

Von Neumann-Morgenstern Farsightedly Stable Sets in Two-Sided Matching*

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Abstract

We adopt the notion of von Neumann-Morgenstern farsightedly stable sets to predict which matchings are possibly stable when agents are farsighted in one-to-one matching problems. We provide the characterization of von Neumann-Morgenstern farsightedly stable sets: a set of matchings is a von Neumann-Morgenstern farsightedly stable set if and only if it is a singleton set and its element is a corewise stable matching. Thus, contrary to the von Neumann-Morgenstern (myopically) stable sets, von Neumann-Morgenstern farsightedly stable sets cannot include matchings that are not corewise stable ones.

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1 Introduction

Gale and Shapley (1962) have proposed the simple two-sided matching model, known as the marriage problem, in which matchings are one-to-one. There are two disjoint sets of agents, men and women, and the problem is to match agents from one side of the market with agents from the other side where each agent has the possibility of remaining single. They have shown that the set of corewise stable matchings is nonempty. A matching is corewise stable if there is no subset of agents who by forming all their partnerships only among themselves (and having the possibility of becoming single), can all obtain a strictly preferred set of partners.¹ Recently, Ehlers (2007) has characterized the von Neumann-Morgenstern stable sets in one-to-one matching problems. A set of matchings is a von Neumann-Morgenstern stable set if this set satisfies two robustness conditions: (internal stability) no matching inside the set is dominated by a matching belonging to the set; (external stability) any matching outside the set is dominated by some matching belonging to the set. He has shown that the set of corewise stable matchings is a subset of any von Neumann-Morgenstern stable sets.

The notions of corewise stability and of von Neumann-Morgenstern stable set are myopic notions since the agents cannot be farsighted in the sense that individual and coalitional deviations cannot be countered by subsequent deviations. A first attempt to reflect the idea that agents are not myopic is the concept of weak stability due to Klijn and Massó (2003). An individually rational matching is weakly stable if for every blocking pair one of the members can find a more attractive partner with whom she/he forms another blocking pair for the original matching.

In this note, we go further by adopting the notion of von Neumann-Morgenstern farsightedly stable sets to predict which matchings are possibly stable when agents are farsighted. This concept is due to Chwe (1994) who has introduced the notion of indirect dominance into the standard definition of von Neumann-Morgenstern stable sets. The indirect dominance relation captures the fact that farsighted agents consider the end matching that their move(s) may lead to. Thus, a set of matchings is a von Neumann-Morgenstern farsightedly stable set if no matching inside the set is *indirectly* dominated by a matching belonging to the set (internal stability), and any matching outside the set is *indirectly* dominated by some matching belonging to the set (external stability).

Our main result is the characterization of von Neumann-Morgenstern farsightedly stable sets in one-to-one matching problems. A set of matchings is a von Neumann-Morgenstern farsightedly stable set if and only if it is a singleton set and its element is a

¹We refer to Roth and Sotomayor (1990) for a comprehensive overview on two-sided matching problems (marriage problems and college admissions problems).

corewise stable matching. Thus, contrary to the von Neumann-Morgenstern (myopically) stable sets, von Neumann-Morgenstern farsightedly stable sets cannot include matchings that are not corewise stable ones. Since the set of unmatched agents is the same for any two matchings belonging to the core, we have that the set of unmatched agents in any von Neumann-Morgenstern farsightedly stable sets is the same.

Replacing the internal and external stability conditions of von Neumann-Morgenstern farsightedly stable sets by the conditions that internal and external deviations should be deterred, Chwe (1994) has proposed a stability concept, the largest consistent set, that always exists and that contains the von Neumann-Morgenstern farsightedly stable sets.² A set is consistent if both external and internal deviations are deterred. A consistent set does not necessarily satisfies the external stability condition. Only the largest consistent set is guaranteed to satisfy external stability. We show that matchings that do not belong to any von Neumann-Morgenstern farsightedly stable sets (hence, that are not corewise stable matchings) may belong to the largest consistent set. Finally, we provide an example showing that there is no relationship between von Neumann-Morgenstern farsightedly stable sets and the set of weakly stable matchings.

The note is organized as follows. Section 2 introduces one-to-one matching problems and standard notions of stability. Section 3 defines von Neumann-Morgenstern farsightedly stable sets. Section 4 provides the characterization of von Neumann-Morgenstern farsightedly stable sets in one-to-one matching problems. Section 5 concludes.

2 One-to-one matching problems

A one-to-one matching problem consists of a set of N agents divided into a set of men, $M = \{m_1, \dots, m_r\}$, and a set of women, $W = \{w_1, \dots, w_s\}$, where possibly $r \neq s$. We sometimes denote a generic agent by i and a generic man and a generic woman by m and w , respectively. Each agent has a complete and transitive preference ordering over the agents on the other side of the market and the prospect of being alone. Preferences are assumed to be strict. Let P be a preference profile specifying for each man $m \in M$ a strict preference ordering over $W \cup \{m\}$ and for each woman $w \in W$ a strict preference ordering over $M \cup \{w\}$: $P = \{P(m_1), \dots, P(m_r), P(w_1), \dots, P(w_s)\}$, where $P(i)$ is agent i 's strict preference ordering over the agents on the other side of the market and himself (or herself). For instance, $P(w) = m_4, m_1, w, m_2, m_3, \dots, m_r$ indicates that woman w prefers m_4 to m_1 and she prefers to remain single rather than to marry anyone else. We write $m \succ_w m'$ if woman w strictly prefers m to m' , $m \sim_w m'$ if w is indifferent between m and

²Other approaches to farsightedness in coalition formation are suggested by the work of Xue (1998), or Herings, Mauleon, and Vannetelbosch (2004).

m' , and $m \succeq_w m'$ if $m \succ_w m'$ or $m \sim_w m'$. Similarly, we write $w \succ_m w'$, $w \sim_m w'$, and $w \succeq_m w'$. A one-to-one matching market is simply a triple (M, W, P) .

A matching is a function $\mu : N \rightarrow N$ satisfying the following properties: (i) $\forall m \in M$, $\mu(m) \in W \cup \{m\}$; (ii) $\forall w \in W$, $\mu(w) \in M \cup \{w\}$; and (iii) $\forall i \in N$, $\mu(\mu(i)) = i$. We denote by \mathcal{M} the set of all matchings. Given matching μ , an agent i is said to be unmatched or single if $\mu(i) = i$. A matching μ is *individually rational* if each agent is acceptable to his or her mate, i.e. $\mu(i) \succ_i i$ for all $i \in N$. For a given matching μ , a pair $\{m, w\}$ is said to form a blocking pair if they are not matched to one another but prefer one another to their mates at μ , i.e. $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A coalition S is a subset of the set of agents N .³

Definition 1 (enforceability) *Given a matching μ , a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following condition holds: $\forall i \in S$, if $\mu'(i) \neq \mu(i)$, then $\mu'(i) \in S$.*

Notice that the concept of enforceability is independent of preferences.

Definition 2 (direct dominance) *A matching μ is directly dominated by μ' , or $\mu < \mu'$, if there exists a coalition $S \subseteq N$ of agents such that: $\mu' \succ \mu \forall i \in S$, and S can enforce μ' over μ .*

Definition 2 gives us the definition of direct dominance. The direct dominance relation is denoted by $<$. A matching μ is corewise stable if there is no subset of agents who by forming all their partnerships only among themselves, possibly dissolving some partnerships of μ , can all obtain a strictly preferred set of partners. Formally, a matching μ is corewise stable if μ is not directly dominated by any other matching $\mu' \in \mathcal{M}$.⁴ Let $C(<)$ denote the set of corewise stable matchings.⁵ Gale and Shapley (1962) have proved that the set of corewise stable matchings is non-empty. Sotomayor (1996) has provided a

³Throughout the paper we use the notation \subseteq for weak inclusion and \subsetneq for strict inclusion.

⁴Setting $|S| \leq 2$ in the definition of corewise stability, we obtain Gale and Shapley's (1962) concept of pairwise stability that is equivalent to corewise stability in one-to-one matchings with strict preferences. An alternative definition to corewise stability is strong corewise stability. A matching μ is strong corewise stable if μ is not directly weakly dominated by any other matching $\mu' \in \mathcal{M}$. A matching μ is directly weakly dominated by μ' , if there exists a coalition $S \subseteq N$ of agents such that: $\mu' \succeq \mu \forall i \in S$, $\mu' \succ \mu$ for some $i \in S$, and S can enforce μ' over μ .

⁵In one-to-one matchings with strict preferences, the set of corewise stable matchings coincides with the set of strong corewise stable matchings. However, in the College Admission model of Gale and Shapley, the set of strong corewise stable matchings do not coincide with the set of corewise stable matchings. See Roth and Sotomayor (1990).

non-constructive elementary proof of the existence of stable marriages.⁶

Another concept used to study one-to-one matching problems is the von Neumann-Morgenstern stable set (von Neumann and Morgenstern, 1953), a set-valued concept, that imposes both internal and external stability. A set of matchings is a von Neumann-Morgenstern stable set if it satisfies the following two robustness conditions: (internal stability) no matching inside the set is directly dominated by a matching belonging to the set; (external stability) any matching outside the set is directly dominated by some matching belonging to the set.

Definition 3 *A set of matchings $V \subseteq \mathcal{M}$ is a von Neumann-Morgenstern stable set if*

- (i) *For all $\mu, \mu' \in V$, $\mu \not\succeq \mu'$;*
- (ii) *For all $\mu' \in \mathcal{M} \setminus V$ there exists $\mu \in V$ such that $\mu > \mu'$.*

Definition 3 gives us the definition of a von Neumann-Morgenstern stable set $V(<)$. Let $\mathcal{V}(<)$ be the set of all von Neumann-Morgenstern stable sets. Ehlers (2007) has studied the properties of the von Neumann-Morgenstern stable sets in one-to-one matching problems. He has shown that the set of corewise stable matchings is a subset of von Neumann-Morgenstern stable sets. Example 1 illustrates his main result.

Example 1 (Ehlers, 2005) *Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let P be such that:*

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(w_1)$	$P(w_2)$	$P(w_3)$
w_1	w_2	w_3	m_2	m_3	m_1
w_2	w_3	w_1	m_3	m_1	m_2
m_1	m_2	m_3	w_1	w_2	w_3
w_3	w_1	w_2	m_1	m_2	m_3

Let

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{pmatrix}, \mu' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_3 & w_1 \end{pmatrix}, \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_3 & w_1 & w_2 \end{pmatrix}.$$

It can be shown that the set of corewise stable matchings is $C = \{\mu'\}$ and $V = \{\mu, \mu', \mu''\}$ is the unique von Neumann-Morgenstern stable set. \square

⁶Roth and Vande Vate (1990) have demonstrated that, starting from an arbitrary matching, the process of allowing randomly chosen blocking pairs to match will converge to a corewise stable matching with probability one in the marriage problem. In relation, Jackson and Watts (2002) have shown that if preferences are strict, then the set of stochastically stable matchings coincides with the set of corewise stable matchings.

In Example 1 the matchings μ and μ'' belong to the unique von Neumann-Morgenstern stable set because μ' does not directly dominate neither μ nor μ'' even though μ and μ'' are not individually rational matchings (either all women or all men prefer to become single). However, farsighted women may decide first to become single in the expectation that further marriages will be formed leading to μ' . The women prefer μ' to μ and once everybody is divorced, men and women prefer μ' to the situation where everybody is single. A similar reasoning can be made for μ'' with the roles of men and women reversed. Then, we may say that (i) μ' farsightedly dominates μ , (ii) μ' farsightedly dominates μ'' , and (iii) $V = \{\mu, \mu', \mu''\}$ is not a reasonable candidate for being a von Neumann-Morgenstern farsightedly stable set.

The concept of weak stability due to Klijn and Massó (2003) is a first interesting attempt to reflect the idea that agents are not myopic. A matching μ is weakly stable if for all μ' such that $\mu' > \mu$ with μ' being enforced over μ by S , there exists μ'' such that $\mu'' > \mu$ with μ'' being enforced over μ by coalition T such that:

- (i) $T \setminus S \neq \emptyset$, $S \setminus T \neq \emptyset$, and $T \cap S \neq \emptyset$;
- (ii) $\mu''(i) \succ_i \mu'(i)$ for some $i \in T \cap S$.

Since any corewise stable matching is a weakly stable matching, the set of weakly stable matchings is non-empty.⁷ A matching μ is said to be weakly efficient if there is no other matching μ' such that $\mu'(i) \succ_i \mu(i)$ for all $i \in N$. Klijn and Massó (2003) have shown that any corewise stable matching is weakly efficient. However, there are weakly stable matchings that are not weakly efficient as shown in Example 2.

Following Zhou (1994), Klijn and Massó (2003) have defined the notion of bargaining set for one-to-one matching problems as follows. An objection against a matching μ is a pair (S, μ') where μ' is a matching that can be enforced over μ by S and such that $\mu'(i) \succ_i \mu(i)$ for all $i \in S$. A counterobjection against an objection (S, μ') is a pair (T, μ'') where μ'' is a matching that can be enforced over μ by T such that:

- (i) $T \setminus S \neq \emptyset$, $S \setminus T \neq \emptyset$, and $T \cap S \neq \emptyset$;
- (ii) $\mu''(i) \succeq_i \mu(i)$ for all $i \in T \setminus S$ and $\mu''(i) \succeq_i \mu'(i)$ for all $i \in T \cap S$.

An objection (S, μ') against a matching is justified if there is no counterobjection against (S, μ') . The bargaining set is the set of matchings that have no justified objections. The

⁷Setting $|S| \leq 2$ and $|T| \leq 2$ in the above definition of weak stability, we obtain the original concept of *weak stability* introduced by Klijn and Massó (2003). Both definitions are equivalent in one-to-one matchings since essential coalitions are pairs of agents.

set of corewise stable matchings is a subset of the bargaining set. Klijn and Massó (2003) have shown that the bargaining set in one-to-one matching markets coincides with the set of matchings that are both weakly stable and weakly efficient.

Example 2 (Klijn and Massó, 2003) Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let P be such that:

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(w_1)$	$P(w_2)$	$P(w_3)$
w_3	w_1	w_2	m_3	m_1	m_2
w_2	w_3	w_1	m_2	m_3	m_1
w_1	w_2	w_3	m_1	m_2	m_3
m_1	m_2	m_3	w_1	w_2	w_3

Let

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{pmatrix}, \mu' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_3 & w_1 \end{pmatrix}, \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_3 & w_1 & w_2 \end{pmatrix}.$$

It can be shown that μ is a weakly stable matching but μ is not weakly efficient since $\mu'(i) \succ_i \mu(i)$ for all $i \in N$. Thus, μ does not belong to the bargaining set. The bargaining set is equal to $\{\mu', \mu''\}$ which is also the set of corewise stable matchings. \square

Example 3 (Knuth, 1976) Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Let P be such that:

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(m_4)$	$P(w_1)$	$P(w_2)$	$P(w_3)$	$P(w_4)$
w_1	w_2	w_3	w_4	m_4	m_3	m_2	m_1
w_2	w_1	w_4	w_3	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_2	m_1	m_4	m_3
w_4	w_3	w_2	w_1	m_1	m_2	m_3	m_4
m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4

Let

$$\mu' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_3 & w_2 & w_4 \end{pmatrix}, \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_4 & w_2 & w_3 & w_1 \end{pmatrix}.$$

It can be shown that μ' and μ'' belong to the bargaining set (see Klijn and Massó, 2003) but none of them is a corewise stable matching. \square

3 Von Neumann-Morgenstern farsighted stability

The indirect dominance relation was first introduced by Harsanyi (1974) but was later formalized by Chwe (1994). It captures the idea that coalitions of agents can anticipate

the actions of other coalitions. In other words, the indirect dominance relation captures the fact that farsighted coalitions consider the end matching that their matching(s) may lead to. A matching μ' indirectly dominates μ if μ' can replace μ in a sequence of matchings, such that at each matching along the sequence all deviators are strictly better off at the end matching μ' compared to the status-quo they face. Formally, indirect dominance is defined as follows.

Definition 4 *A matching μ is indirectly dominated by μ' , or $\mu \ll \mu'$, if there exists a sequence of matchings $\mu^0, \mu^1, \dots, \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of coalitions S^0, S^1, \dots, S^{K-1} such that for any $k \in \{1, \dots, K-1\}$,*

- (i) $\mu^K \succ \mu^{k-1} \forall i \in S^{k-1}$, and
- (ii) coalition S^{k-1} can enforce the matching μ^k over μ^{k-1} .

Definition 4 gives us the definition of indirect dominance. The indirect dominance relation is denoted by \ll . Direct dominance is obtained by setting $K = 1$ in Definition 4. Obviously, if $\mu < \mu'$, then $\mu \ll \mu'$.⁸

Definition 5 *A set of matchings $V \subseteq \mathcal{M}$ is a von Neumann-Morgenstern farsightedly stable set with respect to P if*

- (i) For all $\mu \in V$, there does not exist $\mu' \in V$ such that $\mu' \gg \mu$;
- (ii) For all $\mu' \notin V$ there exists $\mu \in V$ such that $\mu \gg \mu'$.

Definition 5 gives us the definition of a von Neumann-Morgenstern farsightedly stable set $V(\ll)$. Let $\mathcal{V}(\ll)$ be the set of all von Neumann-Morgenstern farsightedly stable sets. Part (i) in Definition 5 is the internal stability condition: no matching inside the set is indirectly dominated by a matching belonging to the set. Part (ii) is the external stability condition: any matching outside the set is indirectly dominated by some matching belonging to the set.

We now reconsider the above examples to show that non corewise stable matchings, that belong either to the von Neumann-Morgenstern stable set or to the set of weakly stable matchings, do not survive the stability requirements imposed by introducing farsightedness into the concept of von Neumann-Morgenstern stable sets.

⁸The only difference between the indirect dominance and indirect weak dominance relations is that, in the definition of indirect weak dominance, the condition (i) $\mu^K \succ \mu^{k-1} \forall i \in S_{k-1}$ is replaced by a weaker condition (i) $\mu^K \succeq \mu^{k-1} \forall i \in S_{k-1}$ and $\mu^K \succ \mu^{k-1}$ for some $i \in S_{k-1}$. Since preferences are strict, both relations are equivalent in one-to-one matching problems.

Example 1 (*continue*) Remember that $C(<) = \{\mu'\}$ is the set of stable matchings and $V(<) = \{\mu, \mu', \mu''\}$ is the unique von Neumann-Morgenstern stable set. It is easy to verify that $\mu' \gg \mu$ and $\mu' \gg \mu''$. Let $\mu^0 = \mu$, $\mu^1 = \emptyset$ (all agents are single), $\mu^2 = \mu'$, $S^0 = \{w_1, w_2, w_3\}$, and $S^1 = N$. We have (i) $\mu^2 \succ \mu^0 \forall i \in S^0$ and S^0 can enforce μ^1 over μ^0 , (ii) $\mu^2 \succ \mu^1 \forall i \in S^1$ and S^1 can enforce μ^2 over μ^1 . Thus, $\mu^2 \gg \mu^0$ or $\mu' \gg \mu$. Similarly, it is easy to verify that $\mu' \gg \mu''$. Hence, $\{\mu, \mu', \mu''\}$ cannot be a von Neumann-Morgenstern farsightedly stable set, nor can $\{\mu, \mu'\}$ or $\{\mu', \mu''\}$ be a von Neumann-Morgenstern farsightedly stable set since internal stability is violated. Moreover, μ does not indirectly dominates μ' and μ'' does not indirectly dominates μ' . It implies that the sets $\{\mu, \mu''\}$, $\{\mu\}$ or $\{\mu''\}$ cannot be von Neumann-Morgenstern farsightedly stable sets as they violate the external stability condition. In fact, $V(\ll) = \{\mu'\}$ is the unique von Neumann-Morgenstern farsightedly stable set. \square

Example 2 (*continue*) Remember that μ is weakly stable. We will show that μ cannot belong to any von Neumann-Morgenstern farsightedly stable sets. We prove it by contradiction. Suppose that μ belong to some $V(\ll)$. Since $\mu \not\gg \mu'$ and $\mu \not\gg \mu''$, μ has to be with μ' and μ'' in $V(\ll)$ in order to satisfy the external stability condition. But, since $\mu \not\gg \mu'$ and $\mu \not\gg \mu''$, $\{\mu, \mu', \mu''\} \subseteq V(\ll)$ would violate internal stability. Hence, $\mu \notin V(\ll)$. In fact, the von Neumann-Morgenstern farsightedly stable sets are $\{\mu'\}$ and $\{\mu''\}$. That is, $\mathcal{V}(\ll) = \{\{\mu'\}, \{\mu''\}\}$. \square

4 Main results

From Definition 5, we have that, for $V(\ll)$ to be a singleton von Neumann-Morgenstern farsightedly stable set, only externally stability needs to be verified. That is, the set $\{\mu\}$ is a von Neumann-Morgenstern farsightedly stable set if and only if for all $\mu' \neq \mu$ we have that $\mu \gg \mu'$.

In order to show our main results we use Lemma 1 that shows that an individually rational matching μ does not indirectly dominate another matching μ' if and only if there exists a pair $\{i, \mu'(i)\}$ that blocks μ . In other words, μ indirectly dominates μ' if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ .

Lemma 1 *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ is individually rational. Then, $\mu \not\gg \mu'$ if and only if there exists a pair $\{i, \mu'(i)\}$ such that both i and $\mu'(i)$ prefer μ' to μ , or there exists i such that $\mu'(i) = i$ and i prefers μ' to μ .*

Proof. Let $B(\mu', \mu)$ be the set of men and women who are strictly better off in μ than in μ' . Accordingly, let $I(\mu', \mu)$ and $W(\mu', \mu)$ be the set of men and women who are indifferent

between μ and μ' and worse off in μ than in μ' , respectively.

(\Rightarrow) For μ to indirectly dominate μ' it must be that i or $\mu'(i)$ get divorced along the path from μ' to μ . But both i and $\mu'(i)$ belong to $W(\mu', \mu)$, and then, they will never divorce. Also, if i is single at μ' and $\mu' \succ \mu$ for i , then i will never accept her/his match in μ . Hence $\mu \not\gg \mu'$.

(\Leftarrow) We will prove it by showing that $\mu \gg \mu'$ if the above condition is not satisfied. Assume that for all pairs $\{i, \mu'(i)\}$ such that $\mu'(i) \neq \mu(i)$, either i or $\mu'(i)$ or both belong to $B(\mu', \mu)$. Notice that all agent i single in μ' that accepts a match with someone else in μ also belong to $B(\mu', \mu)$. Next construct the following sequence of matchings from μ' to μ : μ^0, μ^1, μ^2 (where $\mu^0 = \mu'$, $\mu^1 = \{\mu^1(i) = i, \mu^1(\mu'(i)) = \mu'(i)\}$ for all $i \in B(\mu', \mu)$, and $\mu^1(j) = \mu'(j)$ otherwise}, and $\mu^2 = \mu$), and the following sequence of coalitions S^0, S^1 with $S^0 = B(\mu', \mu)$ and $S^1 = B(\mu', \mu) \cup \{\mu(i) \text{ for } i \in B(\mu', \mu)\}$. Then, coalition S^0 can enforce μ^1 over μ^0 and coalition S^1 can enforce μ^2 over μ^1 . Moreover, $\mu^2 \succ \mu^0$ for S^0 , and $\mu^2 \succ \mu^1$ for S^1 because every mate of $i \in B(\mu', \mu)$ in μ^2 (in μ) also prefers his or her mate in μ^2 to being single in μ^1 . Indeed, for every $i \in B(\mu', \mu)$, either $\mu^2(i) \in B(\mu', \mu)$ and hence both prefer μ^2 to μ^1 , or $\mu^2(i) \in W(\mu', \mu)$. In this last case, $\mu^2(i)$ must have lost his or her mate in μ^0 and $\mu^0(\mu^2(i))$ must belong to $B(\mu', \mu)$ since otherwise $\mu^0(\mu^2(i))$ and $\mu^2(i)$ would form a blocking pair of μ^2 , and this by assumption is not possible. Hence $\mu^2(i)$ must be single in μ^1 . Then, since μ^2 is individually rational, $\mu^2(i)$ must prefer accepting his or her mate in μ^2 than remaining single at μ^1 . So, we have that $\mu \gg \mu'$. ■

Hence, if μ is individually rational and there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ , then $\mu \gg \mu'$.

Lemma 2 *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ' is individually rational. Then $\mu \gg \mu'$ implies that μ is also individually rational.*

Proof. Suppose not. Then, there exists $i \in N$ that prefers to be single than to be married to $\mu(i)$ in μ . Since $\mu \gg \mu'$ and μ' is individually rational, we have that i was either single at μ' or matched to $\mu'(i) \succ_i i$. But then in the sequence of moves between μ' and μ , the first time i has to move she/he was either matched with $\mu'(i)$ or single and, hence, i cannot belong to a coalition S^{k-1} that can enforce the matching μ^k over μ^{k-1} and such that all members of S^{k-1} prefer μ to μ^{k-1} , contradicting the fact that $\mu \gg \mu'$. ■

The next theorem shows that every corewise stable matching is a von Neumann-Morgenstern farsightedly stable set.

Theorem 1 *If μ is a stable matching, $\mu \in C(<)$, then $\{\mu\}$ is von Neumann-Morgenstern farsightedly stable set, $\{\mu\} = V(\ll)$.*

Proof. We only need to verify condition (ii) in Definition 5: for all $\mu' \neq \mu$ we have that $\mu \gg \mu'$. Since $\mu \in C(<)$, we know that $\forall \mu' \neq \mu$, $\nexists i \in M$ and $j \in W$ such that $\mu'(i) = j$ and $\mu' \succ \mu$ for both i and j , and $\nexists i \in N$ such that $\mu'(i) = i$ and $\mu' \succ \mu$ for i . Since μ is individually rational, we have from Lemma 1 that $\mu \gg \mu'$. ■

We have just shown that if $\mu \in C(<)$ then $\{\mu\}$ is a von Neumann-Morgenstern farsightedly stable set.⁹ But, a priori there may be other von Neumann-Morgenstern farsightedly stable sets of matchings. We now show that the only possible von Neumann-Morgenstern farsightedly stable sets are singleton sets whose elements are the corewise stable matchings.

Theorem 2 *If $V(\ll) \subseteq \mathcal{M}$ is a von Neumann-Morgenstern farsightedly stable set of matchings then $V(\ll) = \{\mu\}$ with $\mu \in C(<)$.*

Proof. Notice that if $V(\ll) \subseteq C(<)$, then $V(\ll)$ is a von Neumann-Morgenstern farsightedly stable set only if $V(\ll)$ is a singleton set $\{\mu\}$ with $\mu \in C(<)$. From Theorem 1, we know that for all $\mu' \neq \mu$, $\mu \gg \mu'$. Suppose now that $V(\ll) \not\subseteq C(<)$. Then, either $V(\ll) \cap C(<) = \emptyset$ or $V(\ll) \cap C(<) \neq \emptyset$.

Suppose first that $V(\ll) \cap C(<) \neq \emptyset$. Then, there exists a matching $\mu \in V(\ll) \cap C(<)$, and we know that for all $\mu' \neq \mu$, $\mu \gg \mu'$ since $\{\mu\}$ is a von Neumann-Morgenstern farsightedly stable set. But, then there exists a matching $\mu' \neq \mu \in V(\ll)$ such that $\mu \gg \mu'$, violating the internal stability condition.

Suppose now that $V(\ll) \cap C(<) = \emptyset$. Then, we will show that $V(\ll)$ is not a von Neumann-Morgenstern farsightedly stable set because either the internal stability condition (condition (i) in Definition 5) or the external stability condition (condition (ii) in Definition 5) is violated.

Assume $V(\ll) = \{\mu\}$ is a singleton. Since $\mu \notin C(<)$ there exists a deviating coalition S in μ and a matching $\mu' \in \mathcal{M}$ such that $\mu' \succ_i \mu$ for all $i \in S$ and S can enforce μ' over μ . Then, $\mu \not\gg \mu'$ and the external stability condition is violated.

⁹The result that for all $\mu' \neq \mu$ with $\mu \in C(<)$ we have that $\mu \gg \mu'$ was already shown in Diamantoudi and Xue (2003), as a special case of a general result. Indeed, Diamantoudi and Xue (2003) have proved that in hedonic games (of which marriage problems are a special case) with strict preferences, any partition belonging to the core indirectly dominates any other partition. Here, we provide an alternative proof of their result for one-to-one matching problems with strict preferences.

Assume now that $V(\ll)$ contains more than one matching that do not belong to $C(<)$. Take any matching $\mu_1 \in V(\ll)$. Since $\mu_1 \notin C(<)$, there exists at least a pair of agents $\{i, j\}$ such that $\mu_1(j) \neq i$ (or a single agent $\{i\}$) and a matching $\mu'_1 \in \mathcal{M}$ such that $\mu'_1 \succ \mu_1$ for both i and j (or $\mu'_1 \succ \mu_1$ for i), and $\{i, j\}$ (or i) can enforce μ'_1 over μ_1 , i.e. such that $\mu'_1(j) = i$ (or $\mu'_1(i) = i$). Let $S(\mu_1)$ the set of blocking pairs of μ_1 . Consider the deviation from μ_1 to μ'_1 of the subset of blocking pairs $S'(\mu_1) \subseteq S(\mu_1)$, where $S'(\mu_1)$ contains the maximum number of blocking pairs and is such that the subset $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking pair of μ'_1 . In order for $V(\ll)$ being a von Neumann-Morgenstern farsightedly stable set we need that the following conditions are satisfied:

(i) for any other matching $\mu_2 \in V(\ll)$, $\mu_2 \neq \mu_1$, it should be that $\mu_1 \not\gg \mu_2$ and $\mu_2 \not\gg \mu_1$

(ii) for all $\mu' \notin V(\ll)$ there exists $\mu \in V(\ll)$ such that $\mu \gg \mu'$ (in particular, we need that there exists a matching $\mu_2 \in V(\ll)$ such that $\mu_2 \gg \mu'_1$ for each matching, like μ'_1 , that can be enforced by any subset of blocking pairs of any matching in $V(\ll)$).

We will show that $V(\ll)$ is not a von Neumann-Morgenstern farsightedly stable set because one of the above conditions is not satisfied. Three different cases should be considered.

1. Assume that $S'(\mu_1)$ contains a blocking pair $\{i\}$ that divorces from $\mu_1(i)$. Consider the deviation of $\{i\}$ from μ_1 to μ''_1 where he or she divorces from $\mu_1(i)$, while the other blocking pairs do not move. Then if $\mu_2 \gg \mu''_1$ (in order for $V(\ll)$ satisfying external stability), we also have that $\mu_2 \gg \mu_1$ since i will never marry someone else and becoming worse off than being single. Hence, the internal stability condition is violated and $V(\ll)$ is not a von Neumann-Morgenstern farsightedly stable set. So $V(\ll)$ cannot contain non-individually rational matchings.
2. Assume that $S'(\mu_1)$ contains at least a blocking pair $\{i, j\}$ that are single at μ_1 but married at μ'_1 , i.e., $\mu'_1(i) = j$. Consider the deviation of $\{i, j\}$ from μ_1 to μ''_1 where they get married, while the other blocking pairs do not move. Then if $\mu_2 \gg \mu''_1$ (in order for $V(\ll)$ satisfying external stability), we will show that the internal stability condition is violated. Two sub-cases have to be considered.
 - 2.1. If i and j are still married in μ_2 (or they are married to someone else preferred to j and i respectively), then $\mu_2 \gg \mu_1$ and the internal stability condition is violated.
 - 2.2. On the contrary, assume that one of them, i , has divorced from j (leaving j single in μ_2 like she was in μ_1) to marry to another woman $\mu_2(i)$ preferred to j . But notice that the position (matching) of any other agent different than i

and j in μ_1'' is the same than in μ_1 , since only i and j married at μ_1'' while they were single at μ_1 , and then since $\mu_2 \gg \mu_1''$, we should have that $\{i, \mu_2(i)\} \in S(\mu_1)$. But then, the pair $\{i, j\}$ cannot belong to $S'(\mu_1)$, since j is not the best partner for i . Thus, consider the deviation of $\{i, \mu_2(i)\}$ from μ_1 to μ_1''' where they get married, while the other blocking pairs do not move. Then, if we have that $\mu_2 \gg \mu_1'''$ (in order for $V(\ll)$ satisfying external stability), we also have $\mu_2 \gg \mu_1$ and the internal stability condition is violated.

- 3.** Assume that all blocking pairs $\{i, j\} \in S'(\mu_1)$ are such that at least one of the blocking partners (or both of them) is married at μ_1 with someone else, $\mu_1(j) \neq i, j$ and now at μ_1' they get married $\mu_1'(j) = i$. Assume that in the deviation from μ_1 to μ_1' all blocking pairs in $S'(\mu_1)$ get married so that at μ_1' no other blocking pair exists ($S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking pair of μ_1'). Three sub-cases have to be considered.

3.1. If at μ_2 we have that every blocking pair $\{i, j\} \in S'(\mu_1)$ is such that i and j are still married or they are married to someone else but preferred to j and to i , respectively, then $\mu_2 \gg \mu_1$ and the internal stability condition is violated.

3.2. Assume that at μ_2 no initial blocking pair $\{i, j\} \in S'(\mu_1)$ is still married, and that in each blocking pair $\{i, j\} \in S'(\mu_1)$ we have that i married $\mu_2(i) \neq j$ divorcing from j . Hence, in order to have that $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$ and given Lemma 1 and Lemma 2, we need that there does not exist a pair $\{k, \mu_1'(k)\}$ married at μ_1' (or a single agent k) that blocks μ_2 , and that there exists a pair $\{k, \mu_1(k)\}$ married at μ_1 (or a single agent k) that blocks μ_2 , with μ_2 individually rational. Notice that the only change between μ_1 and μ_1' is that each blocking pair $\{i, j\} \in S'(\mu_1)$ get married leaving $\mu_1(j)$ (and possibly $\mu_1(i)$) single. Then we will prove that whenever $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$, we have that $\mu_1 \gg \mu_2$, violating the internal stability condition. By Lemma 1, we only need to show that there does not exist a pair $\{k, \mu_2(k)\}$ married at μ_2 (or a single agent k) that blocks μ_1 .

Since $\mu_2 \gg \mu_1'$ but $\mu_2 \not\gg \mu_1$, we have that, for each pair $\{i, j\} \in S'(\mu_1)$, whenever i is better off at μ_2 than at μ_1' (and then better than at μ_1), $\mu_2(i)$ is worse off at μ_2 than at μ_1' , because no other blocking pair different from the ones contained in $S'(\mu_1)$ exists at μ_1' (and hence $\mu_2(i)$ is worse off at μ_2 than at μ_1), and some pair $\{\mu_1(j), j\}$ married at μ_1 (with j such that $\mu_1'(j) = i$, and $\{i, j\} \in S'(\mu_1)$) prefers μ_1 to μ_2 and is a blocking pair of μ_2 making $\mu_2 \not\gg \mu_1$. Thus, whenever $\mu_1(j)$ and j are either single or married to someone else at μ_2

they both would prefer μ_1 to μ_2 . Also all the remaining initial blocking partners $\mu_1(i)$ (and $\mu_1(j)$) that have been left by i (and by j) with $\{i, j\} \in S'(\mu_1)$, when single at μ_2 , prefer μ_1 to μ_2 because otherwise μ_1 would not be individually rational. Whenever some initial blocking partner $\mu_1(i)$ (and $\mu_1(j)$) is not single at μ_2 but married to someone else $\mu_2(\mu_1(i))$ (married to $\mu_2(\mu_1(j))$), either $\mu_1(i)$ or $\mu_2(\mu_1(i))$ (either $\mu_1(j)$ or $\mu_2(\mu_1(j))$) or both should prefer μ_1 to μ_2 because otherwise the pair $\{\mu_1(i), \mu_2(\mu_1(i))\}$ (the pair $\{\mu_1(j), \mu_2(\mu_1(j))\}$) would have been also a blocking pair at μ'_1 . So, we have shown that every pair of agents matched (every single agent at μ_2) at μ_2 and not matched (not single) at μ_1 that contains one of the initial deviating players in $S'(\mu_1)$ is such that one of the mates prefers μ_2 to μ_1 while the other prefers μ_1 to μ_2 .

Since $\mu_2 \gg \mu'_1$ we have, by Lemma 1, that there is no blocking pair at μ'_1 , and then, every pair of agents $\{k, l\}$ such that $\mu_2(l) = k$ and $\mu'_1(l) \neq k$ with $\{k, l\} \notin S(\mu_1)$, is such that one of the mates prefers μ_2 to μ'_1 (and hence, prefers μ_2 to μ_1) while the other prefers μ'_1 to μ_2 (and hence, prefers μ_1 to μ_2), and all single agents at μ_2 not in $S(\mu_1)$, that are married at μ'_1 prefer μ'_1 to μ_2 (and hence, prefer μ_1 to μ_2). Moreover, since the pairs $\{i, j\} \in S'(\mu_1)$ are the only blocking pairs at μ_1 , every pair of agents $\{k, l\}$ such that $\mu_2(l) = k$ and $\mu'_1(k) \neq l$ with $\{k, l\} \in S(\mu_1) \setminus S'(\mu_1)$, is such that one of the mates prefers μ_2 to μ_1 while the other prefers μ_1 to μ_2 because otherwise they would have been a blocking pair at μ_1 , and all single agents at μ_2 that belong to $S(\mu_1) \setminus S'(\mu_1)$, that are married at μ'_1 (and hence, are married at μ_1) prefer μ'_1 to μ_2 (and hence, prefer μ_1 to μ_2) because otherwise μ'_1 (and hence, μ_1) would not be individually rational.

Thus, we have proved that when $\mu_2 \gg \mu'_1$ but $\mu_2 \not\gg \mu_1$, we have that there is a pair $\{\mu_1(j), j\}$ (with j such that $\mu'_1(j) = i$, and $\{i, j\} \in S'(\mu_1)$) that prefers μ_1 to μ_2 and that every other pair of agents matched at μ_2 and not matched at μ_1 is such that one of the mates prefers μ_2 to μ_1 while the other prefers μ_1 to μ_2 . So, there does not exist a pair $\{i, \mu_2(i)\}$ matched at μ_2 that blocks μ_1 , and then by Lemma 1 we have $\mu_1 \gg \mu_2$, violating the internal stability condition.

- 3.3.** Finally, consider the case in which μ_2 contains some but not all initial blocking pairs from μ_1 contained in $S'(\mu_1)$. Then, consider the deviation from μ_1 to μ''_1 such that $\mu''_1 \succ \mu_1$ by all the initial blocking pairs belonging to $S''(\mu_1) \subset S'(\mu_1)$ that are still married at μ_2 and that can be enforced by such blocking pairs from μ_1 . Since $\mu_2 \gg \mu''_1$ (in order for $V(\ll)$ satisfying external stability), we will have that $\mu_2 \gg \mu_1$ violating the internal stability condition. ■

We now study the relationship between von Neumann-Morgenstern farsighted stability and the largest consistent set, a concept that has been defined in Chwe (1994) for general social environments, in one-to-one matching problems.

Definition 6 $Z(\ll) \subseteq \mathcal{M}$ is a consistent set if $\mu \in Z(\ll)$ if and only if $\forall \mu'$, S such that S can enforce μ' over μ , $\exists \mu'' \in Z(\ll)$, where $\mu' = \mu''$ or $\mu' \ll \mu''$, such that $\mu(i) \not\prec_i \mu''(i) \forall i \in S$. The largest consistent set $\Gamma(\ll)$ is the consistent set that contains any consistent set.

Chwe (1994) has shown that the largest consistent set exists, is non-empty, and satisfies external stability. Notice that a consistent set does not necessarily satisfies the external stability condition. Only the largest consistent set is guaranteed to satisfy external stability. Chwe (1994) has also shown that the largest consistent set always contains the von Neumann-Morgenstern farsightedly stable sets. That is, if $V(\ll)$ is a von Neumann-Morgenstern farsightedly stable set, then $V(\ll) \subseteq \Gamma(\ll)$.

Example 3 (*continue*) Remember that μ' and μ'' belong to the bargaining set but are not corewise stable. There are ten corewise stable matchings where men (m_1, m_2, m_3, m_4) are matched to women (w_1, w_2, w_3, w_4) , (w_2, w_1, w_3, w_4) , (w_1, w_2, w_4, w_3) , (w_2, w_1, w_4, w_3) , (w_2, w_4, w_1, w_3) , (w_3, w_1, w_4, w_2) , (w_3, w_4, w_1, w_2) , (w_3, w_4, w_2, w_1) , (w_4, w_3, w_1, w_2) , and (w_4, w_3, w_2, w_1) , respectively. We know, by Theorem 1 and Theorem 2, that μ' and μ'' cannot belong to any von Neumann-Morgenstern farsightedly stable sets. We already know that if $\mu \in C(<)$ then $\mu \in \Gamma(\ll)$. We will show that μ' and μ'' belong to the largest consistent set, $\Gamma(\ll)$. We know that if $\mu \in C(<)$ then for all $\hat{\mu} \neq \mu$ we have that $\mu \gg \hat{\mu}$. Moreover, we have that (i) for each $i \in N$ there is $\mu \in C(<)$ such that $\mu(i) = \mu'(i)$, and (ii) for each $i \in N$ there is $\mu \in C(<)$ such that $\mu(i) = \mu''(i)$. Hence, for all μ''' , S such that S can enforce μ''' over μ' , $\exists \mu \in C(<) \subseteq \Gamma(\ll)$, where $\mu''' = \mu$ or $\mu''' \ll \mu$, such that $\mu'(i) \not\prec_i \mu(i) \forall i \in S$. Thus, $\mu' \in \Gamma(\ll)$. Similarly, for all μ''' , S such that S can enforce μ''' over μ'' , $\exists \mu \in C(<) \subseteq \Gamma(\ll)$, where $\mu''' = \mu$ or $\mu''' \ll \mu$, such that $\mu''(i) \not\prec_i \mu(i) \forall i \in S$; and, $\mu'' \in \Gamma(\ll)$. So, the largest consistent set may contain more matchings than the matchings that belong to the von Neumann-Morgenstern farsightedly stable sets. \square

5 Conclusion

In this note we have characterized the von Neumann-Morgenstern farsightedly stable sets in one-to-one matching problems: a set of matchings is a von Neumann-Morgenstern farsightedly stable set if and only if it is a singleton set and its element is a corewise stable matching. Finally, notice that our main result is robust to a recent alternative notion of farsightedness. In Herings, Mauleon and Vannetelbosch (2006) we have shown that replacing the internal stability condition in the von Neumann-Morgenstern stable set by deterrence of external deviations and minimality, leads to a stability concept, called farsightedly stable sets, that contains the von Neumann-Morgenstern farsightedly stable set, and is always non-empty. We have that, if a set only contains a single corewise stable matching, then it is a farsightedly stable set.

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