from each source they are connected to. We show that if the value of the resource to the users is linear, then each resource exhibits an isolated problem. There exists a unique equilibrium. But when the users have concave values, then the network structure matters. The exploitation at each source depends on the centrality of the links connecting the source to the users. The equilibrium is unique and we provide a formula which expresses the quantities at an equilibrium as a function of a network centrality measure. Next we characterize the efficient levels of extractions by cities and outflows from sources. Again, the case of linear values can be broken down source by source. For the case of concave values, we provide a graph decomposition which divides the network into *regions* according to the availability of sources. Then the efficiency problem can be solved region by region.

**Keywords:** tragedy of the commons, networks, Nash equilibrium, efficiency, centrality measures.

**JEL Classification:** C 62, C 72, D 85, Q 20

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# 1 Introduction

Many environmental resources which supply the basic inputs of production are owned collectively. Typical examples of such commons are clean air, carbon-dioxide levels in the atmosphere, pastures, forests, fisheries and water sources. One similarity they share is that the availability (or the fertility) of the resource decreases with use, and in some cases over exploitation may even destroy it completely.

When the individual users ignore the cost their activity imposes on the rest, "The Tragedy of The Commons" occurs. It was brought under the spotlight by Hardin (1968), but the analysis of the problem in specific contexts precedes that<sup>1</sup>. Although the term is used for issues relating to the use of natural resources, it is not far from a moral hazard in teams as modeled in Holmstrom (1982). It has been studied widely since Hardin's article.

In a standard model of commons, there exists a single resource exploited by many users. In real life examples, the most representative commons (e.g. pastures, forests, fisheries and water sources) are local, but numerous. Each site from which a natural resource is extracted are utilized by many, but in most of the cases the beneficiaries also have access to many such sites, which they might share with similar or different users. A lake might supply water to many cities, but cities also receive water from many lakes. A country shares fisheries with its coastal neighbors, but many countries have coasts in multiple seas and oceans. The exploitation decision of a user would be affected by the availability of sites, but also by the other users who operate on these sites. When the sources and users are interconnected, the exploitation of each user from each source will depend on the structure of the connections.

We model a bipartite network, where sources and cities are connected through links. We assume that the average cost of extraction at each source increases with the amount of total exploitation from that source. Then each user imposes a cost on all other users. We distinguish between two cases. One where the users value the resource linearly, and the other where they have concave quadratic valuations.

We look at the water extraction game, where agents decide how much to draw from each source they are connected to. They have a value from consuming the resource, but their marginal cost of extraction increases with each extra unit. We assume that at each source, agents share of the total cost is equal to her share from the total extraction. Meaning that

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<sup>1</sup>See Gordon (1954) for a model of fisheries. Olson (1965) also alludes to the problem as it relates to collective action.

the users at the same source face the same per unit cost at that source.<sup>2</sup>

We show that for the case of linear values, each source exhibits an isolated tragedy of the commons. Players' actions depend on how many other users there are at each source. In the terminology of networks, only the source centered graph matters. The network effects do not permeate through paths of more than two links.

When the users' values are concave, their actions at a source does not only depend on the number of users they share it with. It also depends on the number of sources their neighbors are linked to. And also on the number of users at the sources which their neighbors are linked to. The externalities diffuse through the paths *ad infinitum*. We write the equilibrium conditions as a linear complementarity problem and show uniqueness. To interpret the equilibrium quantities, we define a centrality index (*link centrality*) that captures the spreading effects of each extraction. We provide an interpretation of this index comparing it with the Katz-Bonacich centrality (Katz (1953), Bonacich (1987)).

We next characterize the efficient amounts of extraction for both cases. The linear case can again be divided source by source. There exists a continuum of flows which give efficiency, but in all of them the outflows from the sources are equal. For concave values, the efficient amounts depend on the whole network. Generically, there exists a continuum of efficient flows, which all give the same amounts of extractions to cities and outflows to sources. To calculate these efficient amounts, we decompose the network into *regions*. Each region is a connected subgraph of the original network. They are cut out from the network, according to the ratio of sources to cities in them.

Given a network, we determine a connected subgraph such that all its cities are among the least privileged with respect to sources. The subgraph will contain all the sources that its cities are connected to in the network. The aim is to favor most the poorest in source. We give them exclusive rights to the sources they are connected to. After cutting out this subgraph from the network, we will find a similar subgraph formed by the least privileged cities in the remaining one. We continue until we reach a network where all cities are equal with respect to source availability.

We bridge to branches of economics literature. On one side we study a tragedy of the commons. In a standard model of a common pool resource (Gordon (1954), Weitzman (1974), Funaki and Yamato (1999)) multiple users exploit a single source. One user's consumption

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<sup>2</sup>It is logical that the crowding of the source affects everyone equally. When the fish population decreases, the catch becomes difficult for everyone.

affects others identically. In this paper, we extend this basic model to a network of users and sources. The symmetry between the users is lost (except for exceptional networks like the complete network, the hub, etc.). Given a network, we show how the structure of connections determine users' extraction levels. We also characterize the socially efficient outcomes.

We do not explicitly deal with the question of management of the commons<sup>3</sup>. But we provide a network decomposition such that in each of the subnetworks we obtain, the problem of efficiency is equivalent to the case of one source and many users.

Although we use the metaphor of water, this paper is different from Ambec and Sprumont (2001), because the sources in our model work quite differently from the river in theirs. Moreover, we do not make any cooperative analysis of the problem.

The other related line of literature is the analysis of behavior on networks. We study a bipartite-network as in Corominas-Bosch (2004). She studies the equilibria of a bargaining game in a network of buyers and sellers. The model differs from ours in two basic points. First, both buyers and sellers are active agents, where we only take one side, the users, as strategic. Second, buyers and sellers are bargaining over a single indivisible good. In contrast, we assume that the good transferred between parties is perfectly divisible, allowing a source to supply to many users.

Ballester et al. (2006) analyzes the equilibrium activities at each node of a simple non-directed network. Players create externalities on their neighbors. A player has a single level of activity. Her payoff depends on her activity level and of her neighbours'. They show that the equilibrium levels are given by a network centrality index, which is similar to the Katz-Bonacich centrality. Ballester and Calvó-Armengol (2006) show that the first order equilibrium conditions of games which exhibit cross influences between agents' actions are linear complementarity problems. They study some interesting classes of such games which have a unique equilibrium. In both papers, the agents strategy spaces are subsets of the real line. They choose a real number and a link between two agents shows that they impose externalities on each other. In our model, agents' strategy spaces are multidimensional and a link is not only a qualitative object, but also carries a value.

The basic notation, some of which we borrow from Corominas-Bosch (2004), is introduced

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<sup>3</sup>Seabright (1993) gives a survey of the literature on the management of the commons issue. Faysse (2005) provides a survey of game theoretical models of commons management. On the empirical side, many real life examples have been discussed in Ostrom (1991) and Ostrom et al. (1994, 2002). They also provide theoretical and empirical analysis concerning possible solutions for the tragedy of the commons.

in Section 2. Section 3 defines the payoffs and Section 4 defines the water extraction game. We study the equilibrium in section 5 and characterize the efficient outcomes in Section 6. Section 7 discusses the results. The proofs are given in the appendix

## 2 Notation

There are  $n$  sources  $s_1, \dots, s_n$ , and  $m$  cities  $c_1, \dots, c_m$ . They are embedded in a network that links sources with cities, and cities can acquire their water from the sources they are connected to. We will represent the network as a graph.

A non-directed *bipartite graph*  $g = \langle S \cup C, L \rangle$  consists of a set of *nodes* formed by sources  $S = \{s_1, \dots, s_n\}$ , and cities  $C = \{c_1, \dots, c_m\}$  and a set of *links*  $L$ , each link joining a source with a city. A link from  $s_i$  to  $c_j$  will be denoted as  $(i, j)$ . We say that a node  $s_i$  is *linked* to another node  $c_j$  if there is a link joining the two. We will use  $(i, j) \in g$  and  $(i, j) \in L$  interchangeably, meaning that  $s_i$  and  $c_j$  are connected in  $g$ .

A bipartite graph  $g$  is *connected* if there exists a path linking any two nodes of the graph. Formally, a path linking nodes  $s_i$  and  $c_j$  will be a collection of  $t$  cities and  $t$  sources,  $t \geq 0$ ,  $s_1, \dots, s_t, c_1, \dots, c_t$  among  $S \cup C$  (possibly some of them repeated) such that

$$\{(i, 1), (1, 1), (1, 2), \dots, (t, t), (t, j)\} \in g$$

A *subgraph*  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  of  $g$  is a graph such that  $S_0 \subseteq S, C_0 \subseteq C, L_0 \subseteq L$  and such that each link in  $L$  that connects a source in  $S_0$  with a city in  $C_0$  is a member of  $L_0$ . Hence a node of  $g_0$  will continue to have the same links it had with the other nodes in  $g_0$ . We will write  $g_0 \subseteq g$  to mean that  $g_0$  is a subgraph of  $g$ .

For a subgraph  $g_0$  of  $g$ , we will denote by  $g - g_0$ , the subgraph of  $g$  that results when we remove the set of nodes  $S_0 \cup C_0$  from  $g$ .  $g - g_0$  will be defined as the maximal connected parts of the subgraph induced by the set of nodes  $(S - S_0) \cup (C - C_0)$ .

Given a subgraph  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  of  $g$ , let  $\overleftarrow{g_0}$  be the complete bipartite graph with nodes  $S_0 \cup C_0$ . We call  $\overleftarrow{g_0}$  the *completed graph* of  $g_0$ .

$N_g(s_i)$  will denote the set of cities linked with  $s_i$  in  $g = \langle S \cup C, L \rangle$ , more formally:

$$N_g(s_i) = \{c_j \in C \text{ such that } (i, j) \in g\}$$

and similarly  $N_g(c_j)$  stands for the set of sources linked with  $c_j$ .

For a set  $A$ , let  $|A|$  denote the number of elements in  $A$ . For  $s_i$  in  $S$ , we denote  $|N_g(s_i)|$  by  $m_i(g)$ . Similarly for  $c_j \in C$ , let  $|N_g(c_j)| = n_j(g)$ , be the number of sources connected to  $c_j$ .

An *invasive subgraph*  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  of  $g$  is such that  $g_0$  is connected and,

$$S_0 = \bigcup_{c_j \in C_0} N_g(c_j)$$

An invasive subgraph includes all the sources to which its cities were connected in graph  $g$ . We will denote by  $W(g) = \{g_0 \subseteq g : g_0 \text{ is invasive}\}$  the set of invasive subgraphs in  $g$ .  $W(g) \neq \emptyset$  as  $g$  is an invasive subgraph of itself. In the network  $g_1$  in Figure 1, the subgraph  $g_1^0$  that we encircle is invasive. It includes  $c_1$  and all the sources that  $c_1$  is connected to.

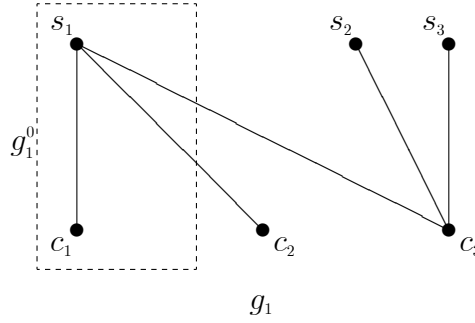


Figure 1

Given a subset of sources  $S_0 \subseteq S$  and a subset of cities  $C_0 \subseteq C$ ,  $\frac{|S_0|}{|C_0|}$  is the average number of sources per city. A *minimally invasive subgraph*  $\hat{g} = \langle \hat{S} \cup \hat{C}, \hat{L} \rangle$  of  $g$  is such that

$$\frac{|\hat{S}|}{|\hat{C}|} < \frac{|S|}{|C|} \text{ and } \langle \hat{S} \cup \hat{C}, \hat{L} \rangle \in \underset{\langle S_0 \cup C_0, L_0 \rangle \in W(g)}{\operatorname{argmin}} \frac{|S_0|}{|C_0|}$$

The first requirement for  $\hat{g}$  to be a minimally invasive subgraph of  $g$  is for it to have a strictly smaller source/city ratio than  $g$ . This means that a graph does not necessarily have a minimally invasive subgraph. For example a complete bipartite graph has no minimally invasive subgraphs. Any invasive subgraph cut out from a complete graph will have a source/city ratio at least as big as the complete graph.

The second requirement is for  $\hat{g}$  to have the smallest source/city ratio among the invasive graphs of  $g$ . A minimally invasive subgraph is invasive and formed by a set of least connected

cities. There should be no cities in  $g$  which are strictly worse than them with respect to source availability.

In Figure 1, the subgraph  $g_1^0$  is not minimally invasive, because the ratio of source to cities in it is 1. But this ratio for the graph  $g_1$  is lower than 1. The subgraph  $g_1^1$  of  $g_1$ , as encircled Figure 2 below, is a minimally invasive subgraph. Its source/city ratio is lower than that of  $g_1$ , and there is no other subgraph of  $g_1$  with a lower ratio.

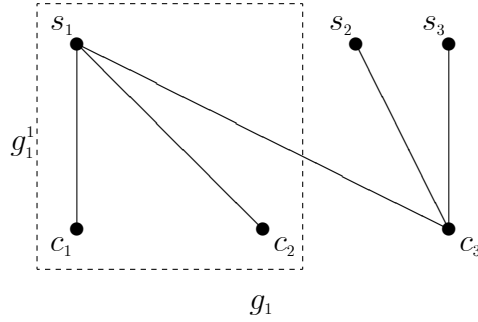


Figure 2

If  $\hat{g}$  is a minimally invasive subgraph of  $g$ , then  $\hat{g}$  cannot have a minimally invasive subgraph of its own. Any invasive subgraph of  $\hat{g}$  is also invasive in  $g$ . If  $\hat{g}$  had a minimally invasive subgraph with a smaller source/city ratio than  $\hat{g}$ , this would have contradicted  $\hat{g}$  having the smallest source/city ratio in  $g$ .

We denote by  $q_{ij} \geq 0$  the amount of water extracted by city  $c_j$  from source  $s_i$ .

## 2.1 Labelling of pairs (i,j)

Let  $\tau : \{1, \dots, n\} \times \{1, \dots, m\} \rightarrow \mathbb{N}_+$ , be a lexicographic order on  $\{1, \dots, n\} \times \{1, \dots, m\}$  such that:

- (i)  $\tau(1, 1) = 1$ ,
- (ii)  $(i, j) \neq (k, l) \Rightarrow \tau(i, j) \neq \tau(k, l)$ ,
- (iii)  $j < l \Rightarrow \tau(i, j) < \tau(k, l)$  for all  $i, k \in \{1, \dots, n\}$ ,
- (iv)  $i < k \Rightarrow \tau(i, j) < \tau(k, j)$  for all  $j \in \{1, \dots, m\}$ ,
- (v) if  $\exists(i, j)$  such that  $\tau(i, j) = y > 1$  then  $\exists(k, l)$  s.t.  $\tau(k, l) = y - 1$ .

$\tau$  orders all possible links such that the links of a city  $j$  are assigned a lower number than any city  $i$ , for  $i > j$ , and the links of a city is ordered according to the indices of the sources they come from. For example for 2 cities and 2 sources, the function  $\tau$  orders the links starting from the first city, and the first source,  $\tau(1, 1) = 1$ . The second ranked link is between the first city and the second source,  $\tau(2, 1) = 2$ . Now, as all links of city  $c_1$  is ranked,  $\tau$  will next rank the link between  $c_2$  and  $s_1$ ,  $\tau(1, 2) = 3$ . Next comes the link between city 2 and source 2,  $\tau(2, 2) = 4$ .

For a network  $g$ , let  $Y(g) = \{y \in \mathbb{N}_+ : y = \tau(i, j) \text{ for some } (i, j) \notin g\}$  be the set of indices that  $\tau$  assigns to links which are not in  $g$ . Assume, without loss of generality that  $|Y(g)| = m \times n - r(g)$ , for some  $1 \leq r(g) \leq m \times n$ , where  $r(g)$  is the number of links in graph  $g$ . For 2 cities and 2 sources, for a graph  $g$ , if the only missing link is  $(1, 2)$ , then  $Y(g) = \{3\}$  and  $r(g) = 3$ .

Observe that  $\tau$  orders all possible links, independent of  $g$ , where as  $Y(g)$  does depend on  $g$ .

We can see how the above definitions work on an example. Suppose that 2 cities and 2 sources, form a completely connected bipartite graph  $g_2$ . For graph  $g_2$ ,  $Y(g_2) = \emptyset$ .

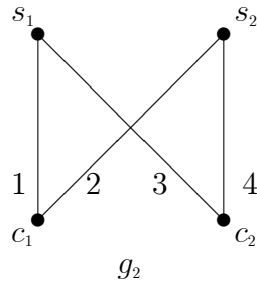


Figure 3

Now we cut the link between  $c_2$  and  $s_1$ , to obtain  $g_3$ .

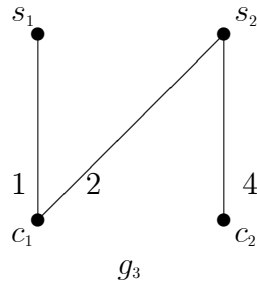


Figure 4

Although link  $(1, 2)$  does not exist in  $g_3$  it is still labelled equally by  $\tau$ .  $\tau(1, 2) = 3$ , meaning that  $Y(g_3) = \{3\}$ .

We will make use of graphs  $g_2$  and  $g_3$  in many examples throughout the paper.

## 2.2 Some useful matrices

Now we define some matrices which we will use during our analysis.

For  $\beta, \gamma \geq 0$ , let  $A = [a_{ij}]_{n \times n}$  be such that,

$$a_{ij} = \begin{cases} 2\beta + \gamma, & \text{for } i = j \\ \gamma & , \text{ for } i \neq j \end{cases}$$

$A$  has  $2\beta + \gamma$  on the diagonal and  $\gamma$  off the diagonal.

$$A = \begin{bmatrix} 2\beta + \gamma & & & \\ & \cdot & \gamma & \\ & & \cdot & \\ & \gamma & & \cdot \\ & & & & 2\beta + \gamma \end{bmatrix}_{n \times n}$$

Let  $B = \beta I_{n \times n}$ , where  $I_{n \times n}$  is the identity matrix of size  $n$ . Using matrices  $A$  and  $B$ , we construct the partitioned matrix  $D = [d_{ij}]_{(m \times n) \times (m \times n)}$  such that:

$$D = \begin{bmatrix} A & & & \\ & \cdot & B & \\ & & \cdot & \\ B & & & \cdot \\ & & & & A \end{bmatrix}_{(m \times n) \times (m \times n)}$$

$D$  has matrix  $A$  on its diagonal and matrix  $B$  off the diagonal. If we want to write it term by term,

$$d_{ij} = \begin{cases} 2\beta + \gamma, & \text{for } i = j \\ \gamma & , \text{ for } i \neq j, \text{ s.t. } (i, j) = (z_1n + z_2, z_1n + z_3) \text{ for } z_1, z_2, z_3 \in \mathbb{N} \\ & \text{s.t. } z_2 \neq z_3, 1 \leq z_2, z_3 \leq n - 1 \text{ and } z_1 \leq m - 1 \\ \beta & , \text{ for } i \neq j, \text{ s.t. } i + j = (1 + z_1)n + 1 + 2z_2, \text{ for } z_1, z_2 \in \mathbb{N} \\ & \text{s.t. } z_1 \leq m - 1, z_2 \leq m \\ 0 & , \text{ otherwise} \end{cases}$$

For example for 2 cities and 2 sources,

$$D_{4 \times 4} = \begin{pmatrix} 2\beta + \gamma & \gamma & \beta & 0 \\ \gamma & 2\beta + \gamma & 0 & \beta \\ \beta & 0 & 2\beta + \gamma & \gamma \\ 0 & \beta & \gamma & 2\beta + \gamma \end{pmatrix}$$

The interpretation, when we use it to find the equilibrium quantities flowing from sources to cities, will be that the column  $z$  and the row  $z$  in  $D$  corresponds to the link  $(i, j)$  in  $g$  such that  $\tau(i, j) = z$ . Hence, column 1 and row 1 corresponds to the link  $(1, 1)$ , column 2 and row 2 corresponds to the link  $(2, 1)$ , column 3 and row 3 corresponds to the link  $(1, 2)$ , and column 4 and row 4 corresponds to the link  $(2, 2)$ .

Let  $D_{-j}$  be the matrix obtained by deleting row  $j$  and column  $j$  from  $D$ . For  $J \subset \mathbb{N}_+$ , let  $D_{-J}$  be the matrix obtained by deleting each row  $j \in J$  and column  $j \in J$  from  $D$ . We will denote  $D_{-Y(g)}$  by  $D_g$ . We obtain  $D_g$  by deleting each row  $y \in Y(g)$  and column  $y \in Y(g)$  from  $D$ . These rows and columns belong to links that are not in  $g$ . Then,  $D_g$  has size  $r(g) \times r(g)$ .

For  $g_2$ , since  $Y(g_2) = \emptyset$ ,  $D_{g_2} = D_{4 \times 4}$ . For  $g_3$ , as  $Y(g_3) = \{3\}$ ,  $D_{g_3}$  is formed by taking out the third column and third row of  $D_{4 \times 4}$ .

$$D_{g_3} = \begin{bmatrix} 2\beta + \gamma & \gamma & 0 \\ \gamma & 2\beta + \gamma & \beta \\ 0 & \beta & 2\beta + \gamma \end{bmatrix}$$

Let  $\overline{B} = 2B$  be the matrix obtained from  $B$  by multiplying it with the scalar 2. Similarly we construct the partitioned matrix  $F = [f_{ij}]_{(m \times n) \times (m \times n)}$  such that:

$$F = \begin{bmatrix} A & & & \\ & \cdot & \overline{B} & \\ & & \cdot & \\ \overline{B} & & & \cdot \\ & & & & A \end{bmatrix}_{(m \times n) \times (m \times n)}$$

$F$  has matrix  $A$  on its diagonal and matrix  $\overline{B}$  off the diagonal. If we want to write it term by term,

$$f_{ij} = \begin{cases} 2\beta + \gamma, & \text{for } i = j \\ \gamma, & \text{for } i \neq j, \text{ s.t. } (i, j) = (z_1 n + z_2, z_1 n + z_3) \text{ for } z_1, z_2, z_3 \in \mathbb{N} \\ & \text{s.t. } z_2 \neq z_3, 1 \leq z_2, z_3 \leq n - 1 \text{ and } z_1 \leq m - 1 \\ 2\beta, & \text{for } i \neq j, \text{ s.t. } i + j = (1 + z_1)n + 1 + 2z_2, \text{ for } z_1, z_2 \in \mathbb{N} \\ & \text{s.t. } z_1 \leq m - 1, z_2 \leq m \\ 0, & \text{otherwise} \end{cases}$$

For example for 2 cities and 2 sources,

$$F_{4 \times 4} = \begin{bmatrix} 2\beta + \gamma & \gamma & 2\beta & 0 \\ \gamma & 2\beta + \gamma & 0 & 2\beta \\ 2\beta & 0 & 2\beta + \gamma & \gamma \\ 0 & 2\beta & \gamma & 2\beta + \gamma \end{bmatrix}$$

Similarly, let  $F_{-j}$  be the matrix obtained by deleting row  $j$  and column  $j$  from  $F$ . Let  $\mathbb{N}_+$  be the set of positive integers. For  $J \subset \mathbb{N}_+$ , let  $F_{-J}$  be the matrix obtained by deleting each row  $j \in J$  and column  $j \in J$  from  $F$ . We will denote  $F_{-Y(g)}$  by  $F_g$ . We obtain  $F_g$  by deleting each row  $y \in Y(g)$  and column  $y \in Y(g)$  from  $F$ . These rows and columns belong to links that are not in  $g$ . Then,  $F_g$  has size  $r(g) \times r(g)$ .

For  $g_1$ , since  $Y(g_2) = \emptyset$ ,  $F_{g_2} = F_{4 \times 4}$ . For  $g_3$ , as  $Y(g_3) = \{3\}$ ,  $F_{g_3}$  is formed by taking out the third column and third row of  $F_{4 \times 4}$ .

$$F_{g_2} = \begin{bmatrix} 2\beta + \gamma & \gamma & 0 \\ \gamma & 2\beta + \gamma & 2\beta \\ 0 & 2\beta & 2\beta + \gamma \end{bmatrix}$$

Before going further we show that for  $\beta, \gamma > 0$ ,  $D_{-J}$  is positive definite, and  $F_{-J}$  is positive semi-definite for any  $J \subset \mathbb{N}_+$ .

**Proposition 1** For  $\beta, \gamma > 0$ ,  $D_{-J}$  is positive definite for any  $J \subset \mathbb{N}_+$ .

**Proposition 2** For  $\beta, \gamma > 0$ ,  $F_{-J}$  is positive semi-definite for any  $J \subset \mathbb{N}_+$ .

Now we define the column vector that shows the quantities flowing at each link. Let  $Q = [e_z]$  be the column vector of quantities extracted such that for  $q_{ij}$ , the quantity extracted from source  $s_i$  by  $c_j$ ,  $e_{\tau(i,j)} = q_{ij}$ . For 2 cities and 2 sources:

$$Q = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix}$$

Let  $Q_{-j}$  be the vector obtained by deleting row  $j$  from  $Q$ . For  $J \subset \mathbb{N}_+$ , let  $Q_{-J}$  be the vector obtained deleting each row  $j \in J$  and column  $j \in J$  from  $Q$ . For  $Y(g) \subset \mathbb{N}$ , let  $Q_g$  be the matrix obtained by deleting each row  $y \in Y(g)$  from  $Q$ . Then  $Q_g$  has size  $r$ .  $Q_g$  is the link by link profile of extractions. For the two graphs given above:

$$Q_{g_1} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix} \qquad Q_{g_2} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{22} \end{bmatrix}$$

Let  $\mathbb{Q}_{(m \times n)}$  be the set of all non-negative real valued column vectors of size  $(m \times n)$ . Let  $\mathbb{Q}_r$  be the set of all non-negative real valued column vectors of size  $r$ .

Given a vector of flows  $Q_g$ , for a city  $c_j$ , we will denote by  $E_j(Q_g)$  the total amount extracted by  $c_j$ . For a source  $s_i$  we will denote by  $O_i(Q_g)$  the total outflow from  $s_i$ .

### 3 Payoffs

We will assume that the utility function of players are additively separable into value and cost of extraction. Hence, for a given  $Q_g \in \mathbb{Q}_r$ ,

$$u_j(Q_g) = v_j(E_j(Q_g)) - \sum_{s_i \in N_g(c_j)} T_{ij}(Q_g),$$

where  $v_j(Q_g)$  is the value obtained from consuming  $E_j(Q_g)$  and  $T_{ij}(Q_g)$  is the cost of extraction by  $c_j$  from source  $s_i$ . We will use quadratic value and cost functions, which will decrease the computational load and help us focus on the effects of the network structure on the equilibrium quantities.

We will assume quadratic costs of extraction, which is uniform for all sources. Hence, for  $\beta > 0$  the total cost of extraction from a given source  $s_i$  would be

$$T_i(Q_g) = \beta(O_i(Q_g))^2$$

We assume that each player pays her share of the cost proportional to her extraction. The cost of extraction  $q_{ij}$  by  $c_j$  from  $s_i$  would be

$$T_{ij}(Q_g) = \beta q_{ij}(O_i(Q_g))$$

We will assume uniform cost functions among sources, but the results would continue to hold as long as the costs are quadratic at each source.

We will analyze two cases, with two different value functions.

### 3.1 Linear Values

For  $\alpha, \beta > 0$ , let

$$u_j(Q_g) = \alpha E_j(Q_g) - \beta \sum_{s_i \in N_g(c_j)} q_{ij} [O_i(Q_g)]$$

for all cities  $c_j \in C$ . As the value function is linear, the utility is separable with respect to each source

$$u_j(Q_g) = \sum_{s_i \in N_g(c_j)} q_{ij} [\alpha - \beta O_i(Q_g)]$$

Then, for all  $c_j \in C$  and for all  $s_i \in N_g(c_j)$ , the marginal utility to  $c_j$  of extraction from  $s_i$  is:

$$\frac{\partial u_j}{\partial q_{ij}} = \alpha - 2\beta q_{ij} - \beta \sum_{c_k \in N_g(s_i) \setminus \{c_j\}} q_{ik}$$

The marginal utility at extraction  $q_{ij}$  depends only on the other levels of extraction at source  $s_i$ .

### 3.2 Concave Values

For  $\alpha, \beta, \gamma > 0$ , let

$$\tilde{u}_j(Q_g) = \alpha E_j(Q_g) - \frac{\gamma}{2} (E_j(Q_g))^2 - \beta \sum_{s_i \in N_g(c_j)} q_{ij} (O_i(Q_g))$$

for all cities  $c_j \in C$ . Now, the utility is not separable with respect to each source. For all  $c_j \in C$  and for all  $s_i \in N_g(c_j)$ , the marginal utility to  $c_j$  of extraction from  $s_i$  is:

$$\frac{\partial \tilde{u}_j}{\partial q_{ij}} = \alpha - (2\beta + \gamma)q_{ij} - \gamma \sum_{s_l \in N_g(c_j) \setminus \{s_i\}} q_{lj} - \beta \sum_{c_k \in N_g(s_i) \setminus \{c_j\}} q_{ik}$$

Neither the marginal utilities are separable source by source. The marginal utility at  $q_{ij}$  does depend on the amounts extracted by  $c_j$  from sources other than  $s_i$ .

## 4 The Water Extraction Game

Given a network  $g$ , each city  $c_j$  maximizes its utility by extracting a non-negative amount of water through its links from the sources in  $N_g(c_j)$ . So, the set of players are the set of cities  $C$ . The set of strategies of a city  $c_j$  is  $\mathbb{Q}_j = \mathbb{Q}_{N_g(c_j)}$ . We denote a representative strategy of  $c_j$  by  $Q_j \in \mathbb{Q}_j$ . Given that there are  $r(g)$  links in  $g$ , the strategy space of the game is  $\mathbb{Q}_g = \prod_{c_j \in C} \mathbb{Q}_j = \mathbb{Q}_{r(g)}$ .

For each city  $j$ , in the water extraction game with linear values, we will assume that each player has utility  $u_j(Q_g)$ . Then a best response  $Q'_j$  of city  $c_j$  to  $Q_g \in \mathbb{Q}_g$  is such that,

$$\text{for all links } (i, j), q'_{ij} = \begin{cases} \frac{\alpha - \beta \sum_{c_k \in N_g(s_i) \setminus \{c_j\}} q_{ik}}{2\beta}, & \text{if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} \geq 0 \\ 0 & , \text{if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} < 0 \end{cases}$$

In the water extraction game with concave values, we assume their utility to be  $\tilde{u}_j(Q_g)$ . Then a best response  $Q'_j$  of city  $c_j$  to  $Q_g \in \mathbb{Q}_g$  is such that,

$$\text{for all links } (i, j), q'_{ij} = \begin{cases} \frac{\alpha - \gamma \sum_{s_l \in N_g(c_j) \setminus \{s_i\}} q_{lj} - \beta \sum_{c_k \in N_g(s_i) \setminus \{c_j\}} q_{ik}}{2\beta + \gamma}, & \text{if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} \geq 0 \\ 0 & , \text{if } \frac{\partial u_j}{\partial q_{ij}}|_{Q_g} < 0 \end{cases}$$

## 5 The Equilibrium

### 5.1 Linear Values

For the linear case, the first order condition for  $q_{ij}$  does not depend on the amounts extracted from sources other than  $s_i$ . Then we can separate the optimization problem source by source. Meaning that equilibrium extractions from a source  $s_i$  depend only on how many players are connected to  $s_i$ .

**Theorem 3** *Water extraction with linear values has a unique Nash equilibrium, such that for any link  $(i, j)$  the equilibrium flow  $q_{ij}^* = \frac{\alpha}{(m_i(g)+1)\beta}$*

**Example** Suppose we have the graph  $g_2$ . Let  $\alpha = \beta = 1$ . Then the equilibrium flows of the water extraction game are  $q_{11}^* = q_{21}^* = q_{12}^* = q_{21}^* = \frac{1}{3}$ .

Suppose the graph was  $g_3$ . Now, at equilibrium  $q_{11}^* = \frac{1}{2}$ , and  $q_{21}^* = q_{22}^* = \frac{1}{3}$ . So, the deletion of the link  $(1, 2)$  does not change the extraction levels on source  $s_2$ .

### 5.2 Concave Values

We will write the equilibrium conditions of the water extraction game with concave values as a linear complementarity problem. Given a matrix  $M \in \mathbb{R}^{t \times t}$  and a vector  $p \in \mathbb{R}^t$ , the linear complementarity problem  $LCP(p; M)$  consists of finding a vector  $z \in \mathbb{R}^t$  satisfying:

$$\begin{aligned} z &\geq 0, \\ p + Mz &\geq 0, \\ z^T(p + Mz) &\geq 0 \end{aligned}$$

Given a graph  $g$ , the first order equilibrium conditions of the game define a  $LCP(-\alpha \mathbf{1}_r; D_g)$  where  $\mathbf{1}$  is a column vector of 1's of size  $r(g)$ .<sup>4</sup>

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<sup>4</sup>The water extraction game with linear values also forms a linear complementarity problem. But it is simpler to find equilibrium flows source by source in that case, rather than work with matrices derived from the network structure.

$$\begin{aligned}
Q_g &\geq 0, \\
-\alpha \mathbf{1}_r + D_g Q_g &\geq 0, \\
Q_g^T (q + D_g Q_g) &\geq 0
\end{aligned}$$

Samelson *et al.* (1958) shows that a linear complementarity problem  $LCP(p; M)$  has a unique solution for all  $p \in \mathbb{R}^t$  if and only if all the principal minors of  $M$  are positive. Positive definite matrices satisfy this condition and we showed in Proposition 1 that  $D_g$  is positive definite. Then the equilibrium conditions have a unique solution.

We further check for the second order conditions for each agent, which reveals that the solution of the  $LCP(-\alpha \mathbf{1}_r; D_g)$  is indeed the equilibrium of the game.

**Theorem 4** *Water extraction with concave values has a unique Nash equilibrium.*

**Example** Suppose we have the graph  $g_2$ . Let  $\alpha = \beta = \gamma = 1$ . Then the link flows at equilibrium are  $q_{11}^* = q_{21}^* = q_{12}^* = q_{22}^* = 0.2$ .

Suppose the graph was  $g_3$ . Now at equilibrium,  $q_{11}^* = 0.2857$ ,  $q_{21}^* = 0.1429$ , and  $q_{22}^* = 0.2857$ . Under concave values of extraction, the deletion of the link  $(1, 2)$  does change the extraction levels on source  $s_2$ , and moreover city  $c_1$  extracts less from the source she shares with city  $c_2$ .

### 5.2.1 The Equilibrium Quantities

Let  $Q_g^*$  be an equilibrium of the water extraction game with concave values. There might be some links in  $g$ , such that they carry zero flow at equilibrium  $Q_g^*$ . Marginal utilities of extractions from those links need not be zero at  $Q_g^*$ .

$$\begin{aligned}
q_{ij}^* > 0 &\Rightarrow \frac{\partial \tilde{u}_j}{\partial q_{ij}} = 0 \\
q_{ij}^* = 0 &\Rightarrow \frac{\partial \tilde{u}_j}{\partial q_{ij}} \leq 0
\end{aligned}$$

To calculate the equilibrium quantities, first we need to weed out the links with zero flow. Let  $\rho : L \rightarrow \mathbb{N}_+$  be a lexicographic order on  $L$  respecting  $\tau$  such that  $\rho$  relabels the  $(i, j)$

pairs from 1 to  $r(g)$  by skipping those links which are not in  $g$ .<sup>5</sup> Now we delete from  $Q_g^*$ , the entries that correspond to links with no flow.

Let  $Z(Q_g^*) = \{z \in \mathbb{N}_+ : z = \rho(i, j) \text{ for some } (i, j) \text{ s.t. } q_{ij}^* = 0\}$ . Let  $|Z(Q_g^*)| = t^*$ , then  $Q_{g-Z(Q_g^*)}^*$  is a vector of size  $r(g) - t^*$  obtained from  $Q_g^*$  by deleting the zero entries. It is the vector of equilibrium quantities for links over which there is a strictly positive flow from a source to a city.

Let's remember the first order conditions. For all  $(i, j) \in g$ ,

$$\frac{\partial \tilde{u}_j}{\partial q_{ij}} = \alpha - (2\beta + \gamma)q_{ij} - \gamma \sum_{s_l \in N_g(c_j) \setminus \{s_i\}} q_{lj} - \beta \sum_{c_k \in N_g(s_i) \setminus \{c_j\}} q_{ik} = 0$$

Then for any equilibrium  $Q_g^*$  of the water extraction game with concave values,

$$D_{g-Z(Q_g^*)} \cdot Q_{g-Z(Q_g^*)}^* = \alpha \cdot \mathbf{1}$$

where  $\mathbf{1}$  is a column vector of 1's of size  $r(g) - t^*$ .

Given a network  $g$  let  $Q_g^*$  be the equilibrium at  $g$ . Then we denote by  $g - Z(Q_g^*)$  the network obtained from  $g$  by deleting the links which have zero flow at  $Q_g^*$ .

**Theorem 5** *Given two networks  $g$  and  $g'$ . Let  $Q_g^*$  and  $Q_{g'}^*$  be the equilibrium of the water extraction game with concave values in  $g$  and  $g'$ , respectively. If  $g - Z(Q_g^*) = g' - Z(Q_{g'}^*)$ , then  $Q_{g-Z(Q_g^*)}^* = Q_{g'-Z(Q_{g'}^*)}^*$ .*

At equilibrium there might be links which carry no flows. For the cities of such links, the marginal utilities of extraction from them are not positive. They are indifferent between having such a link or not. Theorem 3 tells us such links with zero flows play no role while

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<sup>5</sup>Explicitly,  $\rho : L \rightarrow \mathbb{N}_+$  is such that:

- (i)  $\exists (i, j) \in L$  such that  $\rho(i, j) = 1$ ,
- (ii)  $(i, j) \neq (k, l) \Rightarrow \rho(i, j) \neq \rho(k, l)$ ,
- (iii)  $j < l \Rightarrow \rho(i, j) < \rho(k, l)$  for all  $(i, j), (k, l) \in L$ ,
- (iv)  $i < k \Rightarrow \rho(i, j) < \rho(k, j)$  for all  $(i, j), (k, j) \in L$ ,
- (v) if  $\exists (i, j) \text{ s.t. } \rho(i, j) = z > 1$  then  $\exists (k, l) \in L \text{ s.t. } \rho(k, l) = y - 1$ .

determining equilibrium. They are strategically redundant. For an example let's remember graph  $g_1$ .

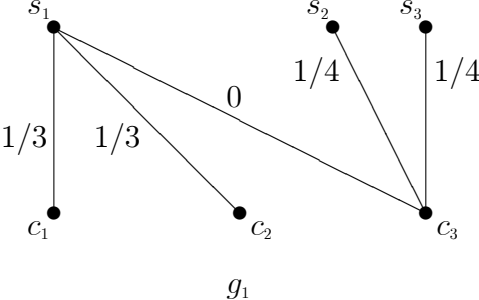


Figure 5

Let  $\alpha = \beta = \gamma = 1$ . Then the link flows at equilibrium are  $q_{11}^* = q_{12}^* = \frac{1}{3}$ ,  $q_{13}^* = 0$  and  $q_{23}^* = q_{33}^* = \frac{1}{2}$ . The cities  $c_1$  and  $c_2$  have no other connections except  $s_1$ . At equilibrium the marginal cost of extraction from source  $s_1$  is higher than  $s_2$  and  $s_3$ . The difference is too large for city  $c_3$  to make any profitable use of the link  $(1, 3)$ . Now we cut the link  $(1, 3)$  and denote the new graph by  $g_1 - (1, 3)$ .

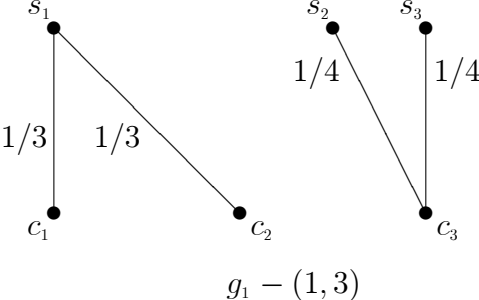


Figure 6

For  $\alpha = \beta = \gamma = 1$ , Theorem 1 tells us that the flows at equilibrium are  $q_{11}^* = q_{12}^* = \frac{1}{3}$  and  $q_{23}^* = q_{33}^* = \frac{1}{2}$ . At the equilibrium in  $g_1$ , the marginal utility to city  $c_3$  from extraction via  $(1, 3)$  was negative. Deleting it does not change the equilibrium quantities on other links.

Now, let's try to give a network interpretation for the flow quantities at equilibrium  $Q_{g-Z}^*(Q_g^*)$  on any given graph  $g$ .

### 5.2.2 Decomposition of $D_{g-Z(Q_g^*)}$

As  $D_{g-Z(Q_g^*)}$  is a symmetric matrix, whose diagonal entries are  $2\beta + \gamma$ , and non-diagonal entries are either 0,  $\gamma$ , or  $\beta$  we can separate it into  $(2\beta + \gamma)I$ , and a symmetric matrix  $G^*$ , where  $I$  is the identity matrix of size  $r(g) - t^*$ . For example for graph  $g_3$  all links have positive flows at equilibrium. Then,

$$G_{g_3}^* = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & \beta \\ 0 & \beta & 0 \end{bmatrix}$$

For any graph  $g$ ,  $G^*$  has diagonal entries as 0 and non-diagonal entries are either 0,  $\gamma$  or  $\beta$ .  $G^*$  can be interpreted as the weighted adjacency matrix of the network obtained from  $g$ , where the active links  $(i, j)$  of  $g$  with  $q_{ij}^* > 0$  at  $Q_g^*$  are the vertices, and the cities and sources in  $g$  are the edges. An edge in the derived matrix has weight  $\beta$  if it is a source and weight  $\gamma$  if it is a city. From now on we will call  $G^*$  *the equilibrium dual of  $g$* . We will use it to denote both the graph derived from  $g$  as explained above and the adjacency matrix of that graph.

Hence,

$$\begin{aligned} D_{g-Z(Q_g^*)} \cdot Q_{g-Z(Q_g^*)}^* &= [(2\beta + \gamma)I + G^*] \cdot Q_{g-Z(Q_g^*)}^* \\ &= (2\beta + \gamma) [I + aG^*] \cdot Q_{g-Z(Q_g^*)}^* \end{aligned}$$

where  $a = \frac{1}{2\beta + \gamma}$ . Remember that  $Q_g^*$  is the solution to  $LCP(-\alpha \mathbf{1}_r; D_g)$ . Then, when we invert  $D_{g-Z(Q_g^*)}$ , the matrix multiplication  $\alpha \cdot [D_{g-Z(Q_g^*)}]^{-1} \mathbf{1}$  will give us a strictly positive vector. Now, for  $a \geq 0$ ,

$$\begin{aligned} [I + aG^*] &= [I - aG^*]^{-1} [I - (aG^*)^2] \\ [I + aG^*]^{-1} &= [I - (aG^*)^2]^{-1} [I - aG^*] \\ &\text{and} \\ [I - (aG^*)^2]^{-1} &= \sum_{k=0}^{\infty} (aG^*)^{2k} \end{aligned}$$

Substituting this into  $D_{g-Z(Q_g^*)} \cdot Q_{g-Z(Q_g^*)}^* = \alpha \cdot \mathbf{1}$ ,

$$\begin{aligned} Q_{g-Z(Q_g^*)}^* &= a\alpha [I - (aG^*)^2]^{-1} [I - aG^*] \cdot \mathbf{1} \\ &= a\alpha \sum_{k=0}^{\infty} (aG^*)^{2k} [I - aG^*] \cdot \mathbf{1} \\ &= a\alpha \left[ \sum_{k=0}^{\infty} (aG^*)^{2k} \cdot \mathbf{1} - \sum_{k=0}^{\infty} (aG^*)^{2k+1} \cdot \mathbf{1} \right] \end{aligned}$$

The last expression is a centrality measure for the network with adjacency matrix  $G^*$ . Although it is not a standard centrality index, we can understand it by comparing it with a known one. For  $a \geq 0$ , and a network adjacency matrix  $G^*$ , let

$$M(G^*, a) = [I - aG^*]^{-1} = \sum_{k=0}^{\infty} (aG^*)^k$$

If  $M(a, G^*)$  is non-negative, its entries  $m_{ij}(G^*, a)$  counts the number of paths in the network, starting at  $i$  and ending at  $j$ , where paths of length  $k$  are weighted by  $a^k$ .

**Definition 1** For a network adjacency matrix  $G$ , and for scalar  $a > 0$  such that  $M(G, a) = [I - aG]^{-1}$  is well-defined and non-negative, the vector Katz-Bonacich centralities of parameter  $a$  in  $G$  is:

$$\mathbf{b}(G, a) = [I - aG^*]^{-1} \cdot \mathbf{1}$$

In a graph with  $z$  nodes, the Katz-Bonacich centrality of node  $i$ ,

$$b_i(G, a) = \sum_{j=1}^z m_{ij}(G, a)$$

counts the total number of paths in  $G$  starting from  $i$ .

Using the Katz-Bonacich centrality as a benchmark, let's define the link centrality of a network of commons  $g$ .

**Definition 2** For scalars  $\alpha, a \geq 0$ , a network of commons  $g$  and its equilibrium dual  $G^*$ , such that  $[I + aG^*]^{-1}$  is well-defined and non-negative, the vector link centralities of in  $g$  is:

$$\mathbf{l}(G, a) = a\alpha [I + aG^*]^{-1} \cdot \mathbf{1}$$

Hence, in the expression

$$a\alpha \left[ \sum_{k=0}^{\infty} (aG^*)^{2k} \cdot \mathbf{1} - \sum_{k=0}^{\infty} (aG^*)^{2k+1} \cdot \mathbf{1} \right]$$

the first summation counts the total number of even paths that start from the corresponding node in  $G^*$ , and the second summation counts the total number of odd paths that start from it.

The first sum tells that the equilibrium extraction from a link is positively related with the number of even length paths that start from it. The links which have an even distance between them are strategical complements. In contrast, the negative sign on the second summation means the equilibrium extraction from a link is negatively related with the number of odd length paths that start from it. The links which have an odd distance between them are strategical substitutes.

For example, in graph  $g_2$ ,

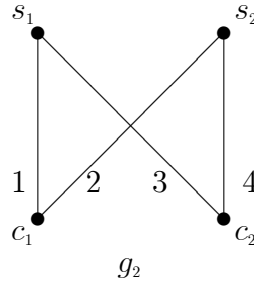


Figure 7

links (1, 1) and (2, 2) are strategical complements. The extraction from source  $s_2$  by city  $c_2$  increases incentives for city  $c_1$  to extract more from source  $s_1$ , because the former increases the marginal cost on  $s_2$ . This makes  $s_1$  a better option. Links (1, 1) and (2, 1) are strategical substitutes, because extraction from one decreases the marginal value of water to city  $c_1$ . This decreases city's incentives to extract more.

In general, the links of a city are substitutes for each other (e.g. (1, 1) and (2, 2) at graph  $g_1$ ). Similarly, the links of a source are substitutes for each other, too (e.g. (1, 1) and (1, 2) at graph  $g_1$ ). If two cities are sharing a source, then their links to sources they don't share are complements (e.g. (1, 1) and (2, 2) at graph  $g_1$ ). Moreover, if a link  $(i_1, j_1)$  is a strategic substitute of a link  $(i_2, j_2)$  and  $(i_2, j_2)$  is a strategic substitute of  $(i_3, j_3)$ , then  $(i_1, j_1)$  and  $(i_3, j_3)$  are strategic complements. Therefore, the strategic effect depends on the parity of the distance between two links.

In the water extraction game the adjacency matrix  $G^*$  does not necessarily have binary entries, neither its non-zero entries are all equal. Each link in  $G^*$  has a weight. While counting the number of paths, these weights are taken into account as well. The extraction by a city  $c_j$  is calculated by summing up the link centralities of the elements in  $N_g(c_j)$ .

## 6 The Efficient Extraction

We will assume that cities have comparable and identical utilities. Such an assumption is not far fetched from reality, in particular for the many commons which are not end products.<sup>6</sup> Indeed in most setups, commons receive their value from being a productive input for firms that supply to a market (Weitzman (1974), Funaki and Yamato(1999)).

### 6.1 Linear Values

When cities value water linearly, the sum of their utilities is

$$U = \sum_{c_j \in C} u_j(Q_g) = \alpha \sum_{(i,j) \in g} q_{ij} - \beta \sum_{s_i \in S} (O_i(Q_g))^2$$

Then the first order condition implies that at an efficient vector of flows  $Q_g^e$ , for any source  $s_i$ ,  $O_i(Q_g^e) = \frac{\alpha}{2\beta}$ .

As the values are linear, it does not matter to whom the water goes, as long as no source's total outflow exceeds the efficient amount.

**Example** Suppose we have the graph  $g_1$ . Let  $\alpha = \beta = 1$ . Then the efficient flows are

$$\{q_{11}^e, q_{21}^e, q_{12}^e, q_{22}^e \geq 0 : q_{11}^e + q_{12}^e = \frac{1}{2} \text{ and } q_{21}^e + q_{22}^e = \frac{1}{2}\}$$

Suppose the graph was  $g_2$ . Now, the efficient flows are

$$\{q_{11}^e, q_{21}^e, q_{22}^e \geq 0 : q_{11}^e = \frac{1}{2} \text{ and } q_{21}^e + q_{22}^e = \frac{1}{2}\}$$

There exists a continuum of flows which give an efficient outcome in both graphs. All the efficient flows lead to the same amounts of outflows from sources.

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<sup>6</sup>Though such a comparison lacks sense for commons which are imperative for their users. To compare a catastrophically dehydrated city with a well provided one would be unacceptable, both in economic and ethical terms.

## 6.2 Concave Values

When cities have concave values, the network structure determines the efficient levels of extraction in a non trivial fashion. The sum of utilities is

$$\tilde{U} = \sum_{c_j \in C} \tilde{u}_j(Q_g) = \alpha \sum_{(i,j) \in g} q_{ij} - \frac{\gamma}{2} \sum_{c_j \in C} (E_j(Q_g))^2 - \beta \sum_{s_i \in S} (O_i(Q_g))^2$$

Then the first order condition that an efficient vector of flows  $Q_g^e$  has to satisfy is,

$$\text{for all } (i, j) \in g \quad \begin{cases} \text{if } q_{ij}^e \neq 0, \text{ then } \alpha = \gamma E_j(Q_g^e) + 2\beta O_i(Q_g^e) \\ \text{if } q_{ij}^e = 0, \text{ then } \alpha < \gamma E_j(Q_g^e) + 2\beta O_i(Q_g^e) \end{cases}$$

Hence given a city  $c_j$ , and 2 different sources  $s_i, s_k \in N_g(c_j)$

$$\begin{aligned} q_{ij}^e, q_{kj}^e &\neq 0 \implies O_i(Q_g^e) = O_k(Q_g^e) \\ q_{ij}^e &= 0 \text{ and } q_{kj}^e \neq 0 \implies O_i(Q_g^e) > O_k(Q_g^e) \end{aligned}$$

Similarly, given a source  $s_i$ , and 2 different cities  $c_j, c_k \in N_g(s_i)$

$$\begin{aligned} q_{ij}^e, q_{ik}^e &\neq 0 \implies E_j(Q_g^e) = E_k(Q_g^e) \\ q_{ij}^e &= 0 \text{ and } q_{ik}^e \neq 0 \implies E_j(Q_g^e) > E_k(Q_g^e) \end{aligned}$$

Observe that this is also a linear complementarity problem with  $LCP(-\alpha \mathbf{1}_r; F_g)$ . But  $F_g$  is positive semi-definite, and not necessarily positive definite. We are not guaranteed a unique solution. Indeed, we will see that, in general, there exists a continuum of solutions to the problem of efficient flows. To solve it, we first characterized the first order conditions above and now we look at the Hessian of the sum  $\tilde{U}$ . The Hessian matrix of  $\tilde{U}$  is so that  $H_{\tilde{U}} = -F_g$ . As  $F_g$  is positive semi-definite,  $H_{\tilde{U}}$  is negative semi-definite. Meaning that any  $Q_g^e$  that satisfies the first order conditions maximizes  $\tilde{U}$ .

**Example** Suppose we have the graph  $g_2$ . Let  $\alpha = \beta = \gamma = 1$ . Observe that  $g_2$  has no minimally invasive subgraphs. Indeed, it is complete. Then the efficient flows are

$$\{q_{11}^e, q_{21}^e, q_{12}^e, q_{22}^e \geq 0 : q_{11}^e + q_{12}^e = \frac{1}{3}, q_{21}^e + q_{22}^e = \frac{1}{3}, q_{11}^e + q_{21}^e = \frac{1}{3} \text{ and } q_{12}^e + q_{22}^e = \frac{1}{3}\}$$

There exists a continuum of flows which give an efficient outcome. The total extractions at each city and the total outflows at each source are the same for all the efficient flow levels.

Now we will find a vector of extractions that satisfies the first order conditions of efficiency. Given a subgraph  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  of  $g$ , consider the efficient amount of extractions and outflows in its completed graph  $\overleftrightarrow{g_0}$ . Clearly the levels are identical across cities and across sources. Let  $\overleftrightarrow{E_0}$  be the efficient amount of total extraction by a city in  $\overleftrightarrow{g_0}$  and  $\overleftrightarrow{O_0}$  the efficient amount of total outflow from a source in  $\overleftrightarrow{g_0}$ . If  $|S_0| = n_0$  and  $|C_0| = m_0$ , then direct calculation shows that

$$\overleftrightarrow{E_0} = \frac{\alpha n_0}{\gamma n_0 + 2\beta m_0} \text{ and } \overleftrightarrow{O_0} = \frac{\alpha m_0}{\gamma n_0 + 2\beta m_0}$$

These values depend only on the source/city ratio. For two graphs  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  and  $g_1 = \langle S_1 \cup C_1, L_1 \rangle$ ,

$$\frac{|S_0|}{|C_0|} = \frac{|S_1|}{|C_1|} \Rightarrow \overleftrightarrow{E_0} = \overleftrightarrow{E_1} \text{ and } \overleftrightarrow{O_0} = \overleftrightarrow{O_1}$$

We will use the efficient levels of the complete graph as benchmarks while calculating the efficient amounts in non-complete bipartite graphs.

The feasible set of flows in a graph  $g_0$  is a subset of the feasible set of flows in its completed graph  $\overleftrightarrow{g_0}$ . Then given efficient levels of extraction  $\overleftrightarrow{E_0}$  and outflow  $\overleftrightarrow{O_0}$  at  $\overleftrightarrow{g_0}$ , if these amounts are possible in  $g_0$ , then they must be efficient for  $g_0$  also.

**Proposition 6** *Let  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  be a connected subgraph of  $g$ . If the extraction of  $\overleftrightarrow{E_0}$  by each city in  $C_0$  is possible without exceeding the outflow  $\overleftrightarrow{O_0}$  in any source in  $S_0$ , then these levels are efficient in  $g_0$ .*

Now we show that if a subgraph  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  of  $g$  has no minimally invasive subgraph, then the extraction of  $\overleftrightarrow{E_0}$  by each city in  $C_0$  is possible without exceeding the outflow  $\overleftrightarrow{O_0}$  in any source in  $S_0$

**Proposition 7** *Let  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  of  $g$  be an invasive subgraph. If  $g_0$  has no minimally invasive subgraphs, then the extraction of  $\overleftrightarrow{E_0}$  by each city in  $C_0$  is possible without exceeding the outflow  $\overleftrightarrow{O_0}$  in any source in  $S_0$ .*

We prove Proposition 4 by induction. We start with a city  $c_j$  of a graph  $g_0$  with no invasive subgraphs. This city must be able to extract  $\overleftrightarrow{E_0}$ , without exceeding the outflow  $\overleftrightarrow{O_0}$  in any of its sources. If not, that city with its sources would have formed a minimally

invasive subgraph in  $g_0$ . Next, we add a new city to this subgraph and iteratively show that such extractions must be possible for all invasive subgraphs of  $g_0$  that contain  $c_j$ . As  $g_0$  is an invasive subgraph of itself, this proves that such extractions are possible in  $g_0$ .

To prove the iteration we manipulate the flows in the following way. Let  $\alpha = \frac{7}{2}, \beta = 1$  and  $\gamma = 1$ . Suppose that our graph is  $g_4$  in Figure 5.

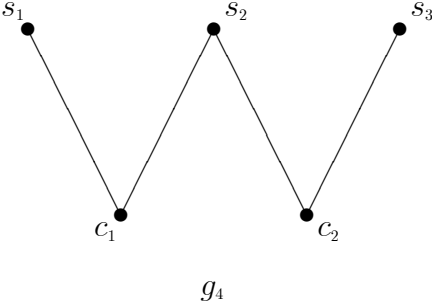


Figure 8

Then according to Proposition 4, extraction 1.5 by each city is possible without exceeding outflow 1.0 in any source. Let's take  $c_1$ . Let's take the vector of flows  $(q_{11}, q_{21}) = (0.5, 1)$ . Then  $c_1$  extracts 1.5 without exceeding 1 at any of its sources. Let's depict those flows on the graph, by writing the quantities that correspond to each link.

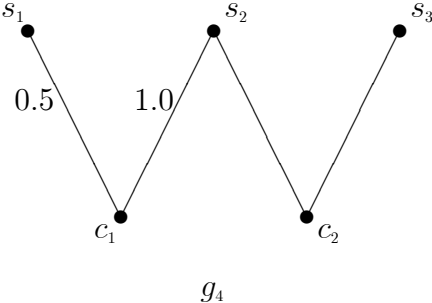


Figure 9

In Figure 6,  $s_2$  supplies 1.0 to  $c_1$ . To extend the argument to the subgraph that contains  $c_2$ , we manipulate the flows, so that the slack in source  $s_1$  can be transferred to city  $c_2$  through the path that connects  $s_1$  with  $c_2$ . Such a change of flows should be possible, because if not,

we could find a minimally invasive subgraph, which leads to a contradiction.

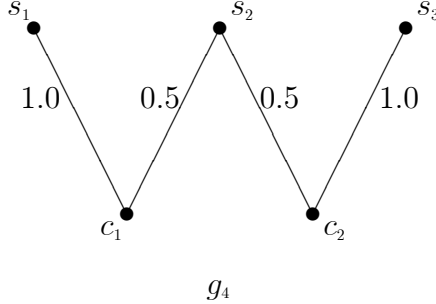


Figure 10

### 6.3 Decomposing the network

Now we will break down the given network  $g$ , so that the commons problem in each subnetwork is independent from the other ones. We will sequentially cut out minimally invasive subgraphs. Hence, they will not have any minimally invasive subgraphs of their own. We will continue until we reach a subgraph which has no minimally invasive subgraphs. Then, given Proposition 4, in each subgraph, the efficient amounts of total extractions at each city and total outflows at each source will be equal to the efficient amounts in their completed graphs.

*Step 1:* Take  $g$ . Suppose  $g = \langle S \cup C, L \rangle$  has no minimally invasive subgraph. Then the efficient total extraction by a city  $c_j$ ,  $E_j(Q_g^e)$ , and the efficient total extraction from a source  $s_i, O_i(Q_g^e)$ , is equal to the extraction of a city in a complete bipartite graph with nodes  $S \cup C$ , and we are done.

Suppose  $g = \langle S \cup C, L \rangle$  has a minimally invasive subgraph  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ . Then, the efficient total extraction by a city  $c_j \in C_0$  is  $\overrightarrow{E}_0$ , and the efficient total extraction from a source  $s_i \in S_0$  is  $\overrightarrow{O}_0$ .

*Step 2:* Now, for the rest of the cities and sources apply *Step 1* to  $g - g_0$ .

In this way we will obtain a series of *regions*  $g_0, g_1 \dots$  of  $g$ , with a non-decreasing source per city ratio. In each of them, the efficient levels of extractions would equal to the levels in their respective completed graphs.

So, given a subgraph  $g_0 = \langle S_0 \cup C_0, L_0 \rangle$  obtained from the above decomposition, the

efficient extraction by a city in  $g_0$  is

$$\overleftarrow{E}_0 = \frac{\alpha n_0}{\gamma n_0 + 2\beta m_0}$$

and the efficient outflow from each source in  $g_0$  is

$$\overleftarrow{O}_0 = \frac{\alpha m_0}{\gamma n_0 + 2\beta m_0}$$

These levels satisfy the first order conditions within each region. Moreover, less resourceful regions have lower amounts of extractions per city and higher amounts of outflows per source. Since there are no flows between different regions the first order conditions hold for graph  $g$  as well.

The idea of redundant links reappears while calculating efficiency. Take two graphs  $g$  and  $g'$  such that their decomposition gives the same subgraphs. The efficient amounts of total extractions at each city and total outflows at each source are the same for both of them.

**Example** Suppose we have the graph  $g_1$ . Let  $\alpha = \beta = \gamma = 1$ . The decomposition would give us two regions,  $g_1^1$  and  $g_1 - g_1^1$ . Then the efficient flows are

$$\{q_{11}^e, q_{12}^e, q_{13}^e, q_{23}^e, q_{33}^e \geq 0 : q_{11}^e = \frac{1}{5}, q_{12}^e = \frac{1}{5}, q_{13}^e = 0, q_{23}^e = \frac{1}{4} \text{ and } q_{33}^e = \frac{1}{4}\}$$

Suppose the graph was  $g_1 - (1, 3)$ . The decomposition leads to the same subgraphs. The efficient flows are

$$\{q_{11}^e, q_{12}^e, q_{23}^e, q_{33}^e \geq 0 : q_{11}^e = \frac{1}{5}, q_{12}^e = \frac{1}{5}, q_{23}^e = \frac{1}{4} \text{ and } q_{33}^e = \frac{1}{4}\}$$

The link  $(1, 3)$  is redundant from an efficiency point of view, just as it was for equilibrium.

## 7 Conclusion

We have analyzed a situation where the tragedy of the commons is embedded in a network. We have shown that when players have concave valuations, their equilibrium actions will depend on the whole structure. The quantity extracted by a user from a source depends on the centrality of the links she has. The centrality index which determines the quantities is calculated using the equilibrium dual of the original network. Then the quantity flowing

from a resource to a city is positively proportional to the total number of even paths and negatively proportional to the total number of odd paths starting from it.

We next characterize the efficient amounts of extractions. Again when players have concave valuations, these amounts depend on the whole network. We find a network decomposition which separates the efficiency problem into subgraphs. These subgraphs, which we call regions, are taken out from the network starting with the one with the lowest source/city ratio. Each region consumes only from its sources. The sources are distributed between regions, so that the less resourceful ones are assigned to the most possible number of sources.

The model we studied can also be used to analyze Cournot competition among firms which are linked through markets. If we think of cities as firms with quadratic costs, and sources as markets with linear demands, the results in this paper shows what the equilibrium quantities would be in such a setup. The efficiency in our story would be equivalent to the profit maximization of a cartel that the suppliers might form.

A further research agenda would be the case where the sources, which can be thought as exporters of the resources, behave strategically as well. Their strategies can be prices they charge and/or quantities they sell through each of their links. The users would be the consumers of the market, buying according to the prices charged. Such a model would be a close approximation of the international petrol and natural gas markets.

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## Appendix

**Proof of Proposition 1** We first show that for the matrix  $D$  we can find a matrix  $R$  with independent columns such that  $D = R^T R$ .<sup>7</sup> We will write columns of  $R$  so that the entries in  $D$  appear in squareroots in  $R$ . For example, let's take  $D_{4 \times 4}$ :

$$D_{4 \times 4} = \begin{bmatrix} 2\beta + \gamma & \gamma & \beta & 0 \\ \gamma & 2\beta + \gamma & 0 & \beta \\ \beta & 0 & 2\beta + \gamma & \gamma \\ 0 & \beta & \gamma & 2\beta + \gamma \end{bmatrix}$$

We write  $R$  as

$$R = \begin{bmatrix} \sqrt{\beta} & 0 & 0 & 0 \\ 0 & \sqrt{\beta} & 0 & 0 \\ 0 & 0 & \sqrt{\beta} & 0 \\ 0 & 0 & 0 & \sqrt{\beta} \\ \sqrt{\gamma} & \sqrt{\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{\gamma} & \sqrt{\gamma} \\ \sqrt{\beta} & 0 & \sqrt{\beta} & 0 \\ 0 & \sqrt{\beta} & 0 & \sqrt{\beta} \end{bmatrix}$$

Then clearly  $D_{4 \times 4} = R^T R$ . Now, we generalize this to all possible  $D$ .

Let  $R = [r_{ij}]_{[3(m \times n)] \times (m \times n)}$  be such that,

$$r_{ij} = \begin{cases} \sqrt{\beta} & , \text{ for } i = j \\ \sqrt{\frac{\gamma}{n}}, & \text{ for } i \neq j, \text{ s.t. } (i, j) = (z_1 n + (m \times n) + z_2, z_1 n + z_3) \text{ for } z_1, z_2, z_3 \in \mathbb{N} \\ & \text{s.t. } 1 \leq z_2, z_3 \leq n \text{ and } z_1 \leq m - 1 \\ \sqrt{\frac{\beta}{m}}, & \text{ for } i \neq j, \text{ s.t. } (i, j) = (z_2 m + 2(m \times n) + z_1 + 1, z_3 n + z_2 + 1), \text{ for } z_1, z_2, z_3 \in \mathbb{N} \\ & \text{s.t. } z_1, z_3 \leq m - 1, \text{ and } z_2 \leq n - 1 \\ 0 & , \text{ otherwise} \end{cases}$$

If we let  $K = \sqrt{\beta} I_{(m \times n) \times (m \times n)}$ ,  $L = \sqrt{\frac{\gamma}{n}} \mathbf{1}_{n \times n}$ , where  $\mathbf{1}_{n \times n}$  is the square matrix of 1's of size  $n$ , and we define  $M = [m_{ij}]_{(m \times n) \times n}$  such that,

---

<sup>7</sup>This is equivalent to checking that  $D$  is positive definite. For other characterizations of positive definiteness see Strang (1988).





**Proof of Proposition 2** For any  $J \subset \mathbb{N}_+$ , we can find a matrix  $R_J$  such that  $F_{-J} = (R_J)^T R_J$ .  $F_{-J}$  has dimension  $m \times n - |J|$ . For any  $J \subset \mathbb{N}_+$ , there exists a bipartite graph  $g(J)$  such that  $F_{-J}$  is derived by deleting from  $F$  the columns and rows that correspond to the links which are missing in  $g(J)$ . Now, let  $R_J = [t_{ij}]_{[3(m \times n - |J|)] \times (m \times n - |J|)}$  be such that,

$$r_{ij} = \begin{cases} \sqrt{\frac{\gamma}{n_{z_1}(g)}}, & \text{for } i \neq j, \text{ s.t. } (i, j) = \left( \sum_{0 \leq k < z_1} n_k(g) + (m \times n - |J|) + z_2, \sum_{0 \leq k < z_1} n_k(g) + z_3 \right) \\ & \text{for } z_1, z_2, z_3 \in \mathbb{N} \text{ s.t. } 1 \leq z_2, z_3 \leq n_{z_1}(g) \text{ and } 1 \leq z_1 \leq m \\ \\ \sqrt{\frac{2\beta}{m_{z_1}(g)}}, & \text{for } i \neq j, \text{ s.t. } (i, j) = \left( \sum_{0 \leq k < z_1} m_k(g) + z_2 + 2(m \times n - |J|), \sum_{k=0}^{z_3} m_{z_3}(g) + 1 \right), \\ & \text{for } z_1, z_2, z_3 \in \mathbb{N} \text{ s.t. } 1 \leq z_1, z_3 \leq m, \text{ and } 1 \leq z_2 \leq m_{z_1}(g) \\ \\ 0 & , \text{ otherwise} \end{cases}$$

If we let  $L_i = \sqrt{\frac{\gamma}{n_i(g)}} \mathbf{1}_{n_i(g)}$ , where  $\mathbf{1}$  is the square matrix of 1. And for  $k \in \{1, \dots, m\}$ , we define  $[m_{ij}^k]_{(m \times n - |J|) \times n_k(g)}$  such that,

$$m_{ij}^k = \begin{cases} \sqrt{\frac{2\beta}{m_{z_1}(g)}}, & \text{for } i \neq j, \text{ s.t. } (i, j) = \left( \sum_{0 \leq k < z_1} m_k(g) + z_2, z_1 + 1 \right), \text{ for } z_1, z_2 \in \mathbb{N} \\ & \text{s.t. } 1 \leq z_1 \leq m, \text{ and } 1 \leq z_2 \leq m_{z_1}(g) \\ \\ 0 & , \text{ otherwise} \end{cases}$$

Then  $R_J$  can be written as a partitioned matrix,

$$R_J = \begin{bmatrix} & & & 0 & & \\ & L_1 & & & & \\ & & \cdot & & 0 & \\ & & & \cdot & & \\ & & & 0 & & \cdot \\ & & & & & L_m \\ M^1 & & \dots & & & M^k \end{bmatrix}_{[3(m \times n - |J|)] \times (m \times n - |J|)}$$

It is straight forward to check that  $F_{-J} = (R_J)^T R_J$ . ■

**Proof of Theorem 1** As the first order condition for  $q_{ij}$  does not depend on the amounts extracted from sources other than  $s_i$ , the optimization problem is separable source by source. Meaning that equilibrium extractions from a source  $s_i$  depend only on how many players are connected to  $s_i$ .

The equilibrium at each source is unique, because for all players at a source  $s_i$ ,

$$q_{ij} = \frac{\alpha - \beta \sum_{c_k \in N_g(s_i) \setminus \{c_j\}} q_{ik}}{2\beta}$$

For all links  $(i, j) \in L$ ,

$$\frac{\partial^2 u_j}{\partial q_{ij}^2} = -2\beta < 0$$

meaning that the second order conditions are satisfied for all cities.

Around each source  $s_i$  there is a symmetric amount of flow at equilibrium, such that

$$q_{ij}^* = \frac{\alpha}{(m_i(g) + 1)\beta}$$

■

**Proof of Theorem 2** Given a graph  $g$ , the equilibrium conditions of the game is a  $LCP(-\alpha \mathbf{1}_r; D_g)$  where  $\mathbf{1}$  is a column vector of 1's of size  $r$ .

$$\begin{aligned} Q_g &\geq 0, \\ -\alpha \mathbf{1}_r + D_g Q_g &\geq 0, \\ Q_g^T (q + D_g Q_g) &\geq 0 \end{aligned}$$

Samelson *et al.* (1958) shows that a linear complementarity problem  $LCP(p; M)$  has a unique solution for all  $p \in \mathbb{R}^t$  if and only if all the principal minors of  $M$  are positive. Positive definite matrices satisfy this condition and we showed in Proposition 1 that  $D_g$  is positive definite. Then the first order equilibrium conditions have a unique solution.

Let's check the second order condition. For city  $c_k$ , denote the Hessian matrix of  $u_k$  by  $H_{u_k} = [h_{ij}]_{n_k(g) \times n_k(g)}$  such that

$$h_{ij} = \begin{cases} -2\beta - \gamma, & \text{for } i = j \\ -\gamma, & \text{for } i \neq j \end{cases}$$

We will show that for any  $z \in \mathbb{N}_+$ , the matrix  $H_z = [h_{ij}]_{z \times z}$  such that

$$h_{ij} = \begin{cases} -2\beta - \gamma, & \text{for } i = j \\ -\gamma, & \text{for } i \neq j \end{cases}$$

is negative definite.

$H_z = -(2\beta + \gamma)H'_z$ , where  $H'_z = [h'_{ij}]_{z \times z}$  such that,

$$h'_{ij} = \begin{cases} 1, & \text{for } i = j \\ \phi, & \text{for } i \neq j \end{cases}, \text{ where } \phi = \frac{\gamma}{2\beta + \gamma}$$

If we denote the determinant of  $H_z$  by  $Det(H_z)$ .

$$Det(H_z) = (2\beta + \gamma)(-1)^n Det(H'_z)$$

Now, we show by induction that for all  $z \in \mathbb{N}_+$ ,  $Det(H'_z) > 0$ .

$Det(H'_1) = 2\beta + \alpha > 0$ . Assume  $Det(H'_{z-1}) > 0$ .

$$Det \begin{pmatrix} 1 & \phi & \dots & \dots & \phi \\ \phi & 1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \phi & \phi & \dots & \dots & 1 \end{pmatrix} = \left(1 - \frac{\phi^2(n-1)}{1 + (n-2)\phi}\right) Det(A'_{z-1})$$

Then,  $H_z$  is negative definite and the water extraction game with concave values has a unique Nash equilibrium. ■

**Proof of Theorem 3** Assume  $Q_{g-Z(Q_g)}^*$ ,  $Q_{g'-Z(Q_{g'})}^*$  are equilibria of the game at  $g$  and  $g'$ , respectively. Let

$$g - Z(Q_g^*) = g' - Z(Q_{g'}^*)$$

Then,

$$D_{g-Z(Q_g^*)} \cdot Q_{g-Z(Q_g^*)}^* = \alpha \cdot \mathbf{1} = D_{g'-Z(Q_{g'}^*)} \cdot Q_{g'-Z(Q_{g'}^*)}^* = D_{g-Z(Q_g^*)} \cdot Q_{g'-Z(Q_{g'}^*)}^*$$

As we showed in Proposition 1  $D_{g-Z(Q_g^*)}$  is positive definite, hence invertible.

$$Q_{g-Z(Q_g)}^* = Q_{g-Z(Q_g'')}^*$$

■

**Proof of Proposition 3** We know that the extraction of  $\overleftrightarrow{E}_0$  and the outflow  $\overleftrightarrow{O}_0$  satisfies the first order conditions in  $\overleftrightarrow{g}_0$ . Since  $g_0$  and  $\overleftrightarrow{g}_0$  have the same set of nodes, they also satisfy the conditions in  $g_0$ . ■

**Proof of Proposition 4** By assumption,  $g_0$  has no minimally invasive subgraphs.

Take a city  $c_j$  in  $g_0$ . Let  $c_j$  extract a total of  $\overleftrightarrow{E}_0$ , such that none of the sources supply more than  $\overleftrightarrow{O}_0$ .  $\overleftrightarrow{E}_0$  and  $\overleftrightarrow{O}_0$  are functions of the source/city ratio. If  $c_j$  is not linked to enough sources to achieve such an extraction, then city  $c_j$  and the sources  $N_g(c_j)$  form a minimally invasive subgraph in  $g_0$ , which is a contradiction with  $g_0$  having no minimally invasive subgraphs.

Now, we are going to show by induction that  $\overleftrightarrow{E}_0$  extraction by a city in  $g_0$  such that no source supplies more than  $\overleftrightarrow{O}_0$  is possible in any invasive subgraph of  $g_0$  that contains  $c_j$ . As  $g_0$  is an invasive subgraph of itself, this will imply that such levels of extraction is possible in  $g_0$ .

We know that it is possible for the invasive subgraph with city  $c_j$  and the sources  $N_g(c_j)$ . Take an invasive subgraph  $g_{k-1}$  of  $g_0$  that contains  $k - 1$  cities including  $c_j$ . Suppose that such levels of extractions are possible in  $g_{k-1}$ . Denote by  $Q_{g_{k-1}}$  such a possible amount of flows in  $g_{k-1}$ .

Now take an invasive subgraph  $g_k$  of  $g_0$  that contains  $k$  cities,  $k - 1$  which were in  $g_{k-1}$  and a fixed city  $c_k$  which was not in  $g_{k-1}$ .

Assume that in  $g_k$ ,  $\frac{|\hat{S}_k|}{|\hat{C}_k|} < \frac{|\hat{S}|}{|\hat{C}|}$ . Then  $g_k$  is a minimally invasive subgraph of  $g_0$ , which is a contradiction.

Then,  $\frac{|\hat{S}_k|}{|\hat{C}_k|} \geq \frac{|\hat{S}|}{|\hat{C}|}$ . Take  $Q_{g_{k-1}}$  which delivers  $\overleftrightarrow{E}_0$  to all cities in  $g_{k-1}$ . As  $g_k$  contains  $g_{k-1}$  we can supply the cities in  $g_{k-1}$  with  $\overleftrightarrow{E}_0$  without exceeding outflow  $\overleftrightarrow{O}_0$  in any source. Now let  $c_k$  extract through its links such that the outflow from each source in  $N_g(c_k)$  is  $\overleftrightarrow{O}_0$ . If the total extraction of  $c_k$  exceeds  $\overleftrightarrow{E}_0$ , then we are done.

If not, denote by  $Q^1$  the flow vector for  $g_k$  such that flows for the links which were already in  $g_{k-1}$  equals to  $Q_{g_{k-1}}$ , and the flows for the links which were not in  $g_{k-1}$  equals to 0. Now,

given that  $c_k \notin g_{k-1}$ , let  $Q^2$  be the flow vector for  $g_k$  such that

$$\begin{aligned} q_{jk}^2 &= \overrightarrow{O_0} - O_i(Q^1), \text{ for } j \in N_g(c_k) \\ q_{jl}^2 &= q_{jl}^1, \text{ for } l \neq k \end{aligned}$$

Since  $\frac{|\hat{S}_k|}{|\hat{C}_k|} \geq \frac{|\hat{S}|}{|\hat{C}|}$ , there must be a source  $s_i$  in  $g_k$  not connected to  $c_k$ , such that its outflow in  $Q^2$  is strictly less than  $\overrightarrow{O_0}$ . Let  $S_k^-$  be the set of sources in  $g_k$  which not connected to  $c_k$  and which have outflows in  $Q^2$  strictly less than  $\overrightarrow{O_0}$ .

$$S_k^- = \{s_i \in g_k : s_i \notin N_g(c_k) \text{ and } O_i(Q^2) < \overrightarrow{O_0}\}$$

Suppose that for any source  $s_i \in S_k^-$  and for all paths

$$P = \{(s_i, c_1), (c_1, s_1), \dots, (c_t, s_t), (s_t, c_k)\}$$

that connects  $s_i$  with  $c_k$ ,  $\exists (c_j, s_j) \in P$  such that  $q_{jj}^2 = 0$ . Given such a path  $P$ , let  $s_P$  denote the source  $s_l$  such that  $(c_l, s_l) \in P$ ,  $q_{ll}^2 = 0$  and there exists no other source  $s_j$  in  $P$ , closer to  $c_k$  than  $s_l$  such that  $(c_j, s_j) \in P$  and  $q_{jj}^2 = 0$ . Let  $\overline{C}_k = \{c_j \in g_k : \exists \text{ a path } P \text{ from } s_i \text{ to } c_k \text{ for some } s_i \in S_k^- \text{ and in } P, c_j \text{ is between } s_P \text{ and } c_k\}$ . Then the invasive subgraph with cities  $\overline{C}_k \cup c_k$  is minimally invasive in  $g_k$ , which is a contradiction.

Then there exists a source  $s_i \in S_k^-$  such that there exists a path

$$P = \{(s_i, c_1), (c_1, s_1), \dots, (c_t, s_t), (s_t, c_k)\}$$

that connects  $s_i$  with  $c_k$  and  $\min_{(c_j, s_j) \in P} q_{jj}^2 \neq 0$ . Let

$$d = \min_{(c_j, s_j) \in P} \{q_{jj}^2, O_i(Q^2)\}$$

Now, given such a path  $P$ , let  $Q^3$  be the flow vector for  $g_k$  such that

$$\begin{aligned} q_{i1}^3 &= q_{i1}^2 + d, \\ q_{jj}^3 &= q_{jj}^2 - d, \\ q_{j(j+1)}^3 &= q_{j(j+1)}^2 + d \\ q_{tk}^3 &= q_{tk}^2 + d \\ q_{ll'}^3 &= q_{ll'}^2, \text{ for all other links } (l, l') \end{aligned}$$

It is possible to make  $c_k$  extract at least  $\overleftarrow{E}_0$  by finding such paths from sources in  $\hat{S}_k^-$  to  $c_k$  and changing the flows as explained above for each path from a source in  $\hat{S}_k^-$  to  $c_k$ . If after using all such paths,  $c_k$  could still not extract  $\overleftarrow{E}_0$ , then we could use the reasoning above to get a contradiction.

Then the desired levels of extractions are possible in  $g_0$ . ■