Contributing or Free-Riding? A Theory of Endogenous Lobby Formation*

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Abstract

We consider a two-stage public good provision game: In the first stage, players simultaneously decide if they join a contribution group or not. In the second stage, players in the contribution group simultaneously offer contribution schemes in order to influence the government's choice on the level of public good provision. Using a communication-based self-enforcing equilibrium concept, *perfectly coalition-proof Nash equilibrium* (Bernheim, Peleg and Whinston, 1987 JET), we show that the the set of equilibrium outcomes is equivalent to an "intuitive" hybrid solution concept *free-riding-proof core*, which is always non-empty but does not necessarily achieve global efficiency. It is not necessarily true that equilibrium lobby group is formed by the highest willingness-to-pay players, nor is a consecutive group with respect to their willingnesses-to-pay. We also show that equilibrium public good provision level shrinks to zero as the economy is replicated.

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1 Introduction

This paper considers a public goods provision problem in two stages with a menu auction. Menu auction game by Bernheim and Whinston (1986) is now commonly employed in public decision problems (political economy models with lobbying), especially in the field of international trade (Grossman and Helpman, 1994). Lobbying for protection within an industry can be considered as a public good provision model by way of lobbying. Since public good provision affect all players positively, there are free-riding motives among players. This makes the lobby formation problem interesting. Our game goes as follows: in the first stage, players decide if they join a contribution group (a lobby), and in the second stage, the participants offers their contribution schemes (menus) to the government, and the government decides how much to produce dependent on the offered contribution schemes and costs of public goods provision. With this game, the questions we ask are: "how does equilibrium lobby group looks like?" and "how efficient is the equilibrium outcome?"

The set of Nash equilibria of our second stage game (a "common agency game" or "menu auction game") by Bernheim and Whinston (1986) is very large and contains many unreasonable equilibria. In order to refine it, Bernheim and Whinston (1986) define a communication-based equilibrium concept, **coalition-proof Nash equilibrium (CPNE)**, and provide a nice characterization of CPNE. In fact, since public good provision involves a coordination problem among players, it clearly makes sense to employ communication-based refinement of Nash equilibria. In order to analyze our two stage game, we employ **perfectly coalition-proof Nash equilibrium (PCPNE)** that is a natural extension of CPNE to dynamic games (Bernheim, Peleg and Whinston, 1987).

We characterize PCPNEs of our game with a new hybrid solution concept by utilizing the core in cooperative game theory. It is not a surprise that there are connections between menu auction outcomes and the core. Laussel and Le Breton (2001) show that in the class of comonotonic games,[1] the generated cooperative games are convex, and the equivalence between CPNE and the core holds. We add a lobby formation stage on Laussel and Le Breton (2001), and characterize PCPNE in order to analyze a participation problem. A **free-**

[1] Preferences are **comonotonic** if for all pair of players i and j, and all pair of actions a and a', if i prefers a to a', then j also prefers a to a'.

riding-proof core allocation for coalition S (FRP-Core allocation for S) is a core allocation achieved by contributor group S, in which no member i of S has an incentive to deviate in expectation of the public good provision level becoming the efficient level for group $S \backslash \{i\}$. A **free-riding-proof core for S (FRP-Core for S)** is collection of FRP-Core allocations for S. The FRP-Core for S is collection of internally stable allocations (no lobby member free-rides given the surplus allocation scheme). Note that it is easily possible to have an empty FRP-Core for S if S is a large coalition. The **free-riding-proof core (FRP-Core)** is the *Pareto-efficient frontier* of the union of FRP-Cores for all $S \subseteq N$. That is, FRP-Core is a collection of internally stable allocations that are not Pareto-dominated by any other internally stable allocations. Our Theorem 1 proves that PCPNE and FRP-Core is equivalent by heavily utilizing the properties of the core in convex games by Shapley (1971).

This equivalence theorem is useful in analyzing PCPNE of our game. We fully analyze the set of FRP-Core allocations of a simple example in which players differ only in their willingnesses-to-pay for a public good, and show that (i) there can be many different equilibrium lobbies, (ii) an equilibrium lobby may not include the highest willingness-to-pay player, and (iii) the members of an equilibrium lobby may not be consecutive in their willingnesses-to-pay.

Then, we also analyze how equilibrium public good provision is affected as economy gets larger. By following Milleron's (1972) notion of replicating a public good economy,[2] we prove that the equilibrium public good provision levels converge to zero as economy gets larger (Theorem 2).

This paper is organized as follows. In the next two subsections, some related literature is discussed briefly. In Section 2, the common agency game is reviewed, then our game and the equilibrium concept, PCPNE, is introduced. In Section 3, we consider the environment without conflict of interests. We define an intuitive hybrid solution concept, free-riding-proof core, and prove the equivalence between PCPNE and the free-riding-proof core (Theorem 1). In Section 4, we provide an example that describes how free-riding-proof core looks like. In Section 5, we consider a replica economy, and shows that the

[2] Muench (1972), Milleron (1972) and Conley (1994) discuss the difficulty of replicate public good economy and offers various possible methods. Milleron's notion of replication is as follows: by splitting endowments with replicates and adjusting preferences so that agents' concerns for the private good are relative to the size of their endowments. This notion is employed by Healy (2007).

public good provision level shrinks to zero as the economy is replicated in a certain way (Theorem 2). Section 6 concludes. Appendix A provides useful properties of the core of convex games and an algorithm that finds a core allocation starting an arbitrary utility, and Appendix B provides involved proofs.

1.1 Related Literature on International Trade

In their seminar paper, Grossman and Helpman (1994) consider an endogenous trade policy formation problem when industries can influence the government's trade policy through lobbying activities by applying common agency game defined by Bernheim and Whinston (1986). In Grossman and Helpman (1994), players/principals are lobbies who represents industries, and an agent is the government. The government cares about social welfare, while it also cares about flexible contribution money provided by lobby groups. Each lobby contributes money to the government in order to influence the government's trade policy for its favor. Each lobby represents one industry, and it prefers a high price for a commodity that is produced by the industry, while prefers low prices for all other commodities.[3] One of their main results is that in equilibrium lobby powers are cancelled out and that the government chooses a free trade (no tariff) policy, it can collect a big amount of contributions from conflicting industries.

Mitra (1999) endogenizes lobby participation using the Grossman-Helpman model. In his model, lobby participation is decided by each industry, and there is no free-riding incentive within the same industry. He shows that Grossman-Helpman's free trade result remains if the government care about social welfare strongly or care about contributions heavily. In contrast, Bombardini (2005) and Paltseva (2006) consider an oligopolistic import competing industry case in which many firms decide lobbying or free-riding. Unlike Grossman and Helpman (1994) and Mitra (1999), these firms in the same industry have no conflict of interests over the government policies like in pure public good provision problem. Introducing firms that differ in amount of specific capital, Bombardini (2005) empirically investigate how protection levels differ across industries dependent of the distributions of firm sizes introducing individual fixed cost of participating in the lobby. She finds that industries characterized by higher firm size dispersion obtains a higher level

[3]This is because lobbies representing industries are ultimately consumers.

of protection. Although her empirical result is very interesting, she assumes that the most efficient lobby group is formed: she assumes that firms enter the lobby in the order of amount of capital: the highest capital firm enters contributing to maximize its benefit, then the second highest firm enters adding contribution to achieve efficiency, and so on until efficiency benefit of adding a firm becomes less than the firm's individual cost of lobby participation. Indeed, we show that in equilibrium, it is not necessary that equilibrium lobby includes the most efficient firm, nor that equilibrium lobby is consecutive. In contrast, assuming symmetric firms and focusing on symmetric outcome among lobby participants in a common agency game, Paltseva (2006) consider Nash equilibrium of lobby participation game to analyze free-riding incentives. Our paper is closest to Paltseva's, but we allow asymmetric players and asymmetric contributions, and characterizes PCPNEs. Due to transferable utilities, we need to employ more sophisticated equilibrium concept than Nash equilibrium in the participation stage if symmetry assumption is dropped. This is the reason that we use PCPNE as our solution concept.

1.2 Related Literature on Public Good Provision

It is well-known that the public good provision is subject to free-riding incentives. Although Samuelson's (1954) view to this problem was pessimistic, Groves and Ledyard (1977) showed that efficient public good provision can be achieved in Nash equilibrium. Although the Groves-Ledyard mechanism does not satisfy individual rationality, Hurwicz (1979), and Walker (1981) succeeded in showing that the Lindahl mechanism is implementable. Subsequently, numerous mechanisms have been proposed to improve the properties of mechanisms. However, they all assume that players have no freedom to make participation decision to the mechanism: players' participation to the mechanism is assumed.

Introducing outside opportunity by a "reversion function" (each outcome is mapped to another outcome in case of no participation), Jackson and Palfrey (2001) analyze the implementation problem including participation of all players when players' participation to a mechanism is voluntary. They extend Maskin monotonicity condition to accommodate voluntary participation condition. Although their reversion function is very general, it assigns the same outcome whoever deviates from the original outcome. Thus, it may not be suitable for a public good provision problem since different players' deviations from participation may generate different outcomes. Taking this consideration into account, Healy (2007) analyze the implementation problem in a public good economy demanding all players' participation in equilibrium of the game. He shows that as economy is replicated in Milleron's sense (1972), the outcomes of any mechanism that satisfies equilibrium participation condition converge to the endowment. Although we also show that the equilibrium public good provision level converges to zero as economy is replicated, we allow some players not to participate in the lobby in equilibrium (and efficiency of public good provision within the lobby group is achieved unlike in Healy, 2007). Thus, Healy's and our results are quite different from each other.

The most related paper to the current work is Saijo and Yamato (1999), which is the first to consider a voluntary participation game with two stages in public good economy without requiring all players' participation in equilibrium. They show a negative result on efficiency of public good provision, and then characterize subgame perfect equilibria in symmetric Cobb-Douglas utility case. In contrast, our domain is quasi-linear utility space, and we fully characterize PCPNE of common agency game with participation decision allowing heterogeneous players.[4]

Le Breton and Salaniè (2003) analyze a common agency problem with asymmetric information on agent's preferences. They show that equilibria can be inefficient even in the case that there is only one player in each interest group.[5] If there are multiple players in each interest group, then the failure in internalizing the benefits of contributions within the group makes contributions even less. In this sense, Le Breton and Salaniè (2003) generate free-riding incentives under compulsory lobby participation. In contrast, we generate "free-riding" is a more obvious way by introducing participation decisions.

[4]Shinohara (2003) considers coalition-proof Nash equilibrium in the voluntary participation game by Saijo and Yamato (1999) with the Lindahl mechanism in the second stage. He shows that there can be multiple coalition-proof Nash equilibria with different sets of players participating in the mechanism in heterogenous player case. One of our results exhibits the same result but with a common agency game in the second stage (thus, payoff allocation within lobby is flexible unlike in Shinohara, 2003).

[5]Laussel and Le Breton (1998) analyze public good case when the agent must sign a contract of participation when all contribution schemes are proposed before knowing her cost type (then Nature plays and the agent chooses an agenda). They show that all equilibria are efficient, and there is no free-riding incentive.

2 The Model

In this section, we consider a case in which all players' interests are in the same direction, while the intensity of their interests can be heterogeneous. We will describe the problem, then propose a hybrid solution concept: free-riding-proof core.

2.1 Public Good Provision Problem with Voluntary Participation

A stylized public good model is defined as follows: Public good is one-dimensional, and public good provision level is denoted by $a \in A = \mathbb{R}_+$.[6] Public good provision cost function $C : A \to \mathbb{R}_+$ is a C^2 function with $C(0) = 0$, $C'(a) > 0$ and $C''(a) > 0$ (for uniqueness: for simplicity). Player i's utility function is quasi linear in private good net consumption x and is written as $v_i(a) - x$, where $v_i : A \to \mathbb{R}_+$ is $v_i(0) = 0$, $v_i'(a) > 0$ and $v_i''(a) \leq 0$. In order to guarantee the existence of solution, we assume the Inada condition on the cost function: $\lim_{a \to 0} C'(a) = 0$ and $\lim_{a \to \infty} C'(a) = \infty$. The only new element is that a consumer has choice between participating in contributing to public good provision and free-riding.

2.2 Lobby Formation Game

In this section, we analyze an equilibrium lobby group and its allocation. Note that we are not only talking about coalition-proof Nash equilibrium allocation in the menu auction stage. We also require that the lobby group formation itself is coalition-proof as well. In order to do so, we first need to define the first stage lobby formation game in an appropriate manner, assuming that the outcome of each possible lobby S is a coalition-proof Nash equilibrium of a common agency game played by S. As an extension of CPNE in strategic form games to extensive form games, Bernheim, Peleg and Whinston (1987) provide a definition of coalition-proof Nash equilibrium for multi-stage games, *perfectly coalition-proof Nash equilibrium (PCPNE)*. The first stage **lobby formation game** is such that N is the set of players,

[6] For our equivalence result (Theorem 1), we only need comonotonic preferences over abstract agenda set A. The extension is straightforward, we chose to use the simple one dimensional public good economy.

and player i's action set is a list $\Sigma_i^1 = \{0, 1\}$: i.e., player i announces her participation decision, where 0 and 1 represent non-participation and participation, respectively. Once action profile $\sigma^1 = (\sigma_1^1, ..., \sigma_n^1) \in \Sigma^1 = \Pi_{j \in N} \Sigma_j^1$ is determined, then in the second stage, lobbying game takes place with the set of active players $S(\sigma^1) = \{i \in N : \sigma_i^1 = 1\}$.

The second stage game is a **common agency game** (or a **menu auction game**) played by participating principals $S(\sigma^1)$ (Bernheim and Whinston, 1986). Thus, $N \backslash S(\sigma^1)$ is the set of passive free-riders. Each player $i \in S(\sigma^1)$ simultaneously offers a contribution scheme $\sigma_i^2 : A \to \mathbb{R}_+$. Given the profile of contribution schemes $\sigma_{S(\sigma^1)}^2$, the government G (an agent) chooses a public good provision level $a \in A$ in order to maximize its net payoff

$$u_G(a; (\sigma_i^2(a))_{i \in S(\sigma^1)}) = \sum_{i \in S(\sigma^1)} \sigma_i^2(a) - C(a),$$

where the first term of the RHS is the contribution revenue and the second term is the cost of public good provision. If the government chooses $a \in A$, then player i gets payoff

$$u_i(a; \sigma_i^2(a)) = v_i(a) - \sigma_i^2(a),$$

for $i \in S(\sigma^1)$, and

$$u_i(a) = v_i(a),$$

for $i \notin S(\sigma^1)$. The government's optimal choice is described by

$$a^*(S, \sigma_{S(\sigma^1)}^2) \in \arg\max_{a \in A} u_G(a; (\sigma_i^2(a))_{i \in S(\sigma^1)}).$$

In the game, the government is not a player: it is just a machine that maximizes its payoff given the contribution schemes.[7]

Now, we will define PCPNE for our two-stage game following Bernheim, Peleg and Whinston (1987). Player i's strategy $\sigma_i = (\sigma_i^1, \sigma_i^2) \in \Sigma_i = \Sigma_i^1 \times \Sigma_i^2$ is such that $\sigma_i^1 \in \Sigma_i^1$ denotes i's lobby participation choice, and $\sigma_i^2 \in \Sigma_i^2$ is a function $\sigma_i^2 : \mathcal{S}(i) \to \Sigma_i^2$ if $\sigma_i^1 = 1$, where $\mathcal{S}(i) = \{S \in 2^N : i \in S\}$.[8]

[7]Strictly speaking, since the government may have multiple optimal policy, we need to introduce a tie-breaking rule. However, it is easy to check the set of truthful equilibria (see below) would not depend on the choice of tie-breaking rules.

[8]For notational simplicity, we *trivially* include second stage strategies by non-participants in the strategy profile. Of course, such a non-participant's second stage strategy σ_i^2 is absolutely irrelevant to the outcome, since the government does not receive money from her.

Each player's payoff function is $u_i : \Sigma \to \mathbb{R}$ that is the same payoff function of lobbying game when lobby group S is determined by $S(\sigma^1)$. For $T \subseteq N$, consider a **reduced game** $\Gamma(T, \sigma_{-T})$ that is a game with players in T by letting players in $N \backslash T$ passive players in Γ, who always play σ_{-T}. We also consider **subgames** for all $\sigma^1 \in \Sigma^1$, and **reduced subgames** $\Gamma(T, \sigma^1, \sigma^2_{-T})$ in similar ways. A **perfectly coalition-proof Nash equilibrium** (**PCPNE**) $(\sigma^*, a^*) = ((\sigma_i^{1*}, \sigma_i^{2*})_{i \in N}, a^*)$ is recursively defined as follows:[9]

(a) In a single player, single stage subgame $\Gamma(\{i\}, \Sigma_i^2, \sigma^1, \sigma^2_{-\{i\}})$, strategy $\sigma_i^{2*} \in \Sigma_i^2$ and the agenda chosen by the agent a^* is a **PCPNE** if σ_i^{2*} maximizes u_i via a^*.

(b-1) Let $(n, 2)$ be the numbers of players and stages of games. Pick any pair of positive integers $(m, r) \leq (n, 2)$ with $(m, r) \neq (n, 2)$.[10] For all $T \subseteq N$ with $|T| \leq m$, assume that PCPNE has been defined for all reduced games $\Gamma(T, \sigma_{-T})$ and their subgames $\Gamma(T, \sigma^1, \sigma^2_{-T})$ (if $r = 1$, then only for all reduced subgames $\Gamma(T, \sigma^1, \sigma^2_{-T})$). Then,

(i) for all reduced games $\Gamma(S, \sigma_{-S})$ and their subgames $\Gamma(S, \sigma^1, \sigma^2_{-S})$ with $|S| = n$, $(\sigma^*, a^*) \in \Sigma \times A$ is **perfectly self-enforcing** if for all $T \subset S$ we have (σ_T^*, a^*) is a PCPNE of reduced game $\Gamma(T, \sigma^*_{S \backslash T}, \sigma_{-S})$, and σ_T^{2*} is a PCPNE of reduced subgame $\Gamma(T, \sigma^1, \sigma^{2*}_{S \backslash T}, \sigma^2_{-S})$, and

(ii) for all $S \subseteq N$ with $|S| = n$, (σ_S^*, a^*) is a **PCPNE** of reduced game $\Gamma(S, \sigma_{-S})$ if (σ_S^*, a^*) is perfectly self-enforcing in reduced game $\Gamma(S, \sigma_{-S})$, and there is no other perfectly self-enforcing σ'_S such that $u_i(\sigma'_S, \sigma_{-S}) \geq u_i(\sigma_S^*, \sigma_{-S})$ for every $i \in S$ with at least one strict inequality.

(b-2) Let $(n, 1)$ be the numbers of players and stages of games. Pick any positive integer $m < n$. For any $T \subseteq N$ with $|T| \leq m$, assume that PCPNE has been defined for all reduced subgames $\Gamma(T, \sigma^1, \sigma^2_{-T})$. Then,

[9] Note that in Bernheim, Peleg and Whinston (1987), the definition of PCPNE is based on strictly improving coalitional deviations. However, we adopt a definition based on weakly improving coalitional deviations, since the theorem on menu auction in Bernheim and Whinston (1986) uses CPNE based on weakly improving deviation. For details on these two definitions, see Konishi, Le Breton and Weber (1999).

[10] The numbers n and t represent the numbers of players and stages of a reduced (sub) game, respectively.

(i) for all reduced subgame $\Gamma(S, \sigma^1, \sigma^2_{-S})$ with $|S| = n$, $(\sigma^*, a^*) \in \Sigma \times A$ is **perfectly self-enforcing** if for all $T \subset S$ we have (σ^{2*}_T, a^*) is a PCPNE of reduced subgame $\Gamma(T, \sigma^1, \sigma^{2*}_{S \setminus T}, \sigma^2_{-S})$, and

(ii) for all $S \subseteq N$ with $|S| = n$, (σ^{2*}_S, a^*) is a **PCPNE** of reduced game $\Gamma(S, \sigma^1, \sigma_{-S})$ if (σ^{2*}_S, a^*) is perfectly self-enforcing in reduced subgame $\Gamma(S, \sigma^1, \sigma_{-S})$, and there is no other perfectly self-enforcing $\sigma^{2\prime}_S$ such that $u_i\left(\sigma^1, \sigma^{2\prime}_S, \sigma^2_{-S}\right) \geq u_i\left(\sigma^1, \sigma^{2*}_S, \sigma^2_{-S}\right)$ for every $i \in S$ with at least one strict inequality.

For any $T \subseteq N$ and any strategy profile σ, let $PCPNE(\Gamma(T, \sigma_{-T}))$ denote the set of PCPNE strategy profiles on T for the game $\Gamma(T, \sigma_{-T})$. For any strategy profile (σ, a), a strategic coalitional deviation (T, σ'_T, a') from (σ, a) is **credible** if $(\sigma'_T, a') \in PCPNE(\Gamma(T, \sigma_{-T}))$. A PCPNE is *a strategy profile that is immune to any credible coalitional deviation.* An **outcome allocation** for (σ^*, a^*) is a list $(S, a^*, u) \in 2^N \times A \times \mathbb{R}^N \times \mathbb{R}$, where $S = S(\sigma^{1*})$ and (u, u_G) is the resulting utility allocation for players.

There are two remarks on PCPNE. First, if a coalition T wants to deviate in the first stage, within the reduced game $\Gamma(T, \sigma_{-T})$ (thus keeping the outsiders' strategy profile fixed), it can orchestrate the whole plan of the deviation by assigning a new CPNE to each subgame so that the target allocation (by the deviation) would be attained as PCPNE of the reduced game $\Gamma(T, \sigma_{-T})$.

Second, note that the definition of PCPNE coincides with *coalition-proof Nash equilibrium* (CPNE) in the (static) second stage. Thus, a CPNE needs to be assigned to each subgame. There are useful characterizations of CPNE of common agency game in the literature. Bernheim and Whinston (1986) introduced a concept of truthful strategies, where τ_i is **truthful relative to** $\bar{a}$ if and only if for all $a \in A$ either $v_i(a) - \sigma^2_i(a) = v_i(\bar{a}) - \sigma^2_i(\bar{a})$, or $v_i(a) - \sigma^2_i(a) < v_i(\bar{a}) - \sigma^2_i(\bar{a})$ and $\sigma^2_i(a) = 0$. A **truthful Nash equilibrium** (σ^{2*}_S, a^*) is a Nash equilibrium such that σ^{2*}_i is truthful relative to $a^* \in A$ for all $i \in S$. Bernheim and Whinston (1986) showed that (i) *every truthful equilibrium is a CPNE*, and that (ii) *the set of truthful equilibria and that of CPNE in utility space are equivalent*, and provided a nice characterization of CPNE in utility space. Laussel and Le Breton (2001) further analyzed CPNE in utility space. One of many results in Laussel and Le Breton (2001) provided a beautiful characterization of CPNE under a special (yet very useful) property, a **comonotonic payoff property**: $u_i(a) \geq u_i(a')$ if and only if $u_j(a) \geq u_j(a')$ for all $i, j \in S$ and all $a, a' \in A$. This property applies to an interesting class of problems including the public good provision problem.

Fact. (Laussel and Le Breton, 2001) Consider a common agency problem $\Gamma = (A, S, (\Sigma_i^2, v_i)_{i \in S}, C)$ with a comonotonic payoff property. Then, in all CPNEs of the common agency game, G obtains $u_G = \max_{a \in A} -C(a)$ (no rent property), and the set of CPNE in utility space is equivalent to the core of characteristic function game $(\tilde{V}(T))_{T \subseteq S}$, where $\tilde{V}(T) = V(T) - u_G = \max_{a \in A} \left(\sum_{i \in S} v_i(a) - C(a) \right) - u_G$.[11]

2.3 An Intuitive Hyblid Solution: FRP-Core

We will define an intuitive hybrid solution concept, *free-riding-proof core (FRP-core)*, which is the set of Foley-core allocations[12] that are immune to free-riding incentives and is Pareto-optimal in a constrained sense. The FRP-core is always nonempty in the public good provision problem.

A public good provision problem determines two things: (i) which group provides public good and how much, and (ii) how to allocate the benefits from providing public good among the members of the group (or how to share the cost). Let $S \subseteq N$ with $S \neq \emptyset$. For $T \subseteq S$, let

$$V(S) \equiv \max_{a \in A} \left[\sum_{i \in S} v_i(a) - C(a) \right],$$

and

$$a^*(S) \equiv \arg\max_{a \in A} \left[\sum_{i \in S} v_i(a) - C(a) \right].$$

An **allocation for** S is (S, a, u) such that $u \in \mathbb{R}_+^N$, $\sum_{i \in S} u_i \leq \sum_{i \in S} v_i(a) - C(a)$, and $u_j = v_j(a)$ for all $j \notin S$ (utility allocation). An **efficient allocation for** S is an allocation (S, a, u) such that $\sum_{i \in S} u_i = V(S)$ with

[11] In the public good provision problem, $u_G = -C(0) = 0$, thus $\tilde{V}(T) = V(T)$ for all $T \subseteq S$. A payoff vector $u_S = (u_i)_{i \in S}$ is in *the core* iff $\sum_{i \in S} u_i = V(S)$, and $\sum_{i \in T} u_i \geq V(T)$ for all $T \subset S$.

[12] The Foley core of our public good economy is the standard core concept assuming that deviating coalitions have to provide public good by themselves. That is, it assumes that there is no spillover of public good across the groups.

$a = a^*(S)$.[13] That is, $N \backslash S$ are passive free-riders, and they do not contribute at all. Given that S is the lobby group, a natural way to allocate utility among the members is to use the core (Foley, 1970). A **core allocation for** S, $(S, a^*(S), u)$, is an efficient allocation for S such that $\sum_{i \in T} u_i \geq V(T)$ holds for all $T \subseteq S$.

However, a core allocation for S may not be immune to free-riding incentives by its members of S. So, we will define a hybrid solution concept of cooperative and noncooperative games. A **free-riding-proof core allocation for** S **(FRP-core allocation for** S**)** is a core allocation $(S, a^*(S), u)$ for S such that

$$u_i \geq v_i(a^*(S \backslash \{i\})) \text{ for all } i \in S.$$

A FRP-core allocation for S is immune to unilateral deviations by the members of S. *Note that, given the nature of public good provision problem, we can allow a coalitional deviation from S at no cost (since one person deviation is the most profitable).* Let $Core^{FRP}(S)$ be the set of all free-riding-proof core allocations for S. Note that $Core^{FRP}(S)$ may be empty for large group S, while for small groups it is nonempty (especially, for singleton groups it is always nonempty). We collect free-riding-proof core allocations for all S, and take their Pareto frontiers: the set of **free-riding-proof core (FRP-core)** is defined as

$$\begin{aligned} Core^{FRP} &= \left\{(S, a^*(S), u) \in \cup_{S' \in 2^N} Core^{FRP}(S') : \right. \\ &\quad \left. \forall T \in 2^N, \forall u' \in Core^{FRP}(T), \exists i \in N \text{ with } u_i > u'_i \right\}. \end{aligned}$$

That is, an element of $Core^{FRP}$ is a free-riding-proof core allocation for some S that is not weakly dominated by any other free-riding-proof core allocation for any T. *Note that $Core^{FRP}$ is **not** a subsolution of $Core(N)$: it only achieves constrained efficiency due to free-riding incentives, since we often have $Core^{FRP}(N) = \emptyset$.* Note that there always exists a free-riding-proof core allocation, since for all singleton set $S = \{i\}$, $Core^{FRP}(S)$ is nonempty.

Proposition 1. $Core^{FRP} \neq \emptyset$.

[13] Note that we have $V(S) = W_\Gamma(S) - W_\Gamma(\emptyset)$ in our public good provision problem.

3 The Main Result

Now, we will characterize PCPNE by the FRP-core. In the public good provision problem, the above fact (Laussel and Le Breton, 2001) says that the second stage CPNE outcomes coincide $Core(S)$ of a characteristic function form game $(V(T))_{T\subseteq S}$ with $V(S) = \max_{a\in A}\left(\sum_{i\in T} v_i(a) - C(a)\right)$, since $u_G = 0$.[14] This is nothing but Foley's core in a public good economy (Foley, 1970). This gives us some insight in our two-stage noncooperative game. First, for each subgame characterized by $S' = S(\sigma^{1\prime})$, the utility outcome $u_{S'}$ must be in the core of $(V(T))_{T\subseteq S'}$. Second, given the setup of our lobby formation game in the first stage, if a CPNE outcome u in a subgame S can realize as the equilibrium outcome (on-equilibrium path), it is *necessary* to have $u \in Core^{FRP}(S)$, since otherwise, some member of S would deviate in the first stage obtaining a secured free-riding payoff. This observation is useful in our analysis in the equivalence theorem. With some constructions, we can show the following:

Proposition 2. If an allocation $(S, a^*(S), u)$ is in the FRP-core, then there is a PCPNE σ of which outcome is $(S, a^*(S), u)$.

We postpone the proof Proposition 2 to Appendix B (with useful preliminary analyses in Appendix A), since it is somewhat involved. Here, we only describe how to construct PCPNE σ. First, in defining σ, we need to assign a CPNE utility profile to every subgame which corresponds to a coalition $S \subseteq N$ (although this does not happen in the equilibrium, it matters when deviations are considered). Since the second stage strategy profile is described by utility allocations assigned in each subgame. We partition the set of subgames $\mathcal{S} = \{S \in 2^N : S \neq \emptyset\}$ into three categories: Case 1. on equilibrium path $\mathcal{S}_1 = \{S^*\}$, Case 2. $\mathcal{S}_2 = \{S \in \mathcal{S} : S \cap S^* = \emptyset\}$, and Case 3. $\mathcal{S}_3 = \{S \in \mathcal{S}\backslash\mathcal{S}_1 : S \cap S^* \neq \emptyset\}$. As is shown in Laussel and Le Breton (2001), a CPNE outcome in a subgame S' corresponds to a core allocation for S'. In order to support the on-equilibrium path $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ by a PCPNE, we need to show that there is no credible deviation in the first stage. This require careful assignments of core allocations to all subgames. The way we prove Proposition 2 is by contradiction. Suppose that

[14] Actually, with no rent property, CPNE and strong Nash equilibrium (Aumann, 1959, but with weakly improving deviations) are equivalent in common agency game. See Konishi, Le Breton and Weber (1999).

there is a credible deviation T from S^*, which achieves lobby S' after the deviation. Then, for all members of T, both *profitability of deviation* and *free-riding-proofness* are satisfied. Thus, for all player $i \in T$, the post deviation payoff u'_i must satisfy $u'_i \geq \bar{u}_i = \max\{u^*_i, v_i(S' \backslash \{i\})\}$. The key case is $S' \cap S^* \neq \emptyset$, and we show that if there were such a deviation, there is an allocation $(S', a^*(S'), u') \in Core^{FRP}(S')$ that Pareto-dominates $(S^*, a^*(S^*), u^*)$. This contradicts with the assumption $(S^*, a^*(S^*), u^*) \in Core^{FRP}$. Pareto-domination is shown by using the fact that the utility allocation assigned to subgame S' under σ is a core allocation, and we construct a core allocation by an algorithm that is provided in Appendix A.

Once this direction is proved then the other direction is trivial. Notice that being PCPNE requires free-riding-proofness. Every PCPNE must be a free-riding-proof core allocation for some S. Since $Core^{FRP}$ is the Pareto-frontier of $\cup_{S \subseteq N} Core^{FRP}(S)$, Proposition 2 actually proves that all Pareto-dominated free-riding-proof core allocation for S can be defeated by a free-riding-proof core allocation.

Theorem 1. *An allocation $(S, a^*(S), u)$ is in the FRP-core, if and only if there is a PCPNE σ of which outcome is $(S, a^*(S), u)$.*

Proof. We will show the other direction of Proposition 2: every PCPNE σ generates a free-riding-proof core allocation as its outcome. It is easy to see that the outcome $(S, a^*(S), u)$ of a PCPNE σ is a FRP-core allocation **for** S, since otherwise the resulting allocation will not be a subgame perfect Nash equilibrium. Thus, $(S, a^*(S), u) \in Core^{FRP}(S)$. Suppose to the contrary that $u \notin Core^{FRP}$. Then, there is an free-riding-proof core allocation $(S', a^*(S'), u') \in Core^{FRP}$ with $u' > u$. Consider a coalitional deviation with a grand coalition N by preparing a PCPNE σ' that achieves u'. There is such a σ' by Proposition 2. This implies that there is a credible coalitional deviation from σ. This is a contradiction. Thus, every PCPNE achieves a free-riding-proof core allocation.□

Note that this result crucially depends on "comonotonicity of preferences" (single pure public good), and perfectly nonexcludable public good (free riders can enjoy public good perfectly). Without these assumptions, the above equivalence may not hold. Although FRP-core is much easier to absorb, it may still not be clear how the FRP-core looks like. In the next section, we will use a simple example to illustrate the properties of free-riding-proof core allocations, thus the outcome of PCPNE.

4 Examples: Linear-Utility and Quadratic-Cost Case

Let $v_i(a) = \theta_i a$ for all $i \in N$ and $C(a) = \frac{1}{2}a^2$, where $\theta_i > 0$ is a parameter.[15] With this setup, for group S, the optimal public good provision is determined by the first order condition, $\sum_{i \in S} \theta_i - a = 0$: i.e.,

$$a^*(S) = \sum_{i \in S} \theta_i.$$

Thus, the value of S is written as

$$\begin{aligned} V(S) &= \sum_{i \in S} \theta_i \left(\sum_{i \in S} \theta_i \right) - \frac{1}{2} \left(\sum_{i \in S} \theta_i \right)^2 \\ &= \frac{\left(\sum_{i \in S} \theta_i \right)^2}{2}. \end{aligned}$$

For an outsider $j \in N \backslash S$, the payoff is

$$v_j(a^*(S)) = \theta_j \left(\sum_{i \in S} \theta_i \right).$$

Consider the following example.

Example 1. Let $N = \{1, 3, 5, 11\}$ with $\theta_i = i$ for each $i \in N$.

First we check if the grand coalition $S = N$ is supportable. Then, we have $a^*(N) = \sum_{i \in N} i = 20$, and $V(N) = \frac{20^2}{2} = 200$. However, in order to have free-riding-proofness, we need to give each player the following payoff

[15] Coefficient 1/2 of $C(a)$ function is just matter of normalization. For any $k > 0$ with $C(a) = ka^2$, we get isomorphic results.

at the very least:

$$\begin{aligned} v_{11}(a^*(N\backslash\{11\})) &= (20-11)\times 11 = 99,\\ v_5(a^*(N\backslash\{5\})) &= (20-5)\times 5 = 75,\\ v_3(a^*(N\backslash\{3\})) &= (20-3)\times 3 = 51,\\ v_1(a^*(N\backslash\{1\})) &= (20-1)\times 1 = 19. \end{aligned}$$

The sum of all the above values exceeds the value of the grand coalition $V(N)$. As a result, we can conclude $Core^{FRP}(N) = \emptyset$.

- *Free-riding-proof core for grand coalition N may be empty. Thus, free-riding-proof core may be suboptimal.*

Next, consider $S = \{11, 5\}$. Then, $a^*(S) = 16$, and $V(S) = 128$. In order to check if the free-riding-proof core for S is nonempty, first again check the free-riding-incentives.

$$\begin{aligned} v(a^*(S\backslash\{11\})) &= (16-11)\times 11 = 55,\\ v(a^*(S\backslash\{5\})) &= (16-5)\times 5 = 55. \end{aligned}$$

Thus, if there is a free-riding-proof core allocation $u = (u_{11}, u_5)$ for S, u must satisfy

$$\begin{aligned} u_{11} + u_5 &= 128,\\ u_{11} &\geq 55,\\ u_5 &\geq 55,\\ u_{11} &\geq \frac{11\times 11}{2} = 60.5,\\ u_5 &\geq \frac{5\times 5}{2} = 12.5. \end{aligned}$$

The last two conditions are obtained by the core requirement. Thus, we have[16]

$$Core(\{11,5\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_{11} \geq 60.5,\ u_5 \geq 12.5, \\ \tilde{u}_3 = 48,\ \tilde{u}_2 = 32,\ \tilde{u}_1 = 16 \end{array} \right\},$$

[16] For notational simplicity, without confusion, we abuse notations by dropping irrelevant arguments of allocations. Thus, in this subsection, allocations are utility allocations.

and

$$Core^{FRP}(\{11,5\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_{11} \geq 60.5,\ u_5 \geq 55, \\ \tilde{u}_3 = 48,\ \tilde{u}_2 = 32,\ \tilde{u}_1 = 16 \end{array} \right\}.$$

As is easily seen, $Core^{FRP}(\{11,5\}) \neq \emptyset$, but it is a smaller set than $Core(\{11,5\})$. Thus, we have:

- "*Free-riding-proof constraints may narrow the set of attainable core allocations for a coalition.*"

Note that in this case, only free-riding incentive constraint for player 5 is binding. It is because player 11 can do a lot alone, it is better for her to provide public good alone than free-riding on player 5.□

Now, let us analyze the free-riding-proof core. Since the free-riding-proof core requires Pareto-efficiency on the union of free-riding-proof cores for all subsets S of the players, we first need to find free-riding-proof core for each S. However, in general, it is not an easy task to check if free-riding-proof core for S is empty or not. It is because free-riding-proof core for S requires two almost unrelated requirements: immune to coalitional deviation attempts to be independent, and immune to free-riding incentives. Interestingly, in the linear-utility and quadratic-cost case, an aggregated version of the latter requirements would suffice to check the nonemptiness of free-riding-proof core for S.

Proposition 3. In linear-utility and quadratic cost case, the free-riding-proof core for S is nonempty if and only if S satisfies (aggregated "no free riding condition").

$$\begin{aligned} \Phi(S) &\equiv V(S) - \sum_{i \in S} \theta_i a^*(S \backslash \{i\}) \\ &= \sum_{i \in S} \theta_i a^*(S) - \frac{1}{2}(a^*(S))^2 - \sum_{i \in S} \theta_i a^*(S \backslash \{i\}) \geq 0 \end{aligned}$$

This condition is equivalent to

$$\sum_{i \in S} \theta_i^2 \geq \frac{1}{2} \left(\sum_{i \in S} \theta_i \right)^2.$$

The proof is postponed to Section 7. By utilizing this proposition, we can completely characterize the FRP-core of the public good economy in Example 1.

Example 1. (continued) The free-riding-proof core allocations are attained by groups $\{11, 5, 1\}$, $\{11, 3, 1\}$, $\{11, 5\}$, $\{11, 3\}$, and $\{5, 3\}$.

First by applying Proposition 3, we can easily check for which S, $Core^{FRP}(S) \neq \emptyset$ holds. There are 12 such contribution groups: $\{11, 5, 1\}$, $\{11, 3, 1\}$, $\{11, 5\}$, $\{11, 3\}$, $\{11, 1\}$, $\{5, 3\}$, $\{5, 1\}$, $\{3, 1\}$, $\{11\}$, $\{5\}$, $\{3\}$, and $\{1\}$.

Note that $S = \{11, 5, 3\}$ does not have nonempty free-riding-proof core for S. Let $S = \{11, 5, 3\}$. Then, $a^*(S) = 19$ and $W(S) = 180.5$. Now, $11v(a^*(S\backslash\{11\})) = 88$, $5v(a^*(S\backslash\{5\}) = 70$ and $3v(a^*(S\backslash\{3\})) = 48$. Since $88 + 70 + 48 > 180.5$, there is no free-riding-proof core allocation for $S = \{11, 5, 3\}$. Thus, $\{11, 5, 1\}$ is the group that achieves the highest level of public good provision, and has nonempty free-riding-proof core.[17] This analysis gives an interesting observation:[18]

- *(Even the largest) group that achieves a free-riding-proof core allocation may not be consecutive.*

The intuition of this result is simple. Suppose $\Phi(S)$ is positive (say, $S = \{11, 5\}$). Then by Lemma 1, there is an internally stable allocation for S. Now, we may try to find $S' \supset S$ that still keeps $\Phi(S') \geq 0$. If the value of $\Phi(S)$ is positive yet the value is not so large, then adding high θ player (say, player 3) may make $\Phi(S') < 0$, since adding such a player may increase $a^*(S')$ a lot, making free-riding problem severer. However, if low θ player (say, player 1) is added, the free-rider problem does not become too severe, and $\Phi(S') \geq 0$ may be satisfied relatively easily.

Among the above 12 groups, it is easy to see that groups $\{5, 1\}$, $\{3, 1\}$, $\{11\}$, $\{5\}$, $\{3\}$, and $\{1\}$ do not survive the test of Pareto-domination by free-riding-proof core allocations for other groups. For example, consider

[17] As is seen below, group $\{11, 5, 1\}$ supports some allocations in $Core^{FRP}$.

[18] Although the context and approach are very different, in political science and sociology, formation of such non-consecutive coalitions is of a tremendous interest. For a game theoretical treatment of this line of literature (known and "Gamson's law"), see Le Breton et al. (2007).

$S = \{11, 5\}$ and $u' = (73, 55, 48, 32, 16) \in Core^{FRP}(\{11, 5\})$.[19] Since the payoff of 11 by free-riding is $v_{11}(a) = 11a$, every allocation for the above groups are dominated by the above u'. On the other hand, $\{5, 3\}$ is not dominated, since player 11 gets 88 by free-riding, respectively. Thus, player 11 would not join a deviation (11 can obtain maximum 73 in a free-riding-proof core allocation for $S \ni 11$). Without player 11's cooperation, there is no free-riding core allocation that dominates those of $\{5, 3\}$.

By the same reasons, free-riding-proof core allocations for $S = \{11, 1\}$ are dominated by the one for $S' = \{11, 5\}$. Under $S = \{11, 1\}$, player 5 gets 60, but S' can attain $u' = (63, 65, 48, 32, 16)$.[20] However, free-riding-proof core allocations for $S = \{11, 3, 1\}$ and $\{11, 3\}$ cannot be beaten by the ones for $S' = \{11, 5\}$, since player 5 gets 70 even under $\{11, 3\}$.[21]

Finally, $S = \{11, 5\}$, $\{11, 3\}$. The free-riding-proof core allocations for $S = \{11, 5\}$ is characterized by $u_{11} + u_5 = 128$, $u_{11} \geq 60.5$ and $u_5 \geq 55$, with $u_3 = 48$, $u_2 = 32$ and $u_1 = 16$. Now, consider $S' = \{11, 5, 1\}$. The free-riding-proof core allocations for S' is characterized by $u'_{11} + u'_5 + u'_1 = 144.5$, $u'_1 \geq 66$, $u'_5 \geq 60$ and $u'_1 \geq 16$, with $u'_3 \geq 51$ and $u'_2 \geq 34$. Thus, S' can attain $u'_{11} + u'_5 = 144.5 - 16 = 128.5$ as long as $u'_{11} \geq 66$ and $u'_5 \geq 60$. Thus, if $u \in Core^{FRP}(\{11, 5\})$ satisfies $u_{11} + u_5 = 128$, $60.5 \leq u_{11} \leq 68.5$, and $55 \leq u_5 \leq 62.5$, then u is improved upon by an allocation in $Core^{FRP}(\{11, 5, 1\})$. However, if $u \in Core^{FRP}(\{11, 5\})$ satisfies $u_{11} + u_5 = 128$, $u_{11} > 68.5$, or $u_5 > 62.5$, then u cannot be improved upon by forming group $\{11, 5, 1\}$. Free-riding-proof core allocations for $S = \{11, 3\}$ has a similar property with possible deviations by group $S' = \{11, 3, 1\}$. This phenomenon illustrates another interesting observation:

- *An expansion of group definitely increases the total value of the group, while it gives less flexibility in allocating it since free-riding incentives are strengthened by having more public good. As a result, some unequal free-riding-proof core allocations for the original group may not be improved upon by expanding the group.*

[19] The best allocation for player 11 in $Core^{FRP}(\{11, 5\})$. See the characterization of $Core^{FRP}(\{11, 5\})$ in Example 1. Other players are free-riders, and their payoffs are directly generated from $a^*(\{11, 5\}) = 16$.

[20] Under $S = \{11, 2\}$, player 11 can get at most 62.5 in order to satisfy the free-riding-proofness for player 2 ($v_2(\{11\}) = 22$).

[21] Since $V(\{11, 5\}) = 128$, and player 5 demands at least 70, player 11 can get at most 58. However, $V(\{11\}) = 60.5$. Thus, involving player 5 is not feasible.

In summary, the free-riding-proof core is *union* of the following sets of allocations attained by five different groups.

1. $S = \{11, 5, 1\}$, then $a^*(S) = 17$ and all free-riding-proof core allocations for S are attained:

$$Core^{FRP}(\{11,5,1\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_{11} + \tilde{u}_5 + \tilde{u}_1 = 144.5,\ \tilde{u}_3 = 51,\ \tilde{u}_2 = 34, \\ 66 \le \tilde{u}_{11},\ 60 \le \tilde{u}_5,\ 16 \le \tilde{u}_1 \end{array} \right\}$$

2. $S = \{11, 3, 1\}$, then $a^*(S) = 15$ and all free-riding-proof core allocations for S are attained:

$$Core^{FRP}(\{11,3,1\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_{11} + \tilde{u}_3 + \tilde{u}_1 = 112.5,\ \tilde{u}_5 = 75,\ \tilde{u}_2 = 30, \\ 60.5 \le \tilde{u}_{11},\ 36 \le \tilde{u}_3,\ 14 \le \tilde{u}_1 \end{array} \right\}$$

3. $S = \{11, 5\}$, then $a^*(S) = 16$ and only subset of free-riding-proof core allocations for S can be attained:

$$\begin{array}{rl} & \left\{\tilde{u} \in Core^{FRP}(\{11,5\}) : \tilde{u}_{11} > 68.5,\ \text{or } \tilde{u}_5 > 62.5\right\} \\ = & \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_{11} + \tilde{u}_5 = 128,\ \tilde{u}_3 = 48,\ \tilde{u}_2 = 32,\ \tilde{u}_1 = 16, \\ [68.5 < \tilde{u}_{11} \le 73 \text{ and } 55 \le \tilde{u}_5 < 59.5] \\ \text{or } [62.5 < \tilde{u}_5 \le 67.5 \text{ and } 60.5 \le \tilde{u}_{11} < 65.5] \end{array} \right\} \end{array}$$

4. $S = \{11, 3\}$, then $a^*(S) = 14$ and only subset of free-riding-proof core allocations for S can be attained:

$$\begin{array}{rl} & \left\{\tilde{u} \in Core^{FRP}(\{11,3\}) : \tilde{u}_{11} > 62.5\right\} \\ = & \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_{11} + \tilde{u}_3 = 98,\ \tilde{u}_5 = 70,\ \tilde{u}_2 = 28,\ \tilde{u}_1 = 14, \\ [62.5 < \tilde{u}_{11} \le 65 \text{ and } 33 \le \tilde{u}_3 < 35.5] \end{array} \right\} \end{array}$$

5. $S = \{5, 3\}$, then $a^*(S) = 8$ and all free-riding-proof core allocations for S are attained:

$$Core^{FRP}(\{5,3\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_5 + \tilde{u}_3 = 32,\ \tilde{u}_{11} = 88,\ \tilde{u}_2 = 16,\ \tilde{u}_1 = 8, \\ 15 \le \tilde{u}_5,\ 15 \le \tilde{u}_3 \end{array} \right\}$$

□

Now, we compare the free-riding-proof core allocations with a Nash equilibrium of a voluntary public good provision game. Let us consider a simultaneous move voluntary public good provision game by Bergstrom, Blume and Varian (1986). Each player i chooses her monetary contribution $m_i \geq 0$ to provide public good. The public good provision level is determined by $a(m) = \sqrt{2\sum_{i\in N} m_i}$ reflecting the cost function of public good production. Consider player i. Given that others are contributing M_{-i} together, player i maximizes $\theta_i\sqrt{2(m_i + M_{-i})} - m_i$. Thus, the best response for player i is $m_i^* = \max\left\{\frac{i^2}{2} - M_{-i}, 0\right\}$. This implies that only player 11 contributes, and the public good provision level is 11. Thus, by forming a contribution group in the first stage, it is possible to increase the public good provision level in equilibrium.[22] We can observe that in the last two groups, the levels of public good provision are less than the Nash equilibrium provision level of the standard voluntary contribution game (recall that $a = 11$ by player 11's contribution only is the unique Nash equilibrium):

- *There may be free-riding-proof core allocations that achieve less public good provision than Nash equilibrium one of a simple voluntary contribution game by Bergstrom, Blume and Varian (1986).*

This occurs since in our setup, player 11 can commit to being an outsider in the first stage. In a simultaneous move voluntary contribution game, this cannot happen. However, with any coalitions that support free-riding-proof core allocations, public good provision level exceeds 11. Finally, needless to say, we have:

- *The free-riding-proof core may be a highly nonconvex set.*

5 Replicated Economies

In this section, we will analyze if public good provision and the participation rate go down by replicating an economy. There is a tricky issue in replicating a (pure) public good economy. If we simply the set of consumers are

[22] In relation to this, the readers may wonder about the Lindahl equilibrium allocation for $S = \{11, 5\}$. Unfortunately, this example is not very useful since utility function is quasi-linear. The result would totally dependent on how the profits are distributed as is seen below. The Lindahl prices are $p_{11} = 11$ and $p_5 = 5$ given $\theta_{11} = 11$ and $\theta_5 = 5$, since $a^*(\{11, 5\}) = 16$ means marginal cost is $16(= 11 + 5)$. Since there are pure profits in producing public goods (cost function is strictly convex), we need to specify the way to allocate the profits 128. If they are distributed equally, then both get 64 each as profit share, and this is the only source of their utilities. If they are distributed according to players' willingnesses-to-pay, then players get 88 and 40. In the former case, the free-riding-proof conditions are satisfied, but in the latter case, they are not satisfied.

replicated, the amount of resource in the economy goes to infinity with the same cost function for public good production. Healy (2007) makes each consumer's endowment shrunk proportionaly to the population as the economy is replicated in order to isolate this problem following the way of replication by Milleron (1972).[23] However, consumers' preferences are also modified along replications. We adopt the same preference modification along replication in quasi-linear economy. We shrink each consumer's willingness-to-pay function (thus, utility function, too) proportionally as the economy is replicated. This way of replicating economy is natural in a quasi-linear economy, since aggregated willingness-to-pay and marginal cost functions stays the same. An original economy is a list $E = (N, (v_i)_{i \in N}, C)$. Let $r = 1, 2, 3, ...$ be a natural number. An rth replica of E is a list $E^r = (N^r, (v^r_{i_q})_{i \in N,\ q=1,...,r}, C)$, where $N^r = \cup_{i \in N}\{i^r_1, ..., i^r_r\}$ and $v^r_{i_q}(a) = v^r_i(a) = \frac{1}{r}v_i(a)$ for all $q = 1, .., r$.[24] Let a characteristic function form game generated from E^r be V^r. Each PCPNE of a lobby participation game generated from E^r has a corresponding free-riding-proof core allocation $(S, a^*(S), u^*)$ of characteristic function form game V^r. Note that for all r, all $S \subseteq N^r$, the public good provision level $a = a^*(S)$ is achieved at $\sum_{i_q \in S} WTP^r_{i_q}(a) = MC(a)$ under our assumptions, where $WTP^r_{i_q}(a) = v^{r\prime}_{i_q}(a)$ and $MC(a) = C'(a)$. We need $\sum_{i_q \in S}\left(v^r_{i_q}(a^*(S) - v^r_{i_q}(a^*(S \backslash \{i_q\})\right) \geq C(a^*(S))$ in order to satisfy the free-riding-proofness (the contents of the parenthesis in the LHS is how much each player can pay without sacrificing the free-riding-proofness). Let S contain $q_i(S) \in \{0, ..., r\}$ type i players for all $i \in N$. Then, the above necessary condition for free-riding-proofness is stated as

$$\sum_{i \in N} m_i(S)\left(v^r_i(a^*(S)) - v^r_i(a^*(S \backslash \{i^r\}))\right) \geq C(a^*(S)).$$

or

$$\sum_{i \in N} \frac{m_i(S)}{r}\left(v_i(a^*(S)) - v_i(a^*(S \backslash \{i^r\}))\right) \geq C(a^*(S)).$$

Consider $k \times r$th replication ($k = 1, 2, 3, ...$: $k = 1$ means the original rth replica). This means that each player is divided into k players. Let S^k be a

[23]Conley (1994) used different definition of replicated economy, and investigated convergence of core.

[24]Milleron's (1972) preference modification is described as follows. Let $\succeq_i$ and $\succeq^r_i$ be preference relations in the original and rth replica economy, respectively. Relation $\succeq^r_i$ is generated as follows: $(x, a) \succeq^r_i (x', a')$ iff $(rx, a) \succeq_i (rx', a')$. Then, by setting $a' = 0$, we have $x + v^r_i(a) = x'$ and $rx + v_i(a) = rx'$. This implies $v^r_i(a) = \frac{1}{r}v_i(a)$.

coalition in $k \times r$th replica economy that contains all k replica players of all members of S in rth replica economy. Obviously, $a^*(S)$ in rth replica economy is equivalent to $a^*(S^k)$ in $k \times r$th replica economy. However, although the coefficients satisfy $\frac{m_i(S)}{r} = \frac{m_i(S^k)}{k \times r}$, $a^*(S^k \backslash \{i^{k\times r}\})$ converges to $a^*(S^k) = a^*(S)$ as k goes to infinity. Thus, this inequality would not be satisfied at some point. See Figure 1 for the case of linear-utility and quadratic-cost case with $r = 1$ and $k = 3$. The question is if we can support the same a even after r further goes up. Formally, we have the following result.

Proposition 4. For all $a > 0$, there exists a natural number $\bar{r}(a)$ such that $a^*(S^*) \leq a$ holds for all $(S^*, a^*(S^*), u^*) \in Core^{FRP}(V^r)$ for all $r \geq \bar{r}(a)$.

Proof. We will show that for a given public good provision level $\bar{a} > 0$, there exists an $\bar{r}(\bar{a})$ such that the above necessary condition for free-riding-proofness,

$$\sum_{i \in N} \frac{m_i(S)}{r} \left(v_i(a^*(S)) - v_i(a^*(S \backslash \{i^r\})) \right) \geq C(a^*(S)),$$

fails for all $S \subset N^r$ with $a^*(S) \geq \bar{a}$ and all $r \geq \bar{r}(\bar{a})$. We construct another condition that is necessary to have the above. For each $i \in N$, let $\underline{WTP}_{-i^r}(a) \equiv C(\bar{a}) - WTP_{i^r}(\bar{a})$. Obviously, $\underline{WTP}_{-i^r}(a) + WTP_{i^r}(a) = C(a)$ is satisfied at $a = \bar{a}$. Note that $\underline{WTP}_{-i^r}(a)$ is a constant function, and that $WTP_{i^r}(a) + \underline{WTP}_{-i^r}(a)$ is the lower bound for $\sum_{i_q \in S} WTP^r_{i_q}(a)$ for all $a \leq \bar{a}$, for all S that achieves $a^*(S) \geq \bar{a}$ (since $v_i^{r\prime}(a)$ is weakly decreasing). Thus, we have

$$a^*(S) - a^*(S \backslash \{i^r_q\}) \leq \bar{a} - \bar{a}_{-i^r},$$

where $\underline{WTP}_{-i^r}(\bar{a}_{-i^r}) = C(\bar{a}_{-i^r})$. Note that we are considering r that is large enough to have $\bar{a}_{-i^r} > 0$, in order to be meaningful. Again, since $v_i^{r\prime}(a)$ is weakly decreasing, we have

$$v^r_i(a^*(S)) - v^r_i(a^*(S \backslash \{i^r_q\})) \leq v^r_i(\bar{a}) - v^r_i(\bar{a}_{-i^r})$$

for all S that achieves $a^*(S) \geq \bar{a}$ and all $i \in N$. This implies that we have

$$v_i(a^*(S)) - v_i(a^*(S \backslash \{i^r_q\})) \leq v_i(\bar{a}) - v_i(\bar{a}_{-i^r})$$

for all S that achieves $a^*(S) \geq \bar{a}$ and all $i \in N$. Let $m_i^r(\bar{a}) \equiv \frac{C(\bar{a})}{v_i(\bar{a}) - v_i(\bar{a}_{-i^r})}$, and let $M^r(\bar{a}) \equiv \max_{i \in N} m_i^r(\bar{a})$. Finally, let $r(\bar{a})$ be the minimum integer r such that $M^r(\bar{a}) \geq r$. Then, we have that

$$\sum_{i \in N} \frac{m_i(S)}{r} \left(v_i(a^*(S)) - v_i(a^*(S \backslash \{i^r\}))\right) \geq C(a^*(S)),$$

fails for all $S \subset N^r$ with $a^*(S) \geq \bar{a}$ and all $r > \bar{r}(\bar{a})$. This means that $a^*(S^*) \leq \bar{a}$ holds for all $(S^*, a^*(S^*), u^*) \in Core^{FRP}(V^r)$ for all $r \geq \bar{r}(a)$.$\square$

This proposition immediately implies the following theorem with Theorem 1.

Theorem 2. *The public good provision levels in all PCPNEs shrink to zero as the economy is replicated.*

Although the models and the objectives are very different, this result has similarity to the main result in Healy (2007), since Healy requires that all players participate voluntarily in equilibrium unlike our model. Note also that Theorem 2 (and Proposition 4) relies on continuous public good space unlike Theorem 1.

6 Summary

This paper added players' participation decisions to common agency games. The solution concept we used is a natural extension of coalition-proof Nash equilibrium to a dynamic game, perfectly coalition-proof Nash equilibrium (PCPNE). We considered a special class of common agency games: an environment without conflict of interests (comonotonic preferences) such as public good economies. In this case, we show that PCPNE is equivalent to an intuitive hybrid solution in transferrable utility case, the *free-riding-proof core*, which is the Pareto-frontier of a union of all core allocations for subset of players that are immune to unilateral free-riding incentives. With a simple example, we found that the equilibrium lobby group may not be consecutive (with respect to willingness-to-pay), and public good can be underprovided. Finally, we show that public good provision relative to the size of economy goes down to zero, as the participants of the economy are replicated to large numbers.

In binary public good provision game with voluntary participation, assuming symmetric players, Palfrey and Rosenthal (1984) showed that all pure strategy Nash equilibria are efficient. With asymmetric players, there are many Nash equilibria with different levels of cooperation. Maruta and Okada (2005) analyze evolutionarily stable equilibria among them. Although it is not a binary model, Nishimura and Shinohara (2007) consider a multi-stage voluntary participation game in a *discrete* multi-unit public good problem. They show that Pareto-efficient allocations in subgame perfect Nash equilibrium through a mechanism that determines public good provision unit-by-unit.[25] These discrete models perform very differently from continuous public good provision models especially in large economy. This direction will be pursued by our future work.

Appendix A: Preliminary Analysis on Core of Convex Games

In this appendix, we list a few useful preliminary results on the core of convex games. In our public good domain, the characteristic function game generated from a (public good) economy is convex. Let $V : 2^N \to \mathbb{R}$ with $V(\emptyset) = 0$ be a characteristic function form game. Game V is **convex** if $V(S \cup T) + V(S \cap T) \geq V(S) + V(T)$ for all pairs of subsets S and T of N. The **core** of game V is $Core(N, V) = \{u \in \mathbb{R}^N : \sum_{i \in N} u_i = V(N)$ and $\sum_{i \in S} u_i \geq V(S)$ for all $S \subset N\}$. Shapley (1971) analyzed properties of core of convex game in detail. One of the convenient results for us is the following.

Property 1. (Shapley, 1971) Let $\omega : |N| \to N$ be an arbitrary bijection, and let $u_{\omega(1)} = V(\{\omega(1)\})$, $u_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\})$,..., and $u_{\omega(|N|)} = V(N) - V(N \backslash \{\omega(|N|)\})$. Then, $u = (u_i)_{i \in N} \in Core(N, V)$, and the set of all such allocations forms the set of vertices of $Core(N, V)$.

Now, we consider a kind of reduced game, when outsiders walk away with the payoffs they can obtain by themselves. Let T be a proper subset

[25] Their efficiency result crucially depends on the following assumption: a player who did not participate in the mechanism in early stages can participate in public good provision later on.

of N. A reduced game of V on T is $\tilde{V}_T : 2^T \to \mathbb{R}$ such that $\tilde{V}_T(S) = V(S \cup (N\backslash T)) - V(N\backslash T)$ for all $S \subseteq T$. We have the following result.

Property 2. Suppose that $V : N \to \mathbb{R}$. is a convex game. Let $u_{N\backslash T} = (u_i)_{i \in N\backslash T}$ be a core allocation of a game $V : N\backslash T \to \mathbb{R}$. Then, $u_T \in Core(T, \tilde{V}_T)$ if and only if $(u_T, u_{N\backslash T}) \in Core(N, V)$.

Proof. First, we show that $u_T \in Core(T, \tilde{V}_T)$ if $(u_T, u_{N\backslash T}) \in Core(N, V)$. Since $(u_T, u_{N\backslash T}) \in Core(N, V)$, $\sum_{i \in S \cup (N\backslash T)} u_i \geq V(S \cup (N\backslash T))$ holds for all $S \subset T$. Rewriting this, we have $\sum_{i \in S} u_i \geq V(S \cup (N\backslash T)) - \sum_{i \in N\backslash T} u_i = V(S \cup (N\backslash T)) - V(N\backslash T) = \tilde{V}_T(S)$. Thus, $u_T \in Core(T, \tilde{V})$.

Second, we show that $u_T \in Core(T, \tilde{V}_T)$ implies $(u_T, u_{N\backslash T}) \in Core(N, V)$. Suppose not. Then, there is $S \subset N$ such that $V(S) > \sum_{i \in S} u_i = \sum_{i \in S \cap T} u_i + \sum_{i \in S \cap (N\backslash T)} u_i$. Since $u_T \in Core(T, \tilde{V})$, we have $\sum_{i \in S \cap T} u_i \geq V(S \cup (N\backslash T)) - V(N\backslash T)$. Since V is a convex game, $V(S \cup (N\backslash T)) + V(S \cap (N\backslash T)) \geq V(S) + V(N\backslash T)$, thus, we have $\sum_{i \in S \cap T} u_i \geq V(S) - V(S \cap (N\backslash T))$. Substituting this into our supposition, we have $V(S) > V(S) - V(S \cap (N\backslash T)) + \sum_{i \in S \cap (N\backslash T)} u_i$. However, since $u_{N\backslash T} \in Core(N\backslash T, V)$, $\sum_{i \in S \cap (N\backslash T)} u_i \geq V(S \cap (N\backslash T))$ holds. This is a contradiction.□

Now, we will rewrite core. Let $u = (u_i)_{i \in N}$ be an arbitrary utility vector. Let

$$\begin{aligned} \mathcal{Q}^+(u) &= \{S \in 2^N : \sum_{j \in S} u_j > V(S)\}, \\ \mathcal{Q}^0(u) &= \{S \in 2^N : \sum_{j \in S} u_j = V(S)\}, \\ \mathcal{Q}^-(u) &= \{S \in 2^N : \sum_{j \in S} u_j < V(S)\}. \end{aligned}$$

That is, sets $\mathcal{Q}^+(u)$ and $\mathcal{Q}^-(u)$ denote collections of coalitions that are satisfied and unsatisfied (in strict sense) under utility vector u, respectively. The set $\mathcal{Q}^0(u)$ is collection of coalitions that are just indifferent between deviating and not deviating. Obviously, a utility vector u is in the core ($u \in Core(N, V)$), if and only if $\mathcal{Q}^-(u) = \emptyset$ (or $S \in \mathcal{Q}^+(u) \cup \mathcal{Q}^0(u)$ for all $S \in 2^N$) and $N \in \mathcal{Q}^0(u)$. Let $\eta(S, u) \equiv \frac{V(S) - \sum_{i \in S} u_i}{|S|}$ be *(per capita) shortage of payoff* for coalition S for all $S \in \mathcal{Q}^-(u)$. Let

$$\mathcal{Q}^-_{\max}(u) \equiv \{S \in \mathcal{Q}^-(u) : \eta(S, u) \geq \eta(S', u) \text{ for all } S' \in \mathcal{Q}^-(u)\},$$

and

$$Q^-_{\max}(u) = \cup_{S\in\mathcal{Q}^-_{\max}(u)} S.$$

Using the above definitions, we will construct an algorithm that starts from an arbitrary utility vector u and terminates at a core allocation $\hat{u}$.

Algorithm. Let $u \in \mathbb{R}^N$ and let $V : N \to \mathbb{R}$ be a convex game. Let $u(t)$ be the utility vector at stage $t \geq 0$, and $u(0) = u$ (the initial value).

(a) Suppose $\mathcal{Q}^-(u) = \emptyset$. Then, $2^N\backslash\{\emptyset\} = \mathcal{Q}^0(u) \cup \mathcal{Q}^+(u)$. If $N \in \mathcal{Q}^0(u(0))$ then the algorithm terminates immediately. Otherwise, $\sum_{i\in N} u_i > V(N)$ holds, and we reduce u_is by the same speed simultaneously and continuously for $i \in N\backslash(\cup_{S\in\mathcal{Q}^0(u)} S)$ as t increases.[26] Since all elements in $\mathrm{Q}^0(u)$ stay in $\mathrm{Q}^0(u(t))$ as the process continues, while some of elements of $\mathcal{Q}^+(u(t))$ start switching to $\mathcal{Q}^0(u(t))$, $\mathcal{Q}^0(u(t))$ monotonically expands in the process. At some stage $t = \hat{t}$, $N \in \mathcal{Q}^0(u(\hat{t}))$ occurs. Then we terminate the process. The final outcome is $\hat{u} = u(\hat{t})$.

(b) Suppose $\mathcal{Q}^-(u) \neq \emptyset$. Start with $u(0) = u$. There are two phases:

   i. Phase 1 ($t \in [0, \tilde{t}]$). For all $i \in Q^-_{\max}(t)$, increase u_is by the same amount simultaneously and continuously. Terminate the algorithm when $Q^-_{\max}(u(t)) = \emptyset$ (or $\mathcal{Q}^-(u(t)) = \emptyset$), and call such $t = \tilde{t}$.[27]

   ii. Phase 2 ($t \in (\tilde{t}, \hat{t}]$). Now, $\mathcal{Q}^-(u(t)) = \emptyset$. Then, we repeat the procedure in (a), and we reach at a final outcome $\hat{u} = u(\hat{t})$ when $N \in \mathcal{Q}^0(u(\hat{t}))$ occurs.

[26] Note that $N\backslash(\cup_{Q\in\mathcal{Q}^0(u)} Q) = \emptyset$ implies $N \in \mathcal{Q}^0(u)$. This follows from the definition of convex game. We show that if $T, T' \in \mathcal{Q}^0(u)$, then $T \cup T' \in \mathcal{Q}^0(u)$ when $\mathcal{Q}^-(u) = \emptyset$ as is assumed. By the definition of convex game, $V(T \cup T') + V(T \cap T') \geq V(T) + V(T')$ $= \sum_{i\in T\cup T'} u_i + \sum_{i\in T\cap T'} u_i$ holds. Since $V(T \cap T') \in \mathcal{Q}^0(u) \cup \mathcal{Q}^+(u)$, $\sum_{i\in T\cap T'} u_i \geq V(T \cap T')$. This implies $V(T \cup T') \geq \sum_{i\in T\cup T'} u_i$. Since $\mathcal{Q}^-(u) = \emptyset$, $T \cup T' \in \mathcal{Q}^0(u)$. This argument implies that $(S' \cap S^*)\backslash(\cup_{Q\in\mathcal{Q}^0(u_{S'\cap S^*})} Q) = \emptyset$ implies $S' \cap S^* \in \mathcal{Q}^0(u)$, and the process terminates.

[27] This process guarantees that a player $i \in Q^-_{\min}(u(t))$ (at some stage $t \in [0, \tilde{t}]$) must belongs to some $S' \in \mathcal{Q}^0(u(\tilde{t}))$ at the end of phase 1.

Let $Q^0(u) \equiv \cup_{S\in\mathcal{Q}^0(u)} S$, and define

$$\begin{aligned} W &\equiv \{i \in N : \exists t \geq 0 \text{ with } i \in Q^-_{\max}(u(t)) \text{ in phase 1 of case (b)}\}, \\ I &\equiv \{i \in N : i \in Q^0(u(0)) \text{ in case (a), or } i \in Q^0(u(\tilde{t}))\backslash W \text{ in phase 2 of case (b)}\}, \\ L &\equiv \{i \in N : i \notin Q^0(u(0)) \text{ in case (a), or } i \notin Q^0(u(\tilde{t})) \text{ in phase 2 of case (b)}\}. \end{aligned}$$

These sets will be shown to be collections of players who gained, kept intact, and lost in their payoffs in the above algorithm relative to the initial value u, respectively. By the construction of the algorithm, the following Lemma is straightforward.

Lemma 1. Consider the above algorithm. In phase (i) of case (b), $Q^-_{\max}(u(t))$ monotonically expands as t increases for $t \in [0, \tilde{t})$. This phase terminates with $\mathcal{Q}^-(u(\tilde{t})) = \emptyset$. Moreover, $W = \lim_{t\to\tilde{t}} Q^-_{\max}(u(t))$, and for all $S \in \lim_{t\to\tilde{t}} \mathcal{Q}^-_{\max}(u(t))$, $S \subseteq W$ and $S \in \mathcal{Q}^0(u(\tilde{t}))$ hold.

Proof. As t increases, the payoffs of all members of $Q^-_{\max}(u(t))$ increases by the same speed, thus for any $S \in \mathcal{Q}^-_{\max}(u(t))$, $\eta(S, u(t))$ decreases with the same speed. Note that for all other coalitions $T \notin \mathcal{Q}^-_{\max}(u(t))$, $\eta(T, u(t))$ decreases with a slower pace (if $T \cap Q^-_{\max}(u(t)) \neq \emptyset$) or stays constant (if $T \cap Q^-_{\max}(u(t)) = \emptyset$). Therefore, $Q^-_{\max}(u(t))$ monotonically expands as t increases. This monotonic utility raising process continues until $\mathcal{Q}^-(u(t)) = \emptyset$ realizes at $t = \tilde{t}$. Since $Q^-_{\max}(u(t))$ monotonically expands, $W = \lim_{t\to\tilde{t}} Q^-_{\max}(u(t))$ holds, and by continuity of $u(t)$ and $\mathcal{Q}^-(u(\tilde{t})) = \emptyset$, for all $S \in \lim_{t\to\tilde{t}} \mathcal{Q}^-_{\max}(u(t))$, $S \subseteq W$ and $S \in \mathcal{Q}^0(u(\tilde{t}))$ hold.□

Lemma 2. Starting from any initial value $u \in \mathbb{R}^N$, this algorithm terminates at a core allocation $\hat{u} \in Core(N, V)$.

Proof. First, we show that case (a) terminates at a core allocation, since the same argument applies to phase 2 of case (b). We can show this statement, if $N\backslash(\cup_{S\in\mathcal{Q}^0(u)} S) \neq \emptyset$ holds whenever $\sum_{i\in N} u_i > V(N)$ holds (otherwise, u is infeasible while the algorithm stops). Suppose that $\sum_{i\in N} u_i > V(N)$, while $N\backslash(\cup_{Q\in\mathcal{Q}^0(u)} Q) = \emptyset$ in case (a). Then, for all $i \in N$, there exists $S \in \mathcal{Q}^0(u)$ with $i \in S$. Then, we can construct a balanced family $\mathcal{B}$ by collecting these Ss (see, say, Ichiishi, 1983). Then, with balanced weight $\{\lambda_S\}_{S\in\mathcal{B}}$ such that $\sum_{S\ni i, S\in\mathcal{B}} \lambda_S = 1$ for all $i \in N$. This implies

$$\sum_{S\ni i, S\in\mathcal{B}} \lambda_S u_i = u_i.$$

Since for all $S \in \mathcal{B}$, $\sum_{j \in S} u_j = V(S)$ by definition, we have

$$\sum_{S \in \mathcal{B}} \lambda_S V(S) = \sum_{i \in N} u_i.$$

By assumption, we have $\sum_{i \in N} u_i > V(N)$, and we can conclude

$$\sum_{S \in \mathcal{B}} \lambda_S V(S) > V(N).$$

This means that the game V is not balanced. This is a contradiction, since convex games are balanced. Thus, in case (a), the algorithm terminates at a feasible allocation. Since $u(t)$ changes continuously, $N \in \mathcal{Q}^0(\hat{u})$ holds, and $\hat{u} \in Core(N, V)$.

Now, by Lemma 1, phase 1 of case (b) terminates with $\mathcal{Q}^-(\tilde{u}) = \emptyset$. Thus, the same argument as case (a) applies to phase 2 of case (b). Thus, $\hat{u} \in Core(N, V)$ in case (b) as well.□

Lemma 3. Set N is partitioned into W, I and L. For all $i \in W$, $\hat{u}_i > u_i$; for all $i \in I$, $\hat{u}_i = u_i$; and for all $i \in L$, $\hat{u}_i < u_i$.

Proof. Note that in phase 2 of case (b), members of W are intact since $W \subseteq \cup_{S \in \mathcal{Q}^0(u(\tilde{t}))} S$. Thus, for all $i \in W$, $\hat{u}_i > u_i$. Given this, the rest is obvious.□

This lemma says that the winners, unaffected players, and losers of the algorithm are identified by sets W, I and L, respectively.

Appendix B: Proofs

Proof of Proposition 2.

First, we construct a strategy profile σ below, which will be shown to support $(S^*, a^*(S^*), u^*)$ as a PCPNE. By definition, we have $u^* \in Core^{FRP}(S^*)$. In defining σ, we need to assign a CPNE utility profile to every subgame S' (although this does not happen in the equilibrium, it matters when deviations are considered). Then, we show that there is no credible and profitable deviation from σ by way of contradiction.

A strategy profile in the second stage σ^2 is generated from utility allocations assigned in each subgame (we utilize truthful strategies that support utility outcomes). We partition the set of subgames $\mathcal{S} = \{S' \in 2^N : S' \neq \emptyset\}$ into three categories: Case 1. on equilibrium path $\mathcal{S}_1 = \{S^*\}$, Case 2. $\mathcal{S}_2 = \{S' \in \mathcal{S} : S' \cap S^* = \emptyset\}$, and Case 3. $\mathcal{S}_3 = \{S' \in \mathcal{S} \backslash \mathcal{S}_1 : S' \cap S^* \neq \emptyset\}$. As is shown in Lausell and Le Breton (2001), a CPNE outcome in a subgame S' corresponds to a core allocation for S'. In order to support the on-equilibrium path $(S^*, a^*(S^*), u^*)$, we need to show that there is no credible deviation in the first stage. Since a credible deviation requires both free-riding-proofness and profitability, utility level $\bar{u}_i = \max\{u_i^*, v_i(S' \backslash \{i\})\}$ plays an important role for player i to join a coalitional deviation. We construct a core allocation **for subgame** S' by utilizing utility vector $\bar{u}$ by the algorithm described in Appendix A. Our construction guarantees that if there is $j \in S' \cap S^*$ with $u_j(S') < \bar{u}_j$ $(j \in L)$, then for all $i \in S' \cap S^*$ with $u_i(S') \geq \bar{u}_i$ $(i \in W \cup I)$, there exists $Q \subseteq S' \cap S^*$ with $i \in Q$, $u_{i'}(S') \geq \bar{u}_i$ $(i \in Q \subseteq W \cup I)$, and $V(Q) = \sum_{i' \in Q} u_{i'}(S')$. This property restricts what a credible coalitional deviation can do by taking advantage of others. The construction of a core allocation for each subgame is as follows.

1. We assign $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ to the on-equilibrium subgame S^*.

2. For any S' with $S' \cap S^* = \emptyset$, we assign an extreme point of the core for S' of a convex game (just to assign a concrete core allocation). For an arbitrarily selected order ω over S', we assign payoff vector $u_{\omega(1)} = V(\{\omega(1)\}) - V(\emptyset)$, $u_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\})$,... etc. following Shapley (1971). Call the allocation $\hat{u}_{S'} \in Core(S', V)$ (property 1).

3. For any S' with $S' \cap S^* \neq \emptyset$, we assign a core allocation in the following manner. It requires a few steps. First, we deal with the outsiders. Let $\omega : |S' \backslash S^*| \to S' \backslash S^*$ be an arbitrary bijection, and let $u_{\omega(1)} = V(\{\omega(1)\})$, $u_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\})$,..., and $u_{\omega(|S' \backslash S^*|)} = V(S' \backslash S^*) - V(S' \backslash S^* \backslash \{\omega(|S' \backslash S^*|)\})$. Such a core allocation suppresses the total payoffs of $S' \backslash S^*$ the most (Shapley, 1971). The rest $V(S') - V(S' \backslash S^*)$ goes to $S' \cap S^*$. Consider a reduced game of (S', V) on $S' \cap S^*$ with $u_{S' \backslash S^*}$, $\tilde{V}_{S' \cap S^*} : 2^{S' \cap S^*} \to \mathbb{R}$ such that $\tilde{V}_{S' \cap S^*}(Q) = V(Q \cup (S' \backslash (S' \cap S^*))) - \sum_{j \in S' \backslash S^*} u_j = V(Q \cup (S' \backslash (S' \cap S^*))) - (S' \backslash (S' \cap S^*))$. By property 2, we know that $u_{S' \cap S^*} \in Core(S' \cap S^*, \tilde{V}_{S' \cap S^*})$ if and

only if $(u_{S'\cap S^*}, u_{S'\setminus S^*}) \in Core(S', V)$. For each $i \in S' \cap S^*$, let $\bar{u}_i = \max\{u_i^*, v_i(S'\setminus\{i\})\}$. By the algorithm in Appendix A, we construct a core allocation $\hat{u}_{S'\cap S^*}$ from vector $\bar{u}_{S'\cap S^*} = (\bar{u}_i)_{i\in S'\cap S^*}$ for reduced game $\tilde{V}_{S'\cap S^*}$ of game $V : 2^{S'} \to \mathbb{R}$.

We support these core allocations by truthful strategies. Let $\sigma_i^1 = 1$ for $i \in S^*$, and $\sigma_i^1 = 0$ for $j \notin S^*$. Let $\sigma_i^2(S^*)$ be a truthful strategy relative to $a^*(S^*)$ with $\tau_i(a^*(S^*)) = v_i(a^*(S^*)) - u_i^*$ for all $i \in S^*$. And let $\sigma_i^2(S')$ be a truthful strategy relative to $a^*(S')$ with $\tau_i(a^*(S')) = v_i(a^*(S')) - \hat{u}_i(S')$ for all $i \in S'$. Since every subgame has a core allocation with truthful strategies, it is a CPNE. Thus, if there is a deviation from σ, then it must happen in the first stage. The rest of the proof is done by way of a contradiction.

Suppose to the contrary that coalition T profitably and credibly deviates from the equilibrium σ. Note that in the reduced game by T, it must be a PCPNE deviation given σ_{-T} fixed. In the original equilibrium, S^* is the lobby group. This implies that all $i \in (N\setminus S^*)\setminus T$ play $\sigma_i^1 = 0$ in the first stage and they free-ride, while all $i \in S^*\setminus T$ play $\sigma_i^1 = 1$ in the first stage and they play the same strategies ($\sigma_i^2(S')$ a menu contingent to formed lobby S') in the second stage. Note that all $i \in T\setminus S^*$ play $\sigma_i^{1\prime} = 1$ in the first period after the deviation (by definition), while $i \in T \cap S^*$ may or may not play $\sigma_i^{1\prime} = 1$. Some may choose to free-ride by switching to 0, while others stay in the lobby with adjustment of their strategies in the second stage.

Let S' be the lobby formed by T's deviation: $S' = S(\sigma_{-T}^1, \sigma_T^{1\prime})$. Then, there are five groups of players (see Figure 2):

(i) the members of $S^*\setminus S' \subset T$ free-ride after the deviation,

(ii) the members of $S'\setminus S^* \subset T$ join the lobby,

(iii) the members of $(S^* \cap S')\setminus T \subset S'$ do not change their strategies in any stage (participate in lobbying, while keep the same menu in the second stage),

(iv) the members of $(S^* \cap S') \cap T \subset S'$ change strategies in the second stage,

(v) the members of $N\setminus(S' \cup S^*)$ are outsiders before or after the deviation.

Let the resulting allocation be $(S', a^*(S'), u')$. Since T is a profitable and credible deviation, the members in (i), (ii) and (iv) are better-off after T

deviates. That is,

$$\begin{aligned} v_i(a^*(S')) &\geq u_i^* \text{ for all } i \in S^*\backslash S', \\ u_i' &\geq \bar{u}_i \text{ for all } i \in S'\backslash S^*, \\ u_i' &\geq \bar{u}_i \text{ for all } i \in (S^* \cap S') \cap T, \end{aligned}$$

must hold, where $\bar{u}_i = \max\{u_i^*, v_i^*(a^*(S'\backslash\{i\})\}$.

Given our supposition, we will provide a sequence of claims below.

First note that since members of (ii) exist and are better off, we have $a^*(S') > a^*(S^*)$. It is because (ii) is nonempty, since otherwise, $S' \subset S^*$ holds, and a coalitional deviation cannot be profitable.

Claim 1. $S'\backslash S^* \neq \emptyset$, and $a^*(S') > a^*(S^*)$.

Since in σ, all players use truthful strategies, even after T's deviation, the members in (iii) (outsiders of T) get the same payoff vector $\hat{u}_{(S^*\cap S')\cap T}(S')$ as in the original subgame CPNE for S'. It is because in subgame S' (even after deviation), $a^*(S')$ must be provided since CPNE (core) must be assigned to the subgame. Thus, we have the following for group (iii).

Claim 2. After deviation by T, all $i \in (S^* \cap S')\backslash T \subset S'$ receives exactly $u_i' = \hat{u}_i(S')$.

Note that, since u' needs to be is a CPNE in the second stage of the reduced game by T, we have $\sum_{i\in S'\backslash S^*} u_i' \geq V(S'\backslash S^*)$, (to be in $Core(S')$). By construction of $\hat{u}(S')$, we have $\sum_{i\in S'\backslash S^*} \hat{u}_i = V(S'\backslash S^*)$. Thus, we have the following for group (ii).

Claim 3. $\sum_{i\in S'\backslash S^*} u_i' \geq \sum_{i\in S'\backslash S^*} \hat{u}_i = V(S'\backslash S^*)$.

Now, we consider group (iv). By Claims 2 and 3, the members of (iv) together can get at most

$$\sum_{i\in S'\cap S^*\cap T} u_i' \leq \sum_{i\in S'\cap S^*\cap T} \hat{u}_i,$$

since group (iv) cannot get transfers from groups (ii). Since group (iv) is better off and free-riding-proofness is satisfied for them after the deviation (PCPNE deviation), $u_i' \geq \bar{u}_i = \max\{u_i^*, v_i(a^*(S'\backslash\{i\}))\}$ must be satisfied for all group (iv) members, $i \in S' \cap S^* \cap T$.

Claim 4. Suppose that $L \neq \emptyset$. Then, $L \cap (S' \cap S^* \cap T) = \emptyset$, and $\sum_{i\in S'\cap S^*\cap T} u'_i \geq \sum_{i\in S'\cap S^*\cap T} \hat{u}_i$ holds.

Proof of Claim 4. In case (a), for all $i \in S' \cap S^*$, we have $\hat{u}_i \leq \bar{u}_i$, since there is no winner for case (a) (Lemma 3). Claims 2 and 3 requires $\sum_{i\in S'\cap S^*\cap T} u'_i \leq \sum_{i\in S'\cap S^*\cap T} \hat{u}_i$. However, we need $u'_i \geq \bar{u}_i$ for all $i \in S' \cap S^* \cap T$. Thus, we have $u'_i = \hat{u}_i = \bar{u}_i$ for all $i \in S' \cap S^* \cap T$, and $S' \cap S^* \cap T \subseteq I$.

In case (b) with $L \neq \emptyset$, $W \cap (S' \cap S^* \cap T) \neq \emptyset$ holds (otherwise, by claim 3, there must be $i \in S' \cap S^* \cap T$ with $u'_i < \bar{u}_i$, which is a contradiction with the supposition that T is a credible deviation). Thus, some of the members of W must belong to (iv). However, for all $i \in W$, the winner group, there is $Q \in \mathcal{Q}^0(\hat{u}_{S'\cap S^*}) \backslash (S' \cap S^*)$ with $i \in Q \subseteq W$ by Lemma 1. Since members of group (iii) $j \in S' \cap S^* \backslash T$ take $\hat{u}_j$s with them (Claim 2), for all such Q, $\sum_{j\in Q\cap T} u'_j = \sum_{j\in Q\cap T} \hat{u}_j$ must hold. Therefore, no winner can transfer utility to non-winners within group (iv): $\sum_{i\in W\cap(S'\cap S^*\cap T)} u'_i \geq \sum_{i\in W\cap(S'\cap S^*\cap T)} \hat{u}_i$. For all $i \in L$, $\hat{u}_i < \bar{u}_i$, if group (iv) has such a member, it needs more total payoffs than assigned core allocation ($\sum_{i\in S'\cap S^*\cap T} u'_i > \sum_{i\in S'\cap S^*\cap T} \hat{u}_i$) in order to satisfy necessary condition for profitable and credible deviation ($u'_i \geq \bar{u}_i$). With claims 2 and 3, this cannot happen since $\sum_{i\in S'} u'_i > V(S')$ would be concluded. Thus, $L \cap (S' \cap S^* \cap T) = \emptyset$ must hold. Since members of I cannot transfer utility to anybody, and members of W cannot, too. Therefore, we have $\sum_{i\in S'\cap S^*\cap T} u'_i \geq \sum_{i\in S'\cap S^*\cap T} \hat{u}_i$.□

Claims 2, 3 and 4 immediately imply the following for group (ii).

Claim 5. Suppose that $L \neq \emptyset$ holds. Then, we have

$$\sum_{i\in S'\backslash S^*} u'_i = \sum_{i\in S'\backslash S^*} \hat{u}_i = V(S'\backslash S^*).$$

Thus, we have shown that if $L \neq \emptyset$, then group (ii) can deviate profitably and credibly (together with group (iv)) achieve $u'_{S'\cap S^*}$ with a limited resource $V(S'\backslash S^*)$. Due to profitability of T, $V(S'\backslash S^*) \geq \sum_{i\in S'\backslash S^*} v_i(a^*(S^*))$, we have $a^*(S'\backslash S^*) > a^*(S^*)$. Moreover, due to credibility of T, we have $u'_i \geq v_i(a^*(S'\backslash\{i\}))$ for all $i \in S'\backslash S^*$, which implies $u'_i \geq v_i(a^*(S'\backslash S^*\backslash\{i\}))$. Thus, a deviation by $S'\backslash S^*$ is credible, too. We consider a new allocation that is achieved only by group (ii).

Claim 6. Consider the case where $S'\backslash S^*$ is the lobby group. Then, an allocation $(S'\backslash S^*, a^*(S'\backslash S^*), (u'_i)_{i\in S'\backslash S^*}, (v_j(a^*(S'\backslash S^*))_{j\notin S'\backslash S^*})$ can be achieved only by $S'\backslash S^*$ (u'_T is the deviators' allocation by T), and this allocation Pareto-dominates $(S^*, a^*(S^*), u^*)$.

Proof of Claim 6. First, groups (i) and (v) are better off, since $a^*(S') > a^*(S^*)$. By assumption, group (ii) are better off ($u'_i \geq v_i(a^*(S^*))$ with at least one strict inequality) and have no free-riding incentives ($u'_i \geq v_i(a^*(S'\backslash\{i\})) > v_i(a^*(S'\backslash S^*\backslash\{i\}))$). Thus, the only groups which need investigation are groups (iii) and (iv). We check if there can be $i \in S' \cap S^*$ with $u^*_i > v_i(a^*(S'\backslash S^*))$ despite of $a^*(S'\backslash S^*) > a^*(S^*)$. Since $u^*_i \in Core(S^*)$, and the game V is convex, $u^*_i \leq V(S^*) - V(S^*\backslash\{i\})$ (Shapley, 1971). Since

$$
\begin{aligned}
& V(S^*) - V(S^*\backslash\{i\}) \\
= & \sum_{j\in S^*} v_j(a^*(S^*)) - C(a^*(S^*)) - \left(\sum_{j\in S^*\backslash\{i\}} v_j(a^*(S^*\backslash\{i\})) - C(a^*(S^*\backslash\{i\}))\right) \\
< & v_i(a^*(S'\backslash S^*)) \\
& + \sum_{j\in S^*\backslash\{i\}} v_j(a^*(S^*)) - C(a^*(S^*)) - \left(\sum_{j\in S^*\backslash\{i\}} v_j(a^*(S^*\backslash\{i\})) - C(a^*(S^*\backslash\{i\}))\right) \\
< & v_i(a^*(S'\backslash S^*)).
\end{aligned}
$$

The last inequality holds since $\sum_{j\in S^*\backslash\{i\}} v_j(a) - C(a)$ is maximized at $a = a^*(S^*\backslash\{i\})$. This proves that all members of (iii) and (iv) are better off in $(S'\backslash S^*, a^*(S'\backslash S^*), (u'_i)_{i\in S'\backslash S^*}, (v_j(a^*(S'\backslash S^*))_{j\notin S'\backslash S^*})$. Hence, we conclude that $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ is Pareto-dominated by $(S'\backslash S^*, a^*(S'\backslash S^*), (u'_i)_{i\in S'\backslash S^*}, (v_j(a^*(S'\backslash S^*))_{j\notin S'\backslash S^*}) \in Core^{FRP}(S'\backslash S^*)$, since the members of (ii), $S'\backslash S^*$, have no free-riding incentive.□

The statement of Claim 6 is an apparent contradiction to $(S^*, a^*(S^*), u^*) \in Core^{FRP}$. Thus, we conclude that $L = \emptyset$ holds.

Suppose that case (a) holds. Then, $L = \emptyset$ implies $I = S' \cap S^*$, thus $\hat{u}_i = \bar{u}_i$. Since there is an allocation $u'_{S'\backslash S^*} \geq \bar{u}_{S\backslash S^*}$ with $\sum_{i\in S'\backslash S^*} u'_i = V(S'\backslash S^*)$ for group (ii). This implies that, by property 2, $(u'_{S'\backslash S^*}, \hat{u}_{S'\cap S^*}) \in Core(S')$, and no one has free-riding incentive. Thus, since T can improve upon S^*, this allocation $(S', u'_{S'\backslash S^*}, \hat{u}_{S'\cap S^*}, v_{N\backslash S'}(a^*(S')))$ Pareto improves upon (S^*, u^*). This is a contradiction.

Suppose that case (b) holds with $L = \emptyset$. Then, we have $\hat{u}_i > \bar{u}_i = \max\{u^*_i, v_i(a^*(S'\backslash\{i\}))\}$ for all $i \in S' \cap S^*$. Thus, members of group (iii) are better off and have no free-riding incentive. Players in groups (i), (ii) and

(iv) deviate credibly and profitably by T, they are better-off and have no free-riding incentive for groups (ii) and (iv). Group (v) is better-off by Claim 1. This means that $(S', a^*(S'), (u'_i)_{i\in S'\cap T}, (\hat{u}_i)_{i\in(S'\cap S^*)\setminus T}, (v_j(a^*(S')))_{j\in N\setminus S'}) \in Core^{FRP}(S')$, and Pareto-dominates $(S^*, a^*(S^*), u^*) \in Core^{FRP}$. This is a contradiction. Hence, $(S^*, a^*(S^*), u^*)$ is supportable with a PCPNE σ.□

Proof of Proposition 3

If the above condition is violated, there is no allocation that satisfies no free riding for S. Thus, we only need to show that if the above condition is satisfied then we can find a core allocation that satisfies $\sum_{i\in T} u_i \geq V(T) = \sum_{i\in T} \theta_i a^*(T) - \frac{1}{2}(a^*(T))^2$. To be instructive, we will not explicitly solve $a^*(T)$ for a while. The strategy we take is to construct an allocation, and verify that it is in the core. Let $u_S \in \mathbb{R}^S_+$ be such that for all $i \in S$

$$u_i = \theta_i a^*(S\setminus\{i\}) + \frac{\theta_i}{\sum_{j\in S}\theta_j}\left(\sum_{i\in S}\theta_i a^*(S) - \frac{1}{2}(a^*(S))^2 - \sum_{j\in S}\theta_j a^*(S\setminus\{j\})\right).$$

Notice that the contents of the parenthesis is the aggregated "no free riding" surplus: given the no free riding conditions, the most surplus the lobby group S can distribute for their members. The above formula distribute this surplus proportionally according to members' willingnesses-to-pay θs. Obviously, we have $\sum_{i\in S} u_i = V(S) = \sum_{i\in S}\theta_i a^*(S) - k(a^*(S))^2$, and $u_i \geq \theta_i a^*(S\setminus\{i\})$. Thus, we only need to check condition 2. For a coalition $T \subsetneq S$, we have

$$\begin{aligned}
&\sum_{i\in T} u_i - V(T) \\
=\;& \sum_{i\in T}\theta_i a^*(S\setminus\{i\}) + \frac{\sum_{i\in T}\theta_i}{\sum_{j\in S}\theta_j}\left(\sum_{j\in S}\theta_j a^*(S) - \frac{1}{2}(a^*(S))^2 - \sum_{j\in S}\theta_j a^*(S\setminus\{j\})\right) \\
& - \left(\sum_{i\in T}\theta_i a^*(T) - \frac{1}{2}(a^*(T))^2\right) \\
=\;& \frac{\sum_{i\in T}\theta_i}{\sum_{j\in S}\theta_j}\left(\sum_{j\in S}\theta_j a^*(S) - \frac{1}{2}(a^*(S))^2\right) - \left(\sum_{i\in T}\theta_i a^*(T) - \frac{1}{2}(a^*(T))^2\right) \\
& + \sum_{i\in T}\theta_i a^*(S\setminus\{i\}) - \frac{\sum_{i\in T}\theta_i}{\sum_{j\in S}\theta_j}\sum_{j\in S}\theta_j a^*(S\setminus\{j\}).
\end{aligned}$$

We want this to be nonnegative for all $T \subset S$. Now, we use quadratic cost and linear utility. The first order condition for optimal public good provision is

$$a^*(S) = \sum_{i \in S} \theta_i.$$

Thus, we have

$$\sum_{i \in S} \theta_i a^*(S) - \frac{1}{2}(a^*(S))^2 = \frac{\left(\sum_{i \in S} \theta_i\right)^2}{2},$$

and

$$\theta_i a^*(S \backslash \{i\}) = \theta_i \left(\sum_{j \in S} \theta_j - \theta_i \right).$$

Thus, we have

$$\begin{aligned}
&\sum_{i \in T} u_i - V(T) \\
= \; & \frac{\sum_{i \in T} \theta_i}{2 \sum_{j \in S} \theta_j} \left(\sum_{j \in S} \theta_j \right)^2 - \frac{1}{2} \left(\sum_{i \in T} \theta_i \right)^2 + \sum_{i \in T} \theta_i \sum_{j \neq i, j \in S} \theta_j - \frac{\sum_{i \in T} \theta_i}{\sum_{i \in S} \theta_i} \sum_{i \in S} \theta_i \sum_{j \neq i, j \in S} \theta_j \\
= \; & \frac{1}{2} \left(\sum_{i \in T} \theta_i \right) \left(\sum_{j \in S} \theta_j \right) + \sum_{i \in T} \theta_i \left(\sum_{j \in S} \theta_j - \theta_i \right) - \frac{\sum_{i \in T} \theta_i}{\sum_{i \in S} \theta_i} \sum_{i \in S} \theta_i \left(\sum_{j \in S} \theta_j - \theta_i \right) \\
= \; & \frac{1}{2} \left(\sum_{i \in T} \theta_i \right) \left(\sum_{j \in S} \theta_j \right) + \sum_{i \in T} \theta_i \left(\sum_{j \in S} \theta_j \right) - \sum_{i \in T} \theta_i^2 - \sum_{i \in T} \theta_i \left(\sum_{j \in S} \theta_j \right) + \frac{\sum_{i \in T} \theta_i}{\sum_{i \in S} \theta_i} \sum_{i \in S} \theta_i^2 \\
= \; & \frac{1}{2} \left(\sum_{i \in T} \theta_i \right) \left(\sum_{j \in S} \theta_j \right) - \sum_{i \in T} \theta_i^2 + \frac{\sum_{i \in T} \theta_i}{\sum_{i \in S} \theta_i} \sum_{i \in S} \theta_i^2 \\
= \; & \left(\sum_{i \in T} \theta_i \right) \left[\frac{\sum_{j \in S} \theta_j}{2} - \frac{\sum_{i \in T} \theta_i^2}{\sum_{i \in T} \theta_i} + \frac{\sum_{i \in S} \theta_i^2}{\sum_{i \in S} \theta_i} \right] \\
= \; & \left(\sum_{i \in T} \theta_i \right) \left[\frac{\sum_{j \in S} \theta_j}{2} - \sum_{j \in T} \frac{\theta_j}{\sum_{i \in T} \theta_i} \times \theta_j + \sum_{j \in S} \frac{\theta_j}{\sum_{i \in S} \theta_i} \times \theta_j \right].
\end{aligned}$$

The second term is the only negative term, and it takes maximum absolute value when T is composed by the players with the highest values of θ_j. Let us call such value $\theta_{\max}$. Suppose that $\sum_{i\in S} u_i - V(T) < 0$. Then, by focusing the first two terms, we know $\theta_{\max} > \frac{1}{2}\sum_{i\in S}\theta_i$. However, if it is the case, we have

$$\begin{aligned}
& \frac{\sum_{j\in S}\theta_j}{2} - \sum_{j\in T}\frac{\theta_j}{\sum_{i\in T}\theta_i}\times\theta_j + \sum_{j\in S}\frac{\theta_j}{\sum_{i\in S}\theta_i}\times\theta_j \\
\geq\ & \frac{\sum_{j\in S}\theta_j}{2} - \theta_{\max} + \sum_{j\in S}\frac{\theta_j}{\sum_{i\in S}\theta_i}\times\theta_j \\
\geq\ & \frac{\theta_{\max}}{2} - \theta_{\max} + \frac{\theta_{\max}}{\sum_{i\in S}\theta_i}\times\theta_{\max} \\
>\ & \frac{\theta_{\max}}{2} - \theta_{\max} + \frac{1}{2}\times\theta_{\max} = 0.
\end{aligned}$$

This is a contradiction. Therefore, u is in $Core^{FRP}(S)$.□

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