

Coalition Formation: Landscape Theory and Potential Games (Very Incomplete and Preliminary)

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Abstract

In this article we expand the landscape theory of aggregation (Axelrod and Bennett, 1993) that aims to explain why alliances, coalitions or organizational structures emerge in a given system. Our approach covers a large class of environments where coalition formation is based on strategic considerations. In doing so, we highlight the important link between the landscape theory and potential games (Monderer and Shapley, 1996).

1 Introduction

The landscape theory introduced by Axelrod and Bennett (1993) (AB- henceforth) aims to provide a consistent approach to explain how elements of a system can fit together, to predict which configurations are most likely to emerge and how the system will respond to changes in the relationship between its elements. When applied to political, economic and social situations, landscape theory can be used to analyze a wide variety of aggregation problems, such as international alignments, cooperation agreements of business firms, coalitions of political parties in parliaments, social networks, social cleavages in democracies, etc.

In outlining the fundamentals of their theory, formulated by using the language of international alignments, AB make two basic assumptions, drawn from the recognition of the challenge faced by a national leadership to assess the value of a potential alignment. The first assumption is that a nation is *myopic* in its assessment. In other words, a national leadership evaluates its bilateral link with any other nation independent of all other members in the system. By making only pairwise evaluations, the national leadership may avoid a difficult problem of assessing all combinations of nations at once. The second basic assumption of the

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landscape theory is that adjustments to alignments take place by an **incremental** actions of single nations, which rules out the possibility that a coalition of nations within an alignment will its switch allegiance as a block.

The primitives of landscape theory consist of the following fundamentals:

- n actors (nations);
- an n -dimensional vector s , whose i -th coordinate represents the size or importance of nation i to others;
- an $n \times n$ matrix p , whose typical entry p_{ij} (positive or negative) describes the propensity of nations i and j to work together. A large value of p_{ij} simply indicates that two nations get along and do not have many sources of potential conflict.

Then the configuration π emerges, and in the simplest version of the theory, the countries are partitioned into two mutually exclusive blocks X and Y . AB define the *frustration from status quo* $F_i(\pi)$ of a country i as the sum over all countries the sum of i ' propensity with all members j of alliance Y weighted by the j ' size. That is, if $i \in X$ then $F_i(\pi) = \sum_{j \in Y} s_j p_{ij}$. Obviously, the country i frustration will be minimized if X does not contain the countries with whom i has a negative propensity to align. The *energy* of the configuration π is then determined as the sum of frustrations of individual countries weighted by their size and AB examine stable configurations that yield a local minimum of energy in the landscape. AB also provide a spectacular application of the landscape theory to European alliances prior to World War II. By using the Correlates of War data and estimating the propensity for cooperation based on ethnic and border conflicts, history, etc., AB concluded that there were two stable configurations, one is the partition to the Axis and Allies of World War II, and the other separates USSR, Yugoslavia and Greece from the rest of Europe! AB then later applied their model to explain the European alliances over the last twenty years.

In our paper we examine the explicit connection between landscape theory and the theory of games and extend the model to include a more comprehensive class of coalitional environments. Once the options on which actors can potentially agree upon are there, the coalition formation game aims to explain how actors select the policy option they support given the choices made by the others. The situation is obviously interactive since each actor will balance its intrinsic taste for the competing policies with coalitional considerations. In such setting, a coalition is implicitly defined as a group of actors adopting the same policy.

We generalize the AB landscape theory in three directions. First, we allow for a non binary set of policy options, which is the specific context of international alliances may lead to the formation of more than two blocs. While the landscape theory has mostly focused on a binary setting where “if you are not with us, you are against us”, in many international

conflicts there are often more than two alternatives. If a policy with respect to a regional conflict is determined by an international institution, one can imagine a broad range of positions over a country participation (or lack of thereof) in the resolution of the conflict.

Second, we introduce policy options to be chosen by coalitions and allow for intrinsic preferences for policies. Note that the basic assumption of the landscape theory that heterogeneous players would prefer to have a high propensity for cooperation with other members of the coalition, is based on similarity (or dissimilarity) types and characteristics of every pair of actors. We will refer to this type of diversity as *pairwise hedonic heterogeneity* and distinguish it from other sources of differences (e.g. incomes, preferences, etc.) that play an indirect role in the determination and evaluation of policy.¹

Third, in addition to pairwise hedonic heterogeneity we, actors are impacted by the size of the coalition to which they belong. Large coalitions may be preferred to small ones due to economies of scale and voting power; in international relations, resources are pooled to reach military objectives. We may conceive other settings where the challenges of coordination and communication could make smaller coalitions more desirable than the large ones.

We show that if the payoff functions of the players (the inverse of frustrations in AB) satisfy some mild properties, such that any alignment which maximizes the aggregate payoff of all actors is a Nash stable configuration. In addition to showing the existence of Nash equilibria for a large class of coalition formation games, this result establishes a bridge between landscapes and potential games. Axelrod (1997) points out the relationships with game theory : *"Landscape theory says that the stable configurations are those that are in Nash equilibrium. What landscape theory adds to game theory is a way of characterizing all possible configurations and the dynamics among them. In particular, the idea of descent from less satisfactory patterns to more satisfactory patterns helps one characterize the entire range of possibilities in a manner that is sometimes obscure in game-theoretic treatments"*. In our framework we use the theory of potential games (Monderer and Shapley (1996)), where players jointly maximize the potential function. A global maximum of the potential is a Nash equilibrium (in pure strategies). It is interesting to note that within the class of potential games the reservations expressed by Axelrod do not apply. With respect to myopic learning processes, Monderer and Shapley show that the existence of a potential function is equivalent to the convergence of a certain learning process, and their finite improvement

¹The term "hedonic" represents the the way each player evaluates the composition of her coalition irrespective of its policy choices. That is, each player can order different groups she can belong to assuming they all adopt the same policy, so that hedonic heterogeneity is generated by different rankings. See Drèze and Greenberg (1980), Banerjee, Konishi and Sömnez (2001), Bogomolnaia and Jackson (2002) for comprehensive discussions on hedonic games.

property (FIP) is identical to the second basic assumption of landscape theory also called principle of downward movement to an adjacent configuration.

We would like to point out that a Nash equilibrium of the coalition formation game does not need to be a global maximizer of the potential. It must be however a “local” maximizer in the sense that the potential cannot be raised by a move of a single player, and it does not imply that the potential is maximized in a neighborhood of the point under consideration. Somewhat surprisingly, this is the case in the setting considered by AB, where each player has two pure strategies. In such case a “ball” of radius 1 centered on any profile consists exactly of all the vectors deviating from the center in one coordinate. This justifies the use of local energy minimizer in AB and the equivalence the set of Nash equilibria and the set of local minimizers of energy. This does not need to be true in our generalized version of their game.

Related Literature This paper belongs to the brand of literature examining the forces driving the formation of groups, coalitions and alliances.² The majority of contributions identify gains to the cooperation together with the limitations to these gains and proceed (non cooperative or cooperative) analysis of the process of coalition formation and stable structures.

Our paper is related to the recent developments on hedonic games, where players’ payoffs depends solely on the group to which they belongs. In this context our paper shares some common features with Milchtaich and Winter (2002), share some common features with ours. They assume that each player is identified by a one-dimensional parameter (status) and evaluates every potential group S according to the average value of the status there. This model is in fact, a special case of pairwise hedonic heterogeneity. Banerjee, Konishi and Sonmez (2001) and Bogomolnaia and Jackson (2002) consider the case of hedonic group formation games, where each player evaluates every potential partner and then ranks groups according to the sum of the individual evaluations in the group.

There are some other papers where heterogeneity is described in a pairwise manner. This is the setting considered in most of the cooperative games where players are distinct according to their location in some network or the real geographical space. In these settings the gains from cooperation increase when the distances among the players in the coalition decrease. Le Breton and Weber (1995) consider such a setting in the case where only a two-player coalitions can form. Desmet et al. (2006) consider a nation formation game where pairwise hedonic heterogeneity is described by the matrix of genetic distances between nations as calculated by scholars in population genetics. In an international relations setting Bueno

²See Greenberg (1994) and Le Breton and Weber (2003).

de Mesquita (1975, 1981) constructs a matrix that captures the proximity between pairs of nations according to their alliances on defense issues; it defines “indicators of tightness” which are used as a key determinant to evaluate the war proneness of the international system.

Binary coalition formation games have been widely studied in the economic literature as they represent one of the most common situation of collective action. There was an extensive examination of adoption of new technologies (see, e.g. Farrell and Saloner (1985)). Axelrod et al. (1995) who use landscape theory to predict how business firms form alliances to develop and sponsor technical standards. Dixit (2003) cites the formation of the EMU (European Monetary Union) as one of the most prominent example of the binary choice. Each European country balances the benefits of joining (reduction on transaction costs of trade and investment) and staying out (ability to conduct independent monetary and exchange rate policies to counter domestic shocks). In his simple linear setting, Dixit shows that the formation of the grand coalition can be simultaneously Nash stable and dominated from the perspective of aggregate welfare.

Alliances and wars in international politics have been approached differently by some other authors. Some alternative theories have been proposed and tested. Altfeld and Bueno de Mesquita (1979) develop a rational-choice theory, tested for more than two thousand national decisions, of how decision makers choose sides or neutrality in ongoing wars. to remain neutral or to join an ongoing war on one side and the other. Unlike the landscape theory, which uses alliance behavior only to identify active states, the Altfeld and Bueno de Mesquita model uses alliances to identify the utilities of the states and bases its predictions about alignments in wars on these alliances patterns. Altfeld (1984) develops a simple theory of the process by which national governments choose to form military alliances and proceeds to test a necessary condition derived from the theory on the set of major power alliances in the 19th century. Walt (1988) studies the rationale of balancing and the bandwagon behavior when the player is confronted by a significant external threat. Walt also test some predictions on the set of postwar alliances in the Southwest Asia focusing on the commitments of Iran, Turkey, India and Pakistan.

Pairwise hedonic heterogeneity is also used (implicitly or explicitly) in examination of ethnic conflicts (see, e.g., Esteban and Ray (2005)(2006)(2007), Horowitz (1985), Montalvo and Reynal-Querol (2005), Reynal-Querol (2002)). In the analysis of the conflicts and cooperation among ethnic groups, the major concern is to understand the causes of civil wars or other forms of violent bilateral or multilateral disputes among these groups. The methodology used to analyze the war proneness of the international system can be applied for the

examination of this problem. The current research on this topic emphasizes the role of heterogeneity across groups. If a civil war is viewed as a coalition formation game with two sides, the framework developed here in the spirit of AB theory could be useful as well.

2 The Model and the Result

The primitives of the coalition formation normal form game that we consider consists of a finite set $N = \{1, 2, \dots, n\}$ of players and a set of possible (policy) choices X . A pure strategy for player i is a policy choice x_i . The n -dimensional vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ of policy choices yields partition $\pi = (S^k)_{1 \leq k \leq K}$ of N , where players i and j belong to the same coalition if $x_i = x_j$. We denote by $S(\mathbf{x}, x_i)$ or $S^{k(i)}$ the group in the partition which contains player i .

We assume that the payoff of each player depends upon the coalition to which she belongs and the policy adopted in that coalition. Let $U_i(x, S)$ denote the payoff derived by player i if she is a member of group S supporting policy x . With that general formulation, it is possible to describe the two types of heterogeneity to which we alluded before. We will have preference heterogeneity if the following inequality

$$U_i(x, S) > U_i(y, S) \text{ and } U_j(x, S) < U_j(y, S)$$

holds for some $S \subseteq N$, $x, y \in X$ and $i, j \in S$. We will have hedonic heterogeneity if

$$U_i(x, S) > U_i(x, T) \text{ and } U_j(x, S) < U_j(x, T)$$

for some $x \in X$, $S, T \subseteq N$ and $i, j \in S \cap T$. One important class of utility functions that will be considered below is of the form:

$$U_i(x, S) = V_i(x) + \Psi_i(S)$$

The separability property exhibited by these utility functions allows a meaningful distinction between two types of heterogeneity in the sense that, due to preference heterogeneity, the first term is independent of S , whereas hedonic heterogeneity) is represented by the second term, which is independent of the value of x and reflects how the group externality is evaluated by different players. Within that class, the total effect $\Psi_i(S)$ is often decomposed into two terms $\Phi_i(S)$ and $\Delta(S)$:

$$\Psi_i(S) = \Phi_i(S) + \Delta(S).$$

Notice that the second component has no subscript and this aspect of the group externality is therefore common to all players.

A strategy profile \mathbf{x} consists of a pair (π, \mathbf{x}) where $\pi = \{S^1, \dots, S^K\}$ is a partition of N and $\mathbf{x} = (x^1, \dots, x^K)$ is the vector of policies associated with the partition π : it is a coalition structure together with a specification of policies within each coalition. It is Nash stable³ if

$$U_i(x^{k(i)}, S^{k(i)}) \geq U_i(x^k, S^k \cup \{i\}) \text{ for all } i = 1, \dots, N \text{ and } k = 1, \dots, K.$$

Proposition: *If X is compact and for all $i \in N$, there exists functions V_i, Φ_i and Δ such that :*

$$U_i(x, S) = V_i(x) + \Phi_i(S) + \Delta(S)$$

V_i is upper semi continuous

$$\Phi_i(S) = \sum_{\substack{j \in S \\ j \neq i}} [\Phi_j(S) - \Phi_j(S \setminus \{i\})]$$

and

$$\Delta \text{ is anonymous : } \#S = \#T \Rightarrow \Delta(S) = \Delta(T)$$

then there exist a Nash stable coalition structure.

Proof: Let \mathcal{G} be the set of coalition structures. Since X is assumed to be compact and N is finite, \mathcal{G} is a compact set too. Consider now the function P defined on \mathcal{G} as follows:

$$P((\pi, \mathbf{x})) = \sum_{k=1}^K \left[2 \sum_{j \in S^k} V_i(x^k) + \sum_{j \in S^k} \Phi_j(S^k) + 2 \sum_{r=1}^{\#S^k} \Delta(r) \right]$$

where $\pi = \{S^1, \dots, S^K\}$ and $\mathbf{x} = (x^1, \dots, x^K)$. Since V_i is upper semi-continuous for all $i \in N$, then P is upper semi-continuous, too. This implies that the function P has a maximum over \mathcal{G} . We claim that any such maximum is a Nash stable group structure. Let (π^*, \mathbf{x}^*) be a maximum, and assume, on the contrary, that it is not a Nash equilibrium. This implies that there exists $i \in N$ and $k \neq k(i)$ such that

$$V_i(x^k) + \Phi_i(S^k \cup \{i\}) + \Delta(\#S^k + 1) > V_i(x^{k(i)}) + \Phi_i(S^{k(i)}) + \Delta(\#S^{k(i)}),$$

and, by our assumption on the profile of functions $(\Phi_j)_{j \in N}$,

$$\begin{aligned} & 2V_i(x^k) + \Phi_i(S^k \cup \{i\}) + \left[\sum_{j \in S^k} \Phi_j(S^k \cup \{i\}) - \Phi_j(S^k) \right] + 2\Delta(\#S^k + 1) \quad (1) \\ & > 2V_i(x^{k(i)}) + \Phi_i(S^{k(i)}) + \left[\sum_{\substack{j \in S^{k(i)} \\ j \neq i}} \Phi_j(S^{k(i)}) - \Phi_j(S^{k(i)} \setminus \{i\}) \right] + 2\Delta(\#S^{k(i)}) \end{aligned}$$

³i.e., a Nash equilibrium of the normal form game.

Consider the group structure $(\tilde{\pi}, \tilde{\mathbf{x}})$ where $\#\tilde{\pi} = \#\pi^* = K$ and

$$\tilde{S}^l = \begin{cases} S^l & \text{for all } l \neq k(i), k \\ S^{k(i)} \setminus \{i\} & \text{for } l = k(i) \\ S^k \cup \{i\} & \text{for } l = k \end{cases}$$

From the construction of P , we verify that

$$\begin{aligned} & P(\tilde{\pi}, \tilde{\mathbf{x}}) - P(\pi^*, \mathbf{x}^*) \\ = & \left[2V_i(x^k) + \Phi_i(S^k \cup \{i\}) + \left[\sum_{j \in S^k} \Phi_j(S^k \cup \{i\}) + \sum_{\substack{j \in S^{k(i)} \\ j \neq i}} \Phi_j(S^{k(i)} \setminus \{i\}) \right] + 2\Delta(\#S^k + 1) \right] \\ & - \left[2V_i(x^{k(i)}) + \Phi_i(S^{k(i)}) + \left[\sum_{j \in S^k} \Phi_j(S^k) + \sum_{\substack{j \in S^{k(i)} \\ j \neq i}} \Phi_j(S^{k(i)}) \right] + 2\Delta(\#S^{k(i)}) \right] \end{aligned}$$

Inequality (1) implies that

$$P(\tilde{\pi}, \tilde{\mathbf{x}}) - P(\pi^*, \mathbf{x}^*) > 0,$$

contradicting our assumption that (π^*, \mathbf{x}^*) yields a maximum of P . \square

It is interesting to observe that the proof of the proposition amounts to show that the group formation game is a potential game (Monderer and Shapley (1996))⁴ and therefore has a Nash equilibrium in pure strategies. The potential is the function P and the all task is to discover it whenever it exists. It is important to observe that $P(\pi, \mathbf{x})$ is not, in general, the aggregate payoff attached to the profile (π, \mathbf{x}) . It departs from it in two places: the first and third term of the individual payoff are doubled and the third term $\#S^k \Delta(\#S^k)$ is replaced by $\sum_{r=1}^{\#S^k} \Delta(r)$. Our proposition generalizes both proposition 2 in Bogomolnaia and Jackson (2002) who assume $V_i \equiv 0$ for all $i \in N$, $\Delta \equiv 0$ and a specific function Φ_i and proposition 4.1. in Konishi, Le Breton and Weber (1997) who assume $\Phi_i \equiv 0$ for all $i \in N$.

3 Complements and Extensions

We discuss several side and related topics

⁴This ingenious idea goes back to Rosenthal (1973).

3.1 Axelrod and Bennet's Landscape

Let us now show how the proposition relates to the AB landscape framework. In their model, the primitives consists of a vector $(s_i)_{i \in N}$ describing the sizes of the players and a matrix $D \equiv (d_{ij})_{i,j \in N}$ where the value d_{ij} represents the distance between the players i and j . The propensity for cooperation p_{ij} is actually the inverse of the distance matrix and we may set $p_{ij} = -d_{ij}$. The matrix D is symmetric, i.e., $d_{ij} = d_{ji}$ for all $i, j \in N$. As alluded to in the introduction, AB assume that for a given policy choice, each player i prefers to be part of groups with players j such that d_{ij} is as small as possible.⁵ They further assume that $\#X = 2$ i.e. that they are at most two possible options say to be in favor or opposed to a specific action. each player must chose his/her side and that V_i is constant for all $i \in N$.

Precisely, their model corresponds to the following specification of the payoff function of the players

$$U_i(x, S) = - \left[\sum_{j \in S} s_i s_j d_{ij} \right],$$

or, equivalently,

$$U_i(x, S) = - \left[\sum_{j \in S} s_j d_{ij} \right]$$

$$\Phi_i(S) = \sum_{j \in S} s_i s_j d_{ij}.$$

It is easy to check that the condition on Φ_i that appears in the proposition is verified under this specification. Indeed, since $d_{ij} = d_{ji}$ for all $i, j \in N$, we obtain

$$\sum_{\substack{j \in S \\ j \neq i}} [\Phi_j(S) - \Phi_j(S \setminus \{i\})] = \sum_{\substack{j \in S \\ j \neq i}} s_i s_j d_{ji} = \sum_{\substack{j \in S \\ j \neq i}} s_j s_i d_{ij} = \Phi_i(S)$$

One important aspect of the AB specification is the fact that the potential coincides with a weighted sum of the payoffs. The configurations which are global minimizers of energy are therefore Pareto optimal. As already pointed out this assertion does not hold in general.

The proposition can be generalized in several directions. The main limitation of the current formulation is our assumption that payoffs do not depend upon policies supported by the other coalitions. In that respect, it is possible to extend the proposition to a broader

⁵In addition to this monotonicity property, they assume that each player i ranks any two groups S and T on the basis of the vectors $(d_{ij})_{j \in S}$ and $(d_{ij})_{j \in T}$. One important special case is the following dichotomic model, where $d_{ij} \in \{0, 1\}$ for all $i, j = 1, \dots, N$. That is, each actor considers every potential partner is either friend or enemy without any intermediate qualification.

class of games where such a concern is taken into account. Specifically, we can demonstrate that if

$$U_i(\mathbf{x}) = V_i(x_i) + \Phi_i(S(\mathbf{x}, x_i)) + \Delta(S(\mathbf{x}, x_i)) + \sum_{x \neq x_i} \Gamma(x, S(\mathbf{x}, x))$$

where the fourth term is assumed to be anonymous, then there exists a potential and therefore the game admits a Nash stable structure.

In the case where a policy is multi-dimensional (a reasonable assumption in current international affairs), we could even generalize further the results by disconnecting the group membership across dimensions. Partners on one specific dimension may differ from partners on another dimension and the set of possible strategies needs not be a product set⁶.

Finally, we could also weaken the anonymity hypothesis at the expense of non linearity in the evaluation of the third component of the group externality. For instance, the proposition extends to the following specification:

$$U_i(x, S) = - \left[\sum_{j \in S} s_j d_{ij} \right] + \gamma \left[\sum_{j \in S} s_j \right],$$

where γ is a positive parameter. It amounts to a change in the definition of the distances.

3.2 Systematic Polarity

One major area of application of our result is international politics where there are multiple reasons for explaining coalition formation among countries. These issues appear in many occasions and can be evaluated from different perspectives. In the short run, we may desire to understand how countries position themselves on a scale of possible positions in some particular specific issue or event (the Iraq's war, the Middle East conflict, etc.). There are many game-theoretical models of war. Among those, the model developed and tested by Bueno de Mesquita aims to describe the decision process leading a specific country to initiate a war against another country. To evaluate the opportunity of that particular action, the initiator must predict the outcome of the coalition formation game that will arise after the beginning of the war. We may instead want to use coalition formation models to understand the patterns of cooperation/conflict from a long run perspective. Indeed, countries sign bilateral or multilateral agreements (defense, trade, immigration,...) which act later on as possible constraints or commitments in some particular occasions. Despite obvious and important differences between the two classes of problems, our framework can be used to formulate the strategic environment describing the interactions across countries. We

⁶See Le Breton and Weber (2003).

illustrate this through the seminal work of Bueno de Mesquita. To understand how pairwise hedonic heterogeneity appears in his work, we cite his 1975 paper: “Three aspects of polarity appear to influence the amount of war that occurs between nations. The first concerns the number of poles—or clusters of nations- in the system. The second concerns the degree to which the foreign policies of nations within a single cluster are similar to each other, and the degree to which the foreign policies of nations in different clusters are dissimilar. The third relevant attribute concerns the degree of inequality in the distribution of power—or the potential ability to wage war—among the clusters of nations”.

The first set of attributes is very much related to the generalization provided in our paper as bipolarity, as opposed to multipolarity of the international system. Bipolarity is not assumed and a major part of the work will consist in the determination of the number of poles. The second set of attributes brings us to pairwise hedonic heterogeneity. To see why and how, it is important to examine how Bueno de Mesquita evaluates heterogeneity within each pole and between different poles. As he puts it “When all members of a bloc are highly committed to one another, it is difficult for any one member of the cluster to venture on an independent foreign policy course that involves developing relations with nations outside the bloc”. The pairwise similarity between two nations can be evaluated in a variety of ways and likely the selection of a particular indicator that reflect foreign policy commitments will greatly influence the memberships and structures of the clusters that are identified. Bueno de Mesquita develops an ingenious and original measure based on the degree of similarity in alliance commitments across all possible pairs of nations. Following the coding scheme of the Correlates of War project (COW), he distinguishes between four types of military alliances:

- Defense pacts, in which the signatories agree to come to each other’s mutual defense in case any one signatory is attacked.
- Neutrality or nonaggression pacts, in which the signatories agree not to declare war against each other in the event that a third nation declares war against one of them.
- Ententes, in which the signatories agree to consult each other in the event any one of them is attacked by a third party.
- No alliance at all.

These four types of alliances can be ranked ordinally from the greatest sacrifice of decision-making autonomy to the least sacrifice of such autonomy. For each dyad of nations, Bueno de Mesquita constructs a four-by-four contingency table. The columns (respectively the rows) represent the degree of military commitment between the column (respectively the row) nation and every other active nation in the international system. Once this is done, the degree of similarity can be summarized through the computation of an appropriate

measure of association. Bueno de Mesquita uses Kendall's τ_b score. Once a suitable statistic of association is computed for each dyad $\{i, j\}$, we obtain a square matrix $(\tau_b(i, j))_{1 \leq i, j \leq n}$ which describes the pairwise hedonic heterogeneity among the n nations. With this data, he proceeds to identify the cluster membership of each nation. Interestingly, he also uses the matrix to determine the degree of similarity of alliance commitments among members of the same cluster, or among members of different clusters. By construction $\tau_b(i, j)$ belongs to the interval $[-1, 1]$. It takes the value 1 if the two nations have signed a defense pact and have the same pattern of alliances with any third nation and it takes the value -1 if they did not sign any alliance pact and have opposite patterns of alliances with any third nation. Therefore, a cluster achieves the maximum degree of similarity among its members if they all have mutual defense pacts with each other. Given a partitioning $\pi = (S_k)_{1 \leq k \leq K}$ of the set of nations into K poles, He defines $T(\pi)$ the *indicator of systemic tightness* attached to π as follows

$$T(\pi) \equiv \frac{\sum_{1 \leq k \leq K} \sum_{\{i, j\} \subset S_k} \tau_b(i, j)}{\sum_{1 \leq k \leq K} \frac{n_k(n_k-1)}{2}}$$

He defines similarly an *indicator of systemic discreteness* $D(\pi)$ attached to π by the dissimilarity of the alliance commitments among members of different clusters along the same lines⁷

$$D(\pi) \equiv \frac{(-1) \sum_{1 \leq k \leq K} \sum_{1 \leq l \leq K} \sum_{i \in S_k, j \in S_l} \tau_b(i, j)}{\frac{n(n-1)}{2} - \sum_{1 \leq k \leq K} \frac{n_k(n_k-1)}{2}}$$

Bueno de Mesquita also considers the distribution of power among actors in the international system as a third important component to understanding the occurrence and duration of wars, or other forms of international conflict occurring between nations. On the basis of an extremely rich data set, he proceeds to a statistical analysis of the international system. For the century and a half, the mean discreteness score was 0.411, with the nineteenth-century being 0.597 and the twentieth century mean being 0.203. He also calculates that the average level of tightness for the entire period was 0.713, with the mean in the earlier period equal to 0.725, and the mean for the twentieth century equal to 0.693. As, with the discreteness indicator, there is a substantial differences in the number of poles between the two centuries, with the mean for the earlier period being 2.33, and the mean for the latter period being 5.88. Note however that while the average number of clusters increases with time, the size of the clusters remains about the same. The average cluster in the nineteenth century contained 4.58 members, while in the twentieth century the average cluster had 4.60 members.

⁷The multiplication by -1 is done for the sake of presentation.

While statistical, this analysis is instructive in many respects. First, it offers a decomposition of the set of nations into an endogenous number of poles which is based on a matrix of pairwise hedonic heterogeneity which is quite different from the matrix constructed by AB. Second, up to the normalization operation, it also suggests the use of a potential function to determine the optimal partitioning under the presumption that for nation i being with some nations j such that $\tau_b(i, j)$ is large increases its payoff in the game. Third, it prepares the grounds for the analysis of the coalition formation games resulting from the initiation of a war and the war proneness of the international system.

Bueno de Mesquita uses the tipal method to produce an empirical partition of the countries into poles. We propose below an alternative and new⁸ method which, surprisingly, amounts to the maximization of a potential. Consider the situation where there is a true state of the world described by an equivalence relation \sim over N with the following interpretation : $i \sim j$ iff and only have friendly relationships. It is assumed therefore that \sim is transitive and symmetric. Let us assume for the time being that if two countries have a friendly relationship, the probability p that none of them initiates a war against the other during any single period is larger than $\frac{1}{2}$ while the probability that two countries not having a friendly relationship over a given period of time experience a bilateral war is also equal to p . Consider the data on wars over T periods and let $\chi^t(i, j) = 1$ if countries i and j have not experimented a war against each other during period t and $\chi^t(i, j) = 0$ otherwise. What can be learned from the dyadic data collected over the T periods concerning the true state of the world i.e. the equivalence relation \sim described alternatively by its set of equivalence classes $(S_k)_{1 \leq k \leq K}$. The likelihood of this partition is :

$$\left(\prod_{k=1}^K \prod_{\{i,j\} \subset S_k} p^{t_{ij}} (1-p)^{T-t_{ij}} \right) \left(\prod_{k=1}^K \prod_{l=k+1}^K \prod_{i \in S_k} \prod_{j \in S_l} (1-p)^{t_{ij}} (p)^{T-t_{ij}} \right)$$

where $t_{ij} \equiv \sum_{1 \leq t \leq T} \chi^t(i, j)$. The logarithm of the likelihood is then given by the expression :

$$\sum_{k=1}^K \left[\sum_{\{i,j\} \subset S_k} t_{ij} \log \frac{p}{1-p} - \sum_{l=k+1}^K \sum_{i \in S_k} \sum_{j \in S_l} t_{ij} \log \frac{p}{1-p} \right] + \frac{T \log(1-p)}{2} \sum_{k=1}^K n_k (n_k - 1) + T \log(p) \sum_{k=1}^K \sum_{l=k+1}^K n_k n_l.$$

⁸A similar statistical approach has been developed by Kemeny (1959) and Slater (1961) in the theory of voting to infer the true preference over alternatives from a sequence of binary votes displaying cycles.

Since

$$2 \sum_{k=1}^K \sum_{l=k+1}^K n_k n_l = \left(\sum_{k=1}^K n_k \right)^2 - \sum_{k=1}^K n_k^2$$

we obtain, up to a term independent of the partition

$$\log \frac{p}{1-p} \left(\sum_{k=1}^K \left[\sum_{\{i,j\} \subset S_k} t_{ij} - \sum_{l=k+1}^K \sum_{i \in S_k} \sum_{j \in S_l} t_{ij} \right] - \frac{T \sum_{k=1}^K n_k^2}{2} \right).$$

Furthermore

$$\sum_{\{i,j\} \subset N} t_{ij} = \sum_{k=1}^K \sum_{l=k}^K \sum_{i \in S_k} \sum_{j \in S_l} t_{ij} = \sum_{k=1}^K \sum_{l=k+1}^K \sum_{i \in S_k} \sum_{j \in S_l} t_{ij} + \sum_{k=1}^K \sum_{\{i,j\} \subset S_k} t_{ij}$$

we obtain finally, up to a term independent of the partition

$$\log \frac{p}{1-p} \left(2 \sum_{k=1}^K \sum_{\{i,j\} \subset S_k} t_{ij} - \frac{T \sum_{k=1}^K n_k^2}{2} \right) = \log \frac{p}{1-p} \left(\sum_{k=1}^K \sum_{i \in S_k} \sum_{j \in S_k} t_{ij} - \frac{T \sum_{k=1}^K n_k (n_k - 1)}{2} \right)$$

Since $p > \frac{1}{2}$, we deduce that maximizing the likelihood is equivalent to maximizing

$$\left(\sum_{k=1}^K \sum_{i \in S_k} \sum_{j \in S_k} t_{ij} - \frac{T \sum_{k=1}^K n_k (n_k - 1)}{2} \right).$$

Interestingly, in contrast to the tipal method, this function is a potential for a game where the payoff of nation i for being part of coalition S_k would be

$$\sum_{j \in S_k} t_{ij} - \frac{T n_k}{2}.$$

We recognize the hedonic component, that we already encountered in our analysis of AB, to which is added an anonymous negative externality component. Our theorem covers that general case.

Bueno de Mesquita (1981) develops an expected-utility model to provide rational foundations to the decision by a given nation i of initiating a war against another nation j . His formulation is complicated in the sense that many different elements enter into the calculation of the payoff attached to such decision. The first item is an hedonic term U_{ij} describing the i 's value for j 's policy positions before the war. This parameter is normalized to belong to the interval $[-1, 1]$ with the convention $U_{ii} = 1$. The gross gain of a successful war is then $U_{ii} - U_{ij}$. Expected utility is calculated on the basis of a probability of i succeeding

against j in a bilateral conflict. This variable will depend upon the strengths of the two nations. This would be the end of the story if there were no other nations in the world or if it could be asserted for sure that no other nation will participate to this conflict. Otherwise, the initiator needs to predict what will be the reaction of all other states and subsequently the outcome of a multilateral war. The continuation game following the initiation of a war is a coalition formation game similar in any respects to the one examined by AB, except for the fact that nations may have more than two choices. Bueno de Mesquita divides all other states into five categories so that the leader contemplating the initiation of a war must consider the strength and relative utility of seven types of actors including the defender and himself. The seven types include nations that are friendly toward the defender but not the initiator, nations that are friendly toward the initiator but not the defender, nations that are friendly to both, allied nations not friendly toward either the attacker or the defender, and all nonaligned nations. Expected utility maximization in the case of a multilateral conflicts involves many new terms involving determination of the measurement of the utility of any third country for either i or j . Any indicator of utility must capture the congruence of interests between nations. The congruence of interests may be reflected by the general similarity of their behavior across a variety of dimensions⁹. He argues that the dyadic utility scores $\tau_b(i, j)$ described earlier provide a best estimate of the congruence of interests.

The coalition formation side of this large game is developed in Altfeld and Bueno de Mesquita (1979)¹⁰. The optimization investigated from the perspective of a third party i.e. a player in the coalition formation game (following the initiation of a war between two nations). They use also Bueno de Mequita's dyadic tightness indices to estimate utilities. The econometric version of the choice equation is estimated and tested and the predictions are satisfactory. The problem with that formulation as opposed to AB is that they do not consider the simultaneous coalition formation game played by all third nations but opt instead for an analysis of each third party in isolation. On the other hand, players have three possible choices instead of two as they have an option to remain neutral.

3.3 Dichotomic Heterogeneity

In some situations, it makes sense from the perspective of any participant to the coalition formation game to separate the potential partners into two disjoint groups. The first group is composed of the players which he considers to be "good" partners while those in the

⁹Bueno de Mesquita (1975)(1981) presents and comments many alternative proxies that have been used in the literature.

¹⁰See also Altfeld (1984) and Walt (1988).

second group are considered as "bad" partners. There is no further evaluation of goodness and badness on a scale i.e. no further screening within the goods or within the bads. The all social heterogeneity is then represented by a graph \mathcal{G} with N as set of vertices and with an (undirected) edge between i and j if and only if i and j consider themselves as bad partners. The set N may either represent the set of individuals or the set of types if individuals are already partitioned according to some exogenous characteristics; in the second interpretation, we denote by the proportion of individuals into the society which are of type i . Given such a graph, we can consider the following strategic environment. A bad partner of i is perceived by him as a player with whom a conflict can emerge whenever they belong to the same community. Let π_j be the proportion of persons of type j in the group that we are considering. Then player i in that group may want to minimize the probability of having a conflict with somebody else in the society. Suppose that the person with whom you interact in any of these social matching games is drawn randomly, then the probability of a conflict is equal to the expression

$$\sum_{j=1, j \neq i}^N \pi_j d_{ij} \text{ where } d_{ij} = 1 \text{ if } i \text{ and } j \text{ are bad partners and } d_{ij} = 0 \text{ otherwise}$$

The potential to maximize is

$$-\sum_{i=1}^N \sum_{j=1, j \neq i}^N \pi_i \pi_j d_{ij}$$

When the graph \mathcal{G} is complete i.e. when everybody considers everybody with another type as bad, than the potential becomes

$$-\sum_{i=1}^N \sum_{j=1, j \neq i}^N \pi_i \pi_j = -\left(1 - \sum_{i=1}^N \pi_i^2\right)$$

which is, in fact, a fractionalization index. Of course, if we consider a partition $(S_k)_{1 \leq k \leq K}$ of the society into groups, the pairwise hedonic component of the potential to be maximized will be

$$-\sum_{k=1}^K \sum_{i \in S_k} \sum_{j \in S_k, j \neq i} \pi_i \pi_j d_{ij}$$

In many situations, this force pushing to the formation of small groups is balanced by increasing returns to scale. Suppose that the preferences of the players are lexicographic across dimensions. They first look at the hedonic component and if the two groups are equivalent in that respect, they look for the size of the group with a preference for large groups. With

such a profile of preferences, the following construction delivers a Nash equilibrium. An *independent set*¹¹ of the graph \mathcal{G} is a subset S of vertices such that $d_{ij} = 0$ for all $i, j \in S$. Let $\alpha(\mathcal{G})$ be the *independence number of the graph \mathcal{G}* i.e. the largest size of an independent set of \mathcal{G} . Let S_1 be any subset of N containing $\alpha(\mathcal{G})$ vertices. Then, let $N_2 \equiv N \setminus S_1$ and \mathcal{G}_2 be the graph \mathcal{G} restricted to N_2 . Define S_2 to be any subset of N containing $\alpha(\mathcal{G}_2)$ vertices and continue this process until the set of vertices is entirely covered. The partition $(S_k)_{1 \leq k \leq K}$ that we obtain is a Nash equilibrium. Indeed, nobody from group S_k prefers to migrate to group $S_{k'}$ if $k' > k$ since, by construction, $\#S_{k'} < \#S_k$. Further, since preferences are lexicographic towards the hedonic dimension, nobody from group $S_{k'}$ prefers to migrate to group S_k if $k' > k$ since, if that deviation was profitable, this would contradict the definition of the independence number. There may exist other Nash equilibria due to a lack of coordination but we conjecture that this Nash equilibrium is the unique strong Nash equilibrium of this coalition formation game. It is interesting to have an idea of how many groups will appear in that specific coalition structure. If the partition contains K groups, this implies that we can color the vertices of G by using at most K colors in such a way that two adjacent vertices (here two bad partners) have different colors. The least number of colors to partition the vertices in such a way that no two adjacent vertices have the same color, called the *chromatic number of \mathcal{G}* , serves as a lower bound on the number of groups in these ideal Nash partitions.

Interestingly, these Nash stable coalition structures are very similar to the partition into maximal *cliques* considered in sociometry and social network analysis¹². A clique is a subset S of vertices such that in restriction to S , the graph \mathcal{G} is complete. We would have maximal cliques here, instead of maximal independent sets, if instead of creating a link between i and j if they are bad partners, as we did, we created an edge between i and j if there were good partners¹³. Sociologists are mostly interested by the identification of cohesive areas of a network and the concept of clique is among the more popular notions to perform that task even if it has been disputed and/or generalized along several lines.¹⁴ The *clique number of the graph \mathcal{G}* i.e. the largest size of a clique of \mathcal{G} . It is easy to see that the clique number is less than (or equal to) the chromatic number. Knowing the clique number (with our definition of links) provides therefore information on the number of groups in a stable coalition structure. Most of the time this lower bound is rough. A graph is perfect if the chromatic number of every induced subgraph equals the clique number of that subgraph.

¹¹For all graph-theoretical notions used hereafter, we refer the reader to Bollobas (1998).

¹²In sociology the graph \mathcal{G} is called a *sociogram*.

¹³The initial graph is replaced by its complement.

¹⁴It is impossible to review this enormous literature (see for instance Alba (1973), Borgatti, Everett and Shirey (1990) and Seidman and Foster (1978)).

Lovasz (1972) has demonstrated that a graph is perfect if and only if its complement is perfect¹⁵. Finally note that a graph may have of course many maximal cliques. Moon and Moser (1965) have demonstrated that this number is less than $3^{\frac{n}{3}}$ and that this bound is tight. While important, this defense of the concept of clique is, to the best of our knowledge, purely statistical and not based on any behavioral postulates. Our theorem demonstrates that it can be supported as a Nash stable coalition structure for an appropriate specification of the preference parameters.

If X is finite, the lexicographic preference can be represented by an additive utility function meeting the conditions of our theorem¹⁶. Of course, the theorem applies to a larger class of utility functions. Besides social networks, one additional nice application of our theorem in a situation where the pairwise hedonic heterogeneity is dichotomic, is provided by the analysis of standard-setting alliances of Axelrod et al. (1995). The objective of their model is to predict how business firms form alliances to develop and sponsor technical standards. They postulate that the utility for a firm for joining a particular standard setting alliance increases with the size of the alliance and decrease with the presence of rivals in the alliance¹⁷, especially close *rivals*. Formally¹⁸, they assume that the utility $U_i(A)$ to firm i of joining alliance A is as follows :

$$U_i(A) = \sum_{j \in A} s_j - \left[\alpha \sum_{j \in D_i \cap A} s_j + (\alpha + \beta) \sum_{j \in C_i \cap A} s_j \right]$$

where α and β are two positive parameters, s_j denotes the size of firm j and C_i and D_i denote respectively the *close and distant rivals* of firm i : $C_i \cap D_i = \emptyset$ and $C_i \cup D_i = N \setminus \{i\}$. We can reformulate utility as :

$$U_i(A) = \sum_{j \in A} s_j p_{ij} \text{ where } p_{ij} \equiv \begin{cases} 1 - \alpha & \text{if } j \in C_i \\ 1 - \alpha - \beta & \text{if } j \in D_i \end{cases}$$

In contrast to AB, the propensity p_{ij} of firms i and j to cooperate can take here only two possible values. Axelrod and alii (1995) illustrate and test their theory by estimating the choices of nine computer companies to join one of two alliances sponsoring competing UNIX operating system standards in 1988. They partition the 9 firms into 4 workstation

¹⁵The family of perfect graphs contains many important well-known sets of graphs.

¹⁶In the general case, the groups in a Nash stable coalition structure do not need to be cliques. We may wonder how far we are from such ideal pattern. In that respect, the clustering coefficient which is a measure of the likelihood that two associates of a vertex are associates themselves may be relevant as a higher clustering coefficient indicates a greater "cliquishness".

¹⁷We refer to their paper for a nice discussion of the intensity of rivalry.

¹⁸Strictly speaking, they do not assume $\#X = 2$ but conduct most of their analysis in that case.

specialists and 5 generalists. Pairs of specialists and pairs of generalists are defined as close rivals while generalists and specialists are assumed to be distant rivals. They assume further that $\alpha = \beta = 1$. With 9 firms, there are 256 possible alliance configurations of at most two alliances each. Two of them were Nash coalition structures and the first configuration fits what happened : 8 of 9 memberships are estimated correctly and the proportion of aggregate size that was correctly estimated was very high, namely 97%. They also examine the robustness of these predictions with respect to changes in the rivalry parameters.

3.4 MaxCut

We have demonstrated the existence of Nash stable coalition structures for a class of coalition formation games which fits many diverse environments where these issues arise. We have been silent on the computational aspects attached to the determination of these Nash equilibria and the algorithms to perform this task.

Very surprisingly, when $K = \#X = 2$, the determination of a Nash equilibrium through the maximization of the potential is equivalent to a famous problem in combinatorial optimization known as the MaxCut problem which can be described simply as follows. Given an undirected graph \mathcal{G} on a set of vertices N and nonnegative weights $w_{ij} = w_{ji}$ on the edges (i, j) of the graph, the MaxCut problem is that of finding the subset of vertices S that maximizes the weight of the edges in the cut $(S, N \setminus S)$, that is the weight of the edges with one endpoint in S and the other in $N \setminus S$. It has long been known to be NP-complete. Because, it is unlikely that there exist efficient algorithms for NP-hard maximization problems, a typical approach to solving such a problem is to find a ρ -approximation algorithm, that is a polynomial-time algorithm that delivers a solution of value at least ρ times the optimal value (the constant ρ is sometimes called the performance guarantee of the algorithm). When $K > 2$, the determination of a Nash equilibrium through the maximization of the potential is also equivalent to the following generalized version of the MaxCut problem. Given an undirected graph \mathcal{G} on a set of vertices N and nonnegative weights $w_{ij} = w_{ji}$ on the edges (i, j) of the graph, the generalized MaxCut problem is that of finding a partition $\{S_k\}_{1 \leq k \leq K}$ of N that maximizes the weight of the edges in the partition, that is the weight of the edges with one endpoint in some S_k and the other in S_l with l different from k . Indeed, as for any

given partition $\{S_k\}_{1 \leq k \leq K}$ of N , we have the trivial equality

$$\begin{aligned} \sum_{i=1}^N \sum_{j=1}^N w_{ij} &= \sum_{k=1}^K \sum_{l=1}^K \sum_{i \in S_k} \sum_{j \in S_l} w_{ij} \\ &= \sum_{k=1}^K \sum_{i \in S_k} \sum_{j \in S_k} w_{ij} + \sum_{k=1}^K \sum_{l=1, l \neq k}^K \sum_{i \in S_k} \sum_{j \in S_l} w_{ij} \end{aligned}$$

Since the left hand side does not depend upon the partition, minimizing the sum across groups of their intra-group heterogeneities is equivalent to maximizing the sum across pairs of different groups of their inter-group heterogeneities i.e. minimizing the potential is the same as maximizing the aggregate sum of weights over pairs of players belonging to different groups.

The generalized MaxCut problem is equivalent to the following integer quadratic program

$$Max \sum_{i=1}^N \sum_{j=i+1}^N w_{ij} (1 - \langle y_i, y_j \rangle)$$

under the constraints

$$\begin{aligned} y_i &\in \{0, 1\}^K \text{ for all } i = 1, \dots, N \\ \text{and } \sum_{k=1}^K y_{ik} &\geq 1 \text{ for all } i = 1, \dots, N. \end{aligned}$$

Indeed, since the K -dimensional vectors y_i and y_j are integer valued

$\langle y_i, y_j \rangle = 0$ if and only if i and j belongs to different elements of the partition $\{S_k\}_{1 \leq k \leq K}$

When $K = 2$, the integer quadratic problem can also equivalently formulated as

$$Max \sum_{i=1}^N \sum_{j=i+1}^N w_{ij} (1 - y_i y_j)$$

under the constraints

$$y_i \in \{-1, 1\} \text{ for all } i = 1, \dots, N.$$

Goemans and Williamson (1994) examine a relaxation of the above problem which is extremely ingenious. It consists to replace the above problem by the problem

$$Max \sum_{i=1}^N \sum_{j=i+1}^N w_{ij} (1 - \langle v_i, v_j \rangle) \quad (**)$$

under the constraints

$$v_i \in S_n \text{ for all } i = 1, \dots, N,$$

where S_n is the n -dimensional unit sphere. $y_i \in \{0, 1\}^2$. Let $\{v_1^*, v_2^*, \dots, v_N^*\}$ be an optimal set of vectors solution of the problem (*). Let r be a vector uniformly distributed on the unit sphere S_n and $S \equiv \{i \in N : \langle r, v_i^* \rangle\}$. Let W be the value of the random cut produced in this way and $E[W]$ be its expectation. Goemans and Williamson have demonstrated that for any set of vectors $\{v_1, v_2, \dots, v_N\}$ such that $v_i \in S_n$ for all $i = 1, \dots, N$:

$$E[W] \geq \lambda \sum_{i=1}^N \sum_{j=i+1}^N w_{ij}(1 - \langle v_i, v_j \rangle)$$

where

$$\lambda = \text{Min}_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos \theta} > 0.878$$

This algorithm has a performance guarantee of λ . To show that this algorithm can be implemented in polynomial time, they show that problem (*) is equivalent to a semi-definite program. Using the Cholesky decomposition, they show that problem (*) is equivalent to the semi-definite program

$$\text{Max} \sum_{i=1}^N \sum_{j=i+1}^N w_{ij}(1 - v_{ij})$$

under the constraints

v is a $N \times N$ symmetric positive semi-definite matrix such that $v_{ii} = 1$ for all $i = 1, \dots, N$.

They conclude by exploiting algorithmic results established in semi-definite programming. It is not straightforward to generalize this method to treat the generalized max cut problems for $K \geq 3$.

In the case where the hedonic data is dichotomic and described by a graph, we have seen above that the problem was to discover maximal independent set of maximal cliques. Unfortunately, these problems are also well known to be NP-complete. Vertex coloring is also NP-complete in the general case and approximation algorithms have been developed in graph theory to provide satisfactory solutions.

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