Extended abstract of "Optimizing public goods in networks"

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December 18, 2008

Description of the model

In this paper we consider an exogenous network in which otherwise homogeneous players (nodes) play a public good game, as the one defined *Best shot game* in Galeotti et alii (2008).[1] The action of each node i is an effort x_i and her payoff will depend on the aggregate effort of herself and her neighbors, minus some cost for her own effort. In particular we will restrict to the two specialized actions: $x_i \in \{0, 1\}$.[2] In this way $\vec{x}$, a vector of specialized actions whose length is given by the number of nodes, will characterize any possible configuration of the system. We will consider any setup of incentives such that, in Nash equilibrium (NE), agent i will play action x_i according to the following rule:

$$\begin{cases} x_i = 1 & \text{if } x_j = 0 \text{ for any neighbor } j \text{ of node } i; \\ x_i = 0 & \text{otherwise.} \end{cases} \tag{1}$$

We will study all the NE of the game, that is all those actions in which, for any link, no both nodes of the link put effort 1; but at the same time for any

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[1]The best shot game is a particular case, with restricted strategy profiles, of the models in Bramoullé and Kranton (2007) and Galeotti and Goyal (2008).

[2]This could be justified adopting the proper normalization from Bramoullé and Kranton (2007).

node, if we consider the set of it and its neighborhood, at least one node in this set puts effort 1. The subset of nodes playing 1 in a NE will then be a *maximal independent set* of the network, as it is called in graph theory.

The rule specified in (1) is not *behavioral* and could be justified with a lot of modelling choices with completely rational agents. The kind of situation we have in mind is that of every agent deciding whether to exert or not a fixed costly effort that is beneficial for herself but also for her neighbors, so that a typical situation of free riding incentives arises. It could be the case of farmers or firms adopting new technologies, with an information network and the cost of possible failures. Another application could be that of many municipalities in a given region, the public good could be a library or a fire brigade, where two municipalities are linked if the public good in one of them makes the same public good not desirable in the other one, because of geographical proximity. Finally, since the mechanism we will propose requires low costs of shifting between strategies and repeated interaction, a good application could be that of a big firm encouraging people to share cars in order to minimize parking places. Action 1 would mean 'take the car' and an employee would play 0 if a friend of her gives her a lift. Generally, in any of the previous applications there could be a planner whose objective could reasonably be that of minimizing costs.

Suppose that the planner considers all the possible NE of the game (all the maximal independent sets of the network) and wants to minimize among them the number of nodes exerting effort 1 (i.e. find a maximal independent set of minimal cardinality: MNE), then she could impose the proper action to the agent, and the resulting configuration, being a NE, would be stable without imposing more incentives. Suppose however that the planner may not know such optimal distribution (the theoretical problem is typically NP-complete) or that moreover she may not even know anything about the network. Assuming that we also have a temporal dimension our question is: would it still be possible for the planner to build a mechanism that would incentivate the agents to move towards an optimal MNE? Our answer is only theoretical but positive: at the limit of infinite time such a mechanism exists, and it will lead to a MNE with probability 1.

Main result

The mechanism we study is the following in discrete time, $t = 1, 2, 3, \ldots$. Every time–step is characterized by a configuration $\vec{x}_t$ of nodes' actions satisfying (1) for every node, and hence NE. Suppose then that at time 1 the system is in a NE, so that $x_{i,1} \in \{0, 1\}$ is a best response for every agent

i, as specified in (1). The planner does not know anything about the network, the only thing she observes at any time–step t in time is the action of each player and hence the aggregate number of agents playing 1: call it $M_t = \sum_i x_{i,t}$. What she will do is, at every time step, pick an agent i_t, playing 0, at random with uniform probabilities, and force her to flip her strategy to 1.[3] In consequence of this flip, all the other nodes in the network will change their strategy according to a simple *best response* rule. When the system is stable again, i.e. again in a new NE, the planner will obtain a new configuration $\vec{x}_t^{new}$ and observe the new aggregate quantity of 1's, call it M_t^{new}. The planner will accept the new configuration with probability

$$\begin{cases} 1 & \text{if } M_t^{new} < M_t \ ; \\ t^{-\epsilon(M_t^{new} - M_t)} & \text{otherwise;} \end{cases} \tag{2}$$

where $\epsilon > 0$ is a constant. The second probability in (2) identifies the level of rejection of non–improving changes.

We start by proving that $\vec{x}_t^{new}$ is always a NE, for any t (Lemma 1 here below). If the planner accepts the new configuration, then $\vec{x}_{t+1} = \vec{x}_t^{new}$ and $M_{t+1} = M_t^{new}$, otherwise she will impose reverse incentives so that we will get back to the original configuration,[4] i.e. $\vec{x}_{t+1} = \vec{x}_t$ and $M_{t+1} = M_t$.

In the limit $t \to \infty$ the second probability in (2) goes to 0 and the mechanism will converge to a single NE, call the subset of such possible NE *local minima*.[5] Every MNE is also a local minimum. The question is whether the local minimum in which the process ends is also a MNE with finite probability? The aim of the paper is to show under which conditions the answer is positive.

The structure of the paper will move as follows.

Lemma 1: if we start from a NE and invert the action of one node from 0 to 1, then the best response rule of all the other nodes in the network will imply a new NE.

Lemma 2: if we start from a NE and invert the action of one node from 0 to 1, then the best response rule of all the other nodes in the network will be limited to the neighborhood of order 2 of the original node (i.e. the change is only local).

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[3] This can be easily done through incentives. The reason why the planner is looking for a minimum could be that she is financing all the agents exerting effort, in this case she could raise her contribution to the agent up to the desired threshold level.

[4] This can be done by reverting all incentives to the nodes who changed: they are , by following Lemma 2, restricted to a local neighborhood.

[5] It is also possible that the mechanism, at the limit $t \to \infty$, alternates between more than one single NE, if all them has the same number of 1's. Without loss of generality, such subsets of NE can be simply included among *local minima*.

Lemma 3: it is possible to reach any NE from any other NE with a finite number of the following procedures: flip the action of a single node from 0 to 1 and obtain, by best response of the nodes, a new NE.

Proposition: the probability $\pi(\epsilon)$ that the mechanism ends in a MNE is strictly positive for any $\epsilon > 0$; it is decreasing in ϵ; and finally, there exists an $\bar{\epsilon} > 0$ such that, for any $\epsilon < \bar{\epsilon}$ we have that $\lim_{\epsilon \to 0} \pi(\epsilon) = 1$, independently on the initial conditions.

The lemmas can be proven applying the discrete mathematics of network theory. Lemmas 1 and 2 guarantee also that the proposed mechanism is well defined. The main proposition is obtained applying results from the theory of *simulated annealing*, proposed in Kirkpatrick, Gelatt and Vecchi (1983), which is essentially the increasing rejection probability in the Monte Carlo step described in (2). *Simulated annealing* is a heuristic algorithm that works exactly as described above, in order to find a global minimum of a certain function, avoiding local minima. Theory tells us that, if the number of possible configuration is finite, and it is possible to reach any configuration from any other with basic steps (which is our case), then a generalization of the above proposition holds. A rigorous proof that applies for our model can be found in Geman and Geman (1984).[6] The proof takes various pages, its intuition is that we are analyzing a Markov chain of finite possible configurations (all the NE of the game) which is ergodic for any finite t.

Short considerations

The problem of finding a MNE among all the NE is in general not a trivial one, and the difference between the aggregate number of nodes playing 1 in NE could vary dramatically even in homogeneous networks, as we study in the companion paper Dall'Asta, Pin and Ramezanpour (2008). The star structure is a trivial but dramatic example: there are two NE, one in which the center alone plays 1, another one in which all the spokes do so and the center free rides.

[6]Sections X to XII in Geman and Geman (1984) are devoted to the general case of optimization between a finite number of states. Theorem B is the one we are using. In their notation, what they call *temperature* is $\frac{1}{\epsilon \log t}$, N is the number of possible states of the system (the number of NE in our case), while Δ is the difference between the maximal score and the minimum score of the states (the difference between the maximal number of 1's in NE, and the minimal one). It turns out that $\bar{\epsilon} = \frac{1}{N\Delta}$. They prove moreover that, in presence of more MNE, the probabilities of ending in any one of them are uniform.

The main practical problem in the implementation of the mechanism we propose is clearly the necessity of infinite time. This paper is only theoretical. However, simulated annealing is used in practice in many optimization problems. Consider that for $\epsilon > 0$ the system reaches a local minimum, which can be identified even in finite time (the faster the convergence the higher the ϵ). Noting that the values $\epsilon < \bar{\epsilon}$ are very low, and the algorithm converges very slowly, the choice of a proper heuristic $\epsilon > \bar{\epsilon}$ could be appropriate. This choice would depend on a profit/costs comparison but also, in the case of finite time, on the structure of the network (e.g. the star needs a single flip to move from the bad NE to the MNE).

Appendix A Proof of Lemmas

Consider a finite network and call $x_i \in \{0, 1\}$ the action of node i, so that $\vec{x}$ is the vector of the actions of all the nodes. Call N_i^1 the set of nodes which are first neighbors of node i, N_i^2 those which are second neighbors of node i.

We also need the following definitions. A set of nodes in a network is an *independent set* if, for every link of the network, no both its nodes are in the set. A set C of nodes in a network is a *covering* if, for every node i, $C \cap (\{i\} \cup N_i^1) \neq \emptyset$. A set of nodes in a network is a *maximal independent set* if it is both an independent set and a covering. In our notation a maximal independent set is characterized by those nodes playing 1 in a NE $\vec{x}$.

Proof of Lemmas 1 and 2: suppose that $x_i = 1$ and we flip this action so that $x_i^{new} = 0$. Consider now any node j in N_i^1, it is clear that $x_j = 0$ since $x_i = 1$. For all those $j \in N_i^1$ such that $x_k = 0$ for any $k \in N_j^1 \backslash \{i\}$, we will have $x_j^{new} = 1$. In the case that two such j's that flipped from 0 to 1 will be linked together, by best response only some of them will flip to 1 (and this is the only random part in the best response rule). If j is such that $x_j = 0$ and $x_j^{new} = 1$, it is surely the case that any $k \in N_j^1 \backslash \{i\}$ was playing $x_k = 0$ and remains to $x_k^{new} = 0$. The propagation of best response is then limited to N_i^1.

Note: *best response from* 0 *to* 1 *applies only to nodes that are playing* 0*, are linked to a node which is shifting from* 0 *to* 1*, and that node is the only neighbor they have who is originally playing* 1*.*

Suppose now that $x_i = 0$ and we flip this action so that $x_i^{new} = 1$. The nodes j in N_i^1 who were playing $x_j = 0$ will keep on. Node j in N_i^1 (at least one) who were playing $x_j = 1$ will move to $x_j^{new} = 0$. By previous point this will create a propagation to some $k \in N_j^1$, but not on i. This proves that the propagation of best response is limited to N_i^2 (and ends in a new NE). $\square$

Proof of Lemma 3: we proceed defining intermediate NE $\vec{x}^1$, $\vec{x}^2$,... between any two NE $\vec{x}$ and $\vec{x}'$. $\vec{x}^{n+1}$ will be obtained from $\vec{x}^n$ by flipping one node from 0 to 1 and waiting for best response.

If two NE $\vec{x}$ and $\vec{x}'$ are different, it must be that there is at least one i_1 such that $x_{i_1} = 0$ and $x'_{i_1} = 1$ (it is easy to check that any strict subset of a maximal independent set is not a covering any more). Change the action of that node so that $x^1_{i_1} = x'_{i_1} = 1$, by previous proof this will propagate deterministically to $N^1_{i_1}$ and, for all $j \in N^1_{i_1}$ we will have $x^1_j = x'_j = 0$. Propagation may affect also $N^2_{i_1}$ but we do not care about it.

If still $\vec{x}^1 \neq \vec{x}'$ then take another node i_2 such that $x^1_{i_2} = 0$ and $x'_{i_1} = 1$ (i_2 is clearly not a member of $N^1_{i_1} \cup \{i_1\}$). Pose $x^2_{i_2} = x'_{i_2} = 1$, this will change some other nodes by best response, but not $j \in N^1_{i_1} \cup \{i_1\}$, because any $j \in N^1_{i_1}$ can rely on $x^1_{i_1} = 1$, and then also $x^2_{i_1} = x^1_{i_1} = 1$ is fixed.

We can go on as long as $\vec{x}^n \neq \vec{x}'$, taking a node i_{n+1} for which $x^n_{i_{n+1}} = 0$ and $x'_{i_{n+1}} = 1$. This process will converge to $\vec{x}^n \to \vec{x}'$ in a finite number of steps because:

- when i_{n+1} shifts from 0 to 1, the nodes $j \in \bigcup_{h=1}^n \left(N^1_{i_h} \cup \{i_h\}\right)$ will not change, since it is either a 0 with already a 1 beside (the 1 is some i_h, with $h \leq n$), or a 1 (some i_h) surrounded by frozen 0's;
- by construction it is never the case that $i_{n+1} \in \bigcup_{h=1}^n \left(N^1_{i_h} \cup \{i_h\}\right)$, because for all $j \in \bigcup_{h=1}^n \left(N^1_{i_h} \cup \{i_h\}\right)$ we have that $x^n_j = x'_j$;
- the network is finite. □

In previous proof, the shift from $\vec{x}$ to $\vec{x}'$ is done by construction re–defining the covering of any $\vec{x}^n$ from the covering of $\vec{x}'$, being safe that, by best response, it is also an independent set.

References

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