Cooperation in volatile networks

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Abstract

In this paper, we examine the possibility of cooperation among agents who exhibit a particular version of bounded rationality. Players interact in a network, in which links stochastically decay and link formation opportunities stochastically arise. When players believe that their opponents will stick to the action observed in the last period, assortative or homophilous linking results. Therefore, defection comes at the cost of foregoing links to cooperators in the future. We investigate in which cases implicit punishment through linking decisions can enforce cooperation in the network under a wide range of parameter settings. We find that moderate patience suffices to establish (stochastic) stability of cooperation. In particular, volatility of links in the network facilitates cooperation.

1 Introduction

The Prisoner Dilemma itself and the issue of cooperation in PD-typed situations has been an area of intensive research. Particularly, it has been used as a role model for discussing human social evolution in economics, sociology, political science, and other social sciences (see e.g. Bergstrom (2002)). Following standard results of various folk theorems, cooperation can be implemented among rational agents in infinitely repeated Prisoner's Dilemma games in the case $\delta \to 1$. δ can either be seen as the discount factor or as the continuation probability of the game. One assumes that players are able to perform (complicated) punishment strategies. Further, the simple repeated interaction does not include any spatial structure which is essential for human social interaction. Lastly, the assumption of perfectly rational agents might not mirror the behaviour of humans in real life social interactions. This has led economists to consider models in which boundedly rational agents interact on a spatial structure (e.g. Eshel/Samuelson/Shaked (1998) or Ule (2008)). We are following this line in the present model. The environment is a randomly evolving network: There is stochastic link decay and players receive with a certain probability link formation opportunities. In general, players exhibit boundedly rational behaviour and cannot fully optimise their actions in the game. This can either be interpreted as limited cognitive capacity or else as limited information about the network structure as a whole. In the beginning of every round, existing links in the network from the last period are dissolved with exogenous probability λ. Subsequently, each pair of currently unlinked players receives the opportunity to form a new link with probability $\bar{\eta} \in (0, 1)$. If both players agree, the link is established. Hence our network implements mutual link formation. Once a link is established, both agents play the Prisoner's Dilemma game every period until the link is dissolved. Thus agreeing to a link is similar to committing to a bilateral project which lasts $\frac{1}{\lambda}$ rounds in expectation. In the real world, such a bilateral project (e.g. a joint venture) might be sealed through an incomplete contract so that effort cannot be determined in it. Each player receives with probability $\nu \in (0, 1)$ the chance to revise his action before he plays the PD and plays the same action against all his current neighbours. When players make a linking decision or revise their actions, they perform boundedly rational behaviour: They merely assume that the other players will go on to behave as they did in the last period. This reflects their inability to predict future behaviour of others and their best guess for low probabilities of link revision ν. In fact, each player categorises other players into notorious cooperators and notorious defectors. Similarly, they might not take into account their own possibility to revise their action at some (distant) future round. Their behaviour, therefore, is approximately rational for low values of ν. Given these constraints, players optimise their discounted payoffs. The assumption of boundedly rational behaviour has implication for the optimal decisions: Due to mutual link formation, cooperators will never agree to be linked to a defector. In this way, *assortative* (or *homophilous*) *linking* is introduced, i.e. new links are only established between two currently cooperating or defecting players. There are two driving forces behind optimal action choice in the model: On the one hand, there is a *defection effect*: agents have an incentive to defect when they have the chance to revise their action in order to take advantage of their current (cooperative) partners. However, existing links expire after some time. On the other hand, there is an *attraction effect*: when a player chooses to cooperate, other cooperators will agree to form links with him in the future. Intuitively speaking, when he can attract enough cooperators by cooperating himself, this offsets the incentive to take advantage of current partners. The current distribution of agents with different strategies influences player i's optimal decision through two channels: First, the number of player i's current defective or cooperative neighbours determines the utility from the *defection effect.* Second, the total amount of cooperative and defective players in the network influences his utility from the *attraction effect.* Thus a player revising his action faces the same situation *as if* he would play against rational agents implementing trigger strategies: If an agent switches from cooperation to defection, current projects with cooperators expire (possibly after some delay) and instead new projects are only started with defectors. Punishment is not carried out by sophisticated agents but implemented through the link formation itself.[1] In

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[1] The importance of punishment for cooperative behaviour when there is no exclusion is stressed in the empirical literature, e.g. Fehr/Gachter (2000) present results for public goods games.

this paper, we illuminate the possibility of cooperation in such a model.

One can obtain an intuition of this setup by considering an stylised example: The setup mirrors the market of collaborations on published economics articles: Each completed PD interaction then reflects a published article from the two participants. Occasionally, one gets into contact with suitable economists to form a new project. When players agree to a link formation, they commit to a bilateral project that lasts $\frac{1}{\lambda}$ rounds in expectation. Both economists have to approve a new project so that it is indeed established. When some player is deciding about a project, he can ask former partners of his potential new colleague about his behaviour. Furthermore, performance in projects is publicly observable through working papers and published articles. A player can judge the quality of the article and the contribution of the individual author. The general quality of papers and the general work effort in the society is known. Similarly, players take the general propensity to cooperate in our model into account. Working habits seem to be fairly stable over projects and time. They are only occasionally adjusted. Therefore, in making a decision, players might very well assume that others are just sticking to the behaviour he has observed. Our main findings show that cooperation is stable for a reasonable range of parameters. Specifically, discount rates can differ in certain configurations significantly from 1 and still allow cooperation. Volatile networks and networks with high average degrees facilitate cooperation. This might be somehow surprising having in mind models in which cooperation is enforced through punishment in long run interactions (e.g. trigger strategies in an infinitely repeated 2 player PD). In our model, high volatility has two specific effects: First, the relative advantage of exploiting current partners decreases when existing links decay faster. Second, when new links can be established at higher rates, the implicit punishment of defectors through assortative linking becomes stronger. Our model gives some insights of how cooperation might be informally enforced in “failed” states’ economies, i.e. states in which there are no functioning institutions so that there are only short term projects (contracts) and high rates of partner change.

The paper proceeds as follows: First, the relation of our model to existing literature is addressed. In section 3, the model is formally introduced. Static decision making is analysed in the subsequent section. We show that players will indeed perform assortative linking. Then we characterise the utility from the *defection* as well as the *attraction effect.* The stability of cooperation is discussed in the fifth section: We proof a 'folk theorem' showing that cooperation is possible if the discount factors tend to one. If all players cooperate, no one would unilaterally defect. In this sense, cooperation is (locally) stable. We present numerical analyses of threshold values for which cooperation can be enforced under different parameter values. We find that cooperation is stable under a wide range of parameters, and—quite surprisingly—that more volatile networks facilitate cooperation. Afterwards, we characterise the dynamics of the process. Specifically, we find a condition under which the process converges to full full defection. We show that the sets of fully cooperative and fully defective states form the only two recurrent sets of the process. Section six addresses the issue of stochastic stability. In particular, stochastic stability can be used as a criterion to select between the two recurrent sets of the unperturbed process. It is shown that cooperation is stochastically stable in typical settings for moderately high discount factors δ. Some variations of the model are discussed in the following section. The last section concludes.

2 Literature

The model presented in this paper relates in various ways to existing literature. There is a growing literature on models of cooperation using computer simulations. Ule (2008) studies in chapter six the evolution of a norm that prescribes the exclusion of defectors. Every player plays the PD against all his current neighbours in a network in every round. A link between any two players is established in the beginning of the round iff both players support the link; it decays at the end of the round. There are four possible strategies: Each player can either cooperate or defect and either support links to all other players or only support them to players who cooperated in the last round. When players get the chance to revise their strategy, they choose the strategy that maximises their undiscounted total payoff from the next r rounds assuming the distribution of strategies in the society is static and identical to that from round $t-1$. Ule finds the absorbing states for the unperturbed process analytically. For $r>1$, there are specifications for which states with cooperation are absorbing. Using simulations, Ule provides evidence that there are no other recurrent sets apart from absorbing states (using specific PD parameters). For low outside options, the unperturbed process leads according to simulations almost surely to the absorbing state of full non-exclusive defection. Further, this state seems to be stochastically stable. For high outside options, the state where all players are exclusive cooperators seems to be stochastically stable for $r=4$. Subsequently, Ule also analyses the effects of altruistic preferences on the outcome of the game. As we show later on, there is an interpretation of our model in which agents' decision making process is similar to the one in Ule's model. Using his language, there are only exclusive cooperators and non-exclusive defectors in our model due to assortative linking. Further, action revision and link formation decisions are disentangled in contrast to Ule's model. One could see our model in part as an analytic underpinning of Ule's findings. Besides, we also examine network specifications that foster cooperation in such a setting. In chapter four of his book, Ule considers a model of finitely repeated PD's in a network setting. When the outside option of no links is worse than mutual defection and when the maximal number of links is restricted but not to extremely restricted, then cooperation can be enforced through threat of exclusion in a subgame perfect, linking-proof equilibrium.[2] In particular, agents use (subgame perfect) trigger strategies which implement on the outcome path cooperation for the first rounds and defection for later rounds. If some player deviates from cooperation in one of the early rounds, he is punished by exclusion in the later rounds (thus the payoff from being unlinked must necessarily be lower than the payoff from mutual defection for effective punishment of deviators). Also, due to linking proofness, there has to be some restriction on the total amount of neighbours so that the result holds.

Homophily is an empirically well studied phenomenon, see for instance McPherson/Smith-Lovin/Cook (2001). It describes the finding that people tend to be over proportionally matched in groups or interact with other people who resemble them in some respect. Those features could be traits (such as gender or age), abilities (e.g. measured through education) but also certain kinds of behaviour or beliefs. In our model, new links are established in an homophilous way; for instance, we could interpret cooperators as people with high work ethos. Currarini/Jackson/Pin (2009) study a matching model, that can actually simulate empirical group patterns regarding homophily. They use friendship networks as a benchmark and show how biases in preferences and biases in meetings are together sufficient to replicate the observed data. Fosco/Mengel (2009) provide a model of imitation learning in a network setting which extends a model of Eshel/Samuelson/Shaked (1998). Agents choose links and actions by imitating successful other players in their information radius. They play PD's against all their current neighbours. Fosco and Mengel show that (under certain parameters) networks with a core of cooperators and defectors in the periphery can be stable. Hence assortative (or: homophilous) linking is the result of their dynamics under those parameters. In contrast to these models, agents in our model do not

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[2] Due to mutual link formation, linking-proofness is used as an additional requirement for equilibria, i.e. two players are only unlinked at any point in time if they either could not gain from linking or at least one of them has already established his maximally allowed number of links.

imitate successful agents surrounding them but rather optimise their payoff believing that action choices are fixed. The underlying structure of network formation in our model is similar to the model presented in Ehrhardt/ Marsili/Vega-Redondo (2008): Agents receive with a certain probability a link formation and an action revision opportunity in every round.[3] Further, links decay stochastically. In contrast to the model presented here, their underlying game is a coordination game. If two players have a link formation opportunity, the link is established iff they are currently playing the same action in the coordination game, resembling the idea of homophily. The authors are specifically interested in the interaction between agents' struggle to coordinate and the process of network formation (and hence the resulting network topology). When players have the opportunity to revise their action, they choose one of the actions played most among their neighbours at random (possibly with a slight bias for status quo). Therefore, an unlinked agent can choose any action so that in the ultra long run only states prevail in which all linked agents choose the same action. These states form a unique recurrent set and ergodicity leads to an essentially unique configuration in the ultra long run. In the long run, however, small changes in the underlying parameters and initial conditions hugely influence the overall connectivity and topology of the network. There is also an older literature, in which cooperation is enabled through assortative group formation under replicator dynamics. An overview is given by Bergstrom (2002).

The concept of stochastic stability and using small (possibly vanishing) errors in order to construct ergodic processes for means of analysis were introduced into the literature by Kandori/Mailath/Rob (1993) and Young (1993). A good overview is given in the Young (1998). There are some papers using these ideas in the context of evolution of cooperation or networks. Levine/ Pesendorfer (2007) study evolutionary stable outcomes when players imitate

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[3] Specifically, we are following them in disentangling action revision and link formation choices, in contrast to most of the network formation literature.

other players strategies. In particular, they imitate with high probability a relative best response to the existing strategy profile in the society, with lower probability they randomly imitate an existing strategy and in rare cases they innovate. Players are matched in pairs and play symmetric games, which have cooperative and opportunistic Nash equilibria. Strategies map signals into actions and signals are generated automatically following a probability distribution depending on both players strategies. They show that the evolutionary stable outcome is efficient if there is a signal, which is generated iff both players have the same strategy. Stochastic stability is applied in a network setting by Goyal/Vega-Redondo (2005). They study 2-player coordination games in a network. One of the Nash equilibria of the game is risk dominant, the other is pareto efficient. Further, links are costly and can be established one sided. There are multiple equilibria in the short run, but only complete networks with full coordination on one action are stochastically stable when the costs of linking are not too high. There is a threshold for the linking costs determining on which action players coordinate. For low costs, they coordinate on the risk-dominant action. Vega-Redondo (2006) identifies the stock of social capital with the density and stability of social networks. In his model, each pair of players has a specific potential gain from being linked which is stochastic and updated with a certain probability every round. Crucial for his model is the notion of stability of links: Given any network, he defines hypothetical infinitely repeated PDs between any two neighbours based on the potential gain from the link. If mutual cooperation is stable in this game when information about defection diffuse gradually through the network and agents play grim trigger strategies, then the link is considered to be stable. Given this notion of stability, he constructs an ergodic process, in which in each round with certain probability the gain from a particular link is updated, agents can establish new links through local and global search and unstable links are removed. Due to ergodicity of the process, he can use computer simulation in order to determine the long

(1)	$\lambda \in (0,1)$	Probability of link decay
(2)	$\bar{\eta} \in (0,1)$	Probability of link formation opportunity
(3)	$\nu \in (0,1)$	Probability of action revision opportunity
(4)	PD	Realisation of PD payoffs

Figure 1: Structure of a Round

run reaction of the network to increased volatility of payoffs. In particular, they can show that endogenously the average distance between neighbours decreases so that punishment in the underlying PD is more effective.

3 The Model

Suppose there are $N = \{1, 2, \ldots, n\}$ agents with $n \geq 2$. Following standard notation, G stands for the set of undirected graphs with n vertices. Each vertex represents a player $i \in N$ and edges of the graph represent links among players. The network in round t after links have been deleted or established is denoted as $g_t \in G$. Further, we introduce $g_{ij,t} = g_{ji,t} \in \{0, 1\}$: A link exists between any $i, j \in N$ in g_t if and only if $g_{ij,t} = g_{ji,t} = 1$. Let $A_i = \{c, d\}$ be the set of actions for each individual i and $a_{i,t} \in A_i$ denote the action that player i plays in the Prisoner's Dilemma game at period t. Further, $A = A_1 \times \cdots \times A_n$ denotes the set of possible profiles and $a_t = (a_{1,t}, \ldots, a_{n,t}) \in A$ is the corresponding profile in some period t. Then the set $N_t^c = \{i \in N \,|\, a_{i,t-1} = c\}$ contains all cooperators in a certain round *before possible revision of actions has occurred.* Likewise N_t^d is defined. Also we have $n_t^c = |N_t^c|$ and $n_t^d = |N_t^d|$ as the amount of agents that intend to cooperate or defect in the beginning of round t. In general, small letters represent cardinalities henceforth. By including a subscripts $-i$, we address all agents apart from i, e.g. $N_{-i,t}^c = N_t^c \setminus \{i\}$. The neighbours of i in round t are collected in the set $N_{i,t} = \{j \in N \setminus \{i\} \,|\, g_{ij,t} = 1\}$. Lastly, we have the current cooperating neighbours of i before revision of strategies in $N_{i,t}^c = \{j \in N_{i,t} \,|\, a_{j,t-1} = c\}$; $N_{i,t}^d$ is defined accordingly.

$1 \setminus 2$	c	d
c	R, R	S, T
d	T, S	P, P

Figure 2: Prisoner's Dilemma game

The structure of a single round can be seen in figure 1. At the beginning of the round, every existing link in the network from g_{t-1} is dissolved with probability $\lambda \in (0, 1)$, which is independent for each link. Let us call the network obtained after possible deletion g'_{t-1} where $g'_{ij,t-1} = 0$ if the link between i and j has just been deleted and otherwise $g'_{ij,t-1} = g_{ij,t-1}$. Any two players i, j with $g'_{ij,t-1} = 0$, i.e. who either had no link in the last period or whose link has just decayed, receive a link creation opportunity with probability $\bar{\eta} \in (0, 1)$. If the opportunity arises and both of them approve, a link is established between these players and $g_{ij,t} = 1$. Otherwise $g_{ij,t} = g'_{ij,t-1}$. This creates the new network g_t. Subsequently, each player i receives with probability $\nu \in (0, 1)$ the opportunity to revise his action. Once he has chosen an action, he will play this action until he receives a new revision opportunity. In the end of the round, every player plays the Prisoner's Dilemma game against all his current neighbours in g_t. The payoff-matrix is given in table 2. The payoffs have the following structure: $T > R > P > 0 > S$. This is similar to the standard payoff structure of a PD when we normalise it with respect to having no interaction at all. Here we assume that being exploited by j is worse than having no connection to j at all. Variations of this assumption are discussed in section 7.

Players discount future payoffs with the discount factor $\delta \in (0, 1)$ when they are making a decision. They have limited memory, such that they only know the last rounds action profile a_{t-1} and the current linking situation g_t. Hence their possible strategies for action revision are given by $S_i : A \times G \to \{c, d\}$.[4] They assume that other players will stick to their action forever

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[4] For completion, we assume that the initial action profile a_0 as well as the initial network g_0 are randomly determined.

and might assume the same about themselves as specified later. When they are allowed to revise their action, they maximise their discounted future payoffs. If both, cooperation and defection, lead to the same (expected) utility, one action is chosen at random. A state of the dynamic process $\omega_t = (a_t, g_t)$ contains the actions played in the PD in round t and the current linking situation. The set $\Omega = A \times G$ contains all possible states. A state ω_t is called fully cooperative or fully defective iff $a_t = (c, \ldots, c)$ or $a_t = (d, \ldots, d)$ regardless of the current network g_t. The set $\Omega^c = \{\omega_t \in \Omega \,|\, a_t = (c, \ldots, c)\}$ contains all fully cooperative states. Similarly, Ω^d contains all fully defective states. Full cooperation or full defection are called stable iff Ω^c or Ω^d are recurrent sets of the unperturbed dynamics. In that case, once a fully cooperative state is reached, all following states are fully cooperative and the process never leaves Ω^c. Further, each cooperative state can be reached from any other.

4 Static Decision Making

First, we analyse the linking decisions made by players. Essentially, we will see that there is assortative linking since cooperators only agree to link formation with fellow cooperators. Players believe that their potential partners would stick to their chosen action forever. They might be aware that they themselves can revise their action at some (distant) future time. We provide a sufficient condition on ν so that it is weakly dominant for cooperators to agree to a link formation iff the opponent is a cooperator as well. By a similar condition, it is weakly dominant for defectors to agree to any link formation. Subsequently, we assume that links are indeed established, iff both players would support them—i.e. iff it is weakly dominant for both to agree to the link formation. This assumption is related to the link proofness assumption in the literature (see e.g. Ule (2008) chapter 4). It essentially guarantees that there is no miscoordination in the sense that two players both deny approval to a mutually profitable link and behave optimal since a link formation would have needed both their approvals.

Proposition 1. *Assume that there is low probability of action revision such that* $\nu < \min\{\frac{-[1-\delta(1-\lambda)]S}{P-[1-\delta(1-\lambda)]S}, \frac{[1-\delta(1-\lambda)]P}{[1-\delta(1-\lambda)]P-S}\}$ *and every player* i *believes that other players (except from himself) cannot revise their action. Whenever a cooperative player* i *with* $a_{i,t-1} = c$ *has the opportunity of creating a link to some player* j*, it is weakly dominant for him to agree to the link if and only if* $a_{j,t-1} = c$*. It is weakly dominant for a defective player with* $a_{i,t-1} = d$ *to agree to any link formation. Hence new links would be mutually supported if and only if* $a_{i,t-1} = a_{j,t-1}$*.*

Proof. For showing weak dominance of the different linking decisions for player i, we assume that the opposing player j always supports the possible link formation. Hence player i is indeed decisive for establishing the link.[5] First, one can see that every player i agrees to create a link to any cooperative player j with $a_{j,t-1} = c$ regardless of his own current or future action. This is the case because, by bounded rationality, i assumes that j plays his current action forever and $T > R > 0$ so that i does always better than not establishing the link. Next we consider the case when i is cooperative with $a_{i,t-1} = c$ and j is defective with $a_{j,t-1} = d$. Again i assumes that j sticks to his action forever. Then he receives $S < 0$ from linking with j in every period until he is able to change his action to defection as well. Hence the best payoff i can expect from linking to j occurs when he switches to defection as soon as possible and plays defection until the link is dissolved. Since we want to show that i does not agree to be linked with such a j, the following condition has to hold:

$$\sum_{t=0}^{\infty} \delta^t (1-\lambda)^t [(1-\nu)^{t+1} S + (1-(1-\nu)^{t+1}) P] < 0$$

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[5] If player j would always veto the link formation instead, player i's decision does not influence the result and hence any decision of i is optimal.

In this formula, $(1-\lambda)^t$ reflects the probability that the link is dissolved. Further, player i can already revise his strategy with the probability ν before the game is played in the current round. Solving this condition, we get $\nu < \frac{-[1-\delta(1-\lambda)]S}{P-[1-\delta(1-\lambda)]S}$.[6] Lastly, assume that i is defective with $a_{i,t-1} = d$ and j is defective as well with $a_{j,t-1} = d$. Hence we want to show that player i agrees to establishing a link to j even if he plans to change his action to cooperation at the next possible instance. Similarly to above, we get the following condition:

$$\sum_{t=0}^{\infty} \delta^t (1-\lambda)^t [(1-\nu)^{t+1} P + (1-(1-\nu)^{t+1}) S] > 0$$

Solving for ν we get the second condition $\nu < \frac{[1-\delta(1-\lambda)]P}{[1-\delta(1-\lambda)]P-S}$. We have shown that it is indeed weakly dominant for cooperators to agree to links only with other cooperators and for defectors to agree to any link formation. Therefore, new links would only be supported by both partners if $a_{i,t-1} = a_{j,t-1}$. $\square$

As mentioned above, we assume that new links are established if and only if they would be supported by both partners. Thereby, we have found a *sufficient* condition on ν for fully assortative linking. Whenever it is not the case that $\delta = 1$ and $\lambda = 0$ we can find such a small ν. Specifically, if players are unaware of their own possibility to revise their action at some (distant) future round (or merely do not take it into account), the condition is always fulfilled. Henceforth, we assume that agents are boundedly rational in that sense, as well.

Next we consider a decision between cooperation and defection once an agent i receives the opportunity to revise his action. For characterising the optimal strategy $S_i : A \times G \rightarrow \{c, d\}$, only a subset of the total information are actually used: Player i's expected utility from a certain action depends on the number of his cooperative or defective neighbours and the total number of cooperative or defective players apart from i in the network. We distinguish

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[6] Note that we assume $P > 0 > S$.

two parts of his expected utility: First, he receives some utility from his expiring interaction with current neighbours until these links are dissolved. We call this the *defection effect* since this part of the total expected utility incentivises agents to change their action to defection in order to exploit their current neighbours. Second, each players expected utility also depends on the future willingness of other agents to be linked with him. We call this the *attraction effect* because a cooperator attracts other cooperators and a defector attracts other defectors by assortative linking introduced above. The optimal action maximises the sum of these two expected utilities.

Proposition 2. *From the defection effect, a player who chooses defection receives expected utility* $u^d_{DE} = \frac{n^c_{i,t}T + n^d_{i,t}P}{1-\delta(1-\lambda)}$ *and a player who chooses cooperation receives* $u^c_{DE} = \frac{n^c_{i,t}R + n^d_{i,t}S}{1-\delta(1-\lambda)}$.

Proof. According to the above definition, we have to consider current neighbours of i in g_t only. By bounded rationality, player i assumes that they will simply stick to their action. A link to any such current neighbour is dissolved in any future round with probability λ. Hence we get:

$$u^d_{DE} = \sum_{\tau=0}^{\infty} \delta^\tau (1-\lambda)^\tau [\sum_{j\in N^c_{i,t}} T + \sum_{j\in N^d_{i,t}} P] = \frac{n^c_{i,t}T + n^d_{i,t}P}{1-\delta(1-\lambda)}$$

Similarly, we can find $u^c_{DE} = \frac{n^c_{i,t}R + n^d_{i,t}S}{1-\delta(1-\lambda)}$. □

As in the standard PD problem, we have $u^d > u^c$ so that players would unambiguously decide to defect if there was only the defection effect. Next, we consider the expected utility from the attraction effect:

Proposition 3. *The utility from the attraction affect is given by* $u^d_{AE} = (\xi_1 n^d_{-i,t} - \xi_2 n^d_{i,t})\frac{\bar{\eta}\delta P}{1-\delta(1-\lambda)}$ *for defecting and* $u^c_{AE} = (\xi_1 n^c_{-i,t} - \xi_2 n^c_{i,t})\frac{\bar{\eta}\delta R}{1-\delta(1-\lambda)}$ *for* *cooperating where* ξ_1 *and* ξ_2 *are defined as follows:*

$$\begin{aligned}\xi_1 &= \frac{1-\delta+\lambda\delta}{[1-\delta+\lambda\delta+(1-\bar{\eta})\lambda\delta][1-\delta]}\\ \xi_2 &= -(1-\lambda)\frac{1}{1-\delta(1-\bar{\eta})(1-\lambda)}\end{aligned}$$

Proof. We first focus on the utility for defection through the attraction effect. It is generally given by the following formula:[7]

$$u^d_{AE} = \sum_{\tau=1}^{\infty} \delta^{\tau}\bar{\eta}(n^d_{-i,t} - (1-\lambda)E[n^d_{i,t+\tau-1}|n^d_{i,t}])(\sum_{s=\tau}^{\infty}\delta^{s-\tau}(1-\lambda)^{s-\tau}P)$$

Note that due to the order of decision making, player i can only start to attract people from the next period onwards and hence the sum starts from $\tau = 1$. Let us consider the various terms of the formula. $\bar{\eta} \in (0,1)$ stands for the probability that player i and any given $j \in N \setminus \{i\}$ with whom player i is currently not linked get a link formation opportunity. A new link is thus established if (i) j is a defector as well and (ii) j is not a current neighbour of i. Given that a link to such an outsider is established in round τ, player i receives from him the expected utility $(\sum_{s=\tau}^{\infty}\delta^{s-\tau}(1-\lambda)^{s-\tau}P) = \frac{P}{1-\delta(1-\lambda)}$ similarly to above. Lastly, the term $(n^d_{-i,t} - (1-\lambda)E[n^d_{i,t+\tau-1}|n^d_{i,t}])$ stands for the expected number of defectors outside the neighbourhood of i in the network in round $t+\tau$ before the link formation opportunities arise. In other words, these are the defectors with whom i could potentially establish new links in round $t+\tau$. Hence we have to calculate $E[n^d_{i,t+\tau-1}|n^d_{i,t}]$:

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[7] Strictly speaking, the formula holds only since we assume players ignore their own possibility to revise their action at some (distant) future round. Otherwise it might be optimal to (say) play cooperation for several rounds to attract partners and then revise to defection in order to exploit them. This calculation would have to include the value of establishing one more link and computation is a difficult dynamic programming problem. Bounded rationality in our sense means that players ignore this strategic element—which is approximately rational for low ν.

$$\begin{aligned}
E[n^d_{i,t}|n^d_{i,t}] &= n^d_{i,t} \\
E[n^d_{i,t+1}|n^d_{i,t}] &= E[n^d_{i,t}|n^d_{i,t}](1-\lambda) + \bar{\eta}(n^d_{-i,t} - (1-\lambda)E[n^d_{i,t}|n^d_{i,t}]) \\
&= (1-\bar{\eta})(1-\lambda)n^d_{i,t} + \bar{\eta}n^d_{-i,t} \\
E[n^d_{i,t+2}|n^d_{i,t}] &= (1-\bar{\eta})(1-\lambda)E[n^d_{i,t+1}|n^d_{i,t}] + \bar{\eta}n^d_{-i,t} \\
&= [(1-\bar{\eta})(1-\lambda)]^2 n^d_{i,t} + (1-\bar{\eta})(1-\lambda)\bar{\eta}n^d_{-i,t} + \bar{\eta}n^d_{-i,t}
\end{aligned}$$

Thus we get

$$\begin{aligned}
E[n^d_{i,t+\tau-1}|n^d_{i,t}] &= [(1-\bar{\eta})(1-\lambda)]^{\tau-1}n^d_{i,t} + \sum_{s=0}^{\tau-2}[(1-\bar{\eta})(1-\lambda)]^s\bar{\eta}n^d_{-i,t} \\
&= [(1-\bar{\eta})(1-\lambda)]^{\tau-1}n^d_{i,t} + \frac{1-[(1-\bar{\eta})(1-\lambda)]^{\tau-1}}{1-(1-\bar{\eta})(1-\lambda)}\bar{\eta}n^d_{-i,t}
\end{aligned}$$

for $\tau \geq 2$. When we insert this finding into the above formula, it simplifies to:

$$\begin{aligned}
u^d_{AE} = & \{\sum_{\tau=2}^{\infty}\delta^\tau\bar{\eta}([1-(1-\lambda)\bar{\eta}\frac{1-[(1-\bar{\eta})(1-\lambda)]^{\tau-1}}{1-(1-\bar{\eta})(1-\lambda)}]n^d_{-i,t} \\
& -(1-\lambda)[(1-\bar{\eta})(1-\lambda)]^{\tau-1}n^d_{i,t}) \\
& +\delta\bar{\eta}[n^d_{-i,t} - (1-\lambda)n^d_{i,t}]\}\frac{P}{1-\delta(1-\lambda)}
\end{aligned}$$

Note that the last term equals the $\tau = 1$ term of the sum. Using the formula for geometric series, we get:

$$\begin{aligned}
u^d_{AE} &= \{[\frac{1}{1-\delta} - \frac{(1-\lambda)\bar{\eta}}{[1-(1-\bar{\eta})(1-\lambda)][1-\delta]} \\
&\quad + \frac{(1-\lambda)\bar{\eta}}{[1-(1-\bar{\eta})(1-\lambda)][1-\delta(1-\bar{\eta})(1-\lambda)]}]n^d_{-i,t} \\
&\quad -(1-\lambda)\frac{1}{1-\delta(1-\bar{\eta})(1-\lambda)}n^d_{i,t}\}\frac{\bar{\eta}\delta P}{1-\delta(1-\lambda)} \\
&= \{\frac{1-\delta+\lambda\delta}{[1-\delta+\lambda\delta+(1-\bar{\eta})\lambda\delta][1-\delta]}n^d_{-i,t} \\
&\quad -(1-\lambda)\frac{1}{1-\delta(1-\bar{\eta})(1-\lambda)}n^d_{i,t}\}\frac{\bar{\eta}\delta P}{1-\delta(1-\lambda)} \\
&= (\xi_1 n^d_{-i,t} - \xi_2 n^d_{i,t})\frac{\bar{\eta}\delta P}{1-\delta(1-\lambda)}
\end{aligned}$$

Similarly, the utility from cooperation through the *attraction effect* can be found as $u^c_{AE} = (\xi_1 n^c_{-i,t} - \xi_2 n^c_{i,t})\frac{\bar{\eta}\delta R}{1-\delta(1-\lambda)}$. □

One can see that the utility from defection through *the attraction effect* u^d_{AE} decreases in current defecting neighbours $n^d_{i,t}$. An increase in currently defecting neighbours $n^d_{i,t}$ decreases the number of defectors that are not in the current neighbourhood and hence the number of players i can potentially create new links with. Further, u^d_{AE} increases in the total amount of defecting players in the society $n^d_{-i,t}$. The utility from cooperation through the *attraction effect* u^c_{AE} has similar properties. The total utility for defection is given by $u^d = u^d_{DE} + u^d_{AE}$ and similarly for cooperation by $u^c = u^c_{DE} + u^c_{AE}$. Then we get:

$$\begin{aligned}
u^d &= \{\frac{\bar{\eta}\delta n^d_{-i,t}}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]} \\
&\quad + \frac{n^d_{i,t}}{1-\delta(1-\bar{\eta})(1-\lambda)}\}P + \frac{n^c_{i,t}}{1-\delta(1-\lambda)}T
\end{aligned}$$

And similarly:

$$\begin{aligned} u^c &= \{\frac{\bar{\eta}\delta n^c_{-i,t}}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]} \\ &\quad + \frac{n^c_{i,t}}{1-\delta(1-\bar{\eta})(1-\lambda)}\}R + \frac{n^d_{i,t}}{1-\delta(1-\lambda)}S \end{aligned}$$

A player behaves optimal, if he chooses the action that maximises his utility. Hence cooperation is chosen when $u^c > u^d$; if $u^c = u^d$, both actions are chosen with equal probability and for $u^c < u^d$ defection is chosen.

In order to illuminate the connection of this model to the existing literature of cooperation among rational agents, we consider the special case of highly volatile networks, i.e. when links are almost always dissolve after one round and each player gets a link formation opportunity to almost every other player in the network.

Example 4. For $\lim_{\lambda\to 1}$ and $\lim_{\bar{\eta}\to 1}$, the condition for stability of full cooperation among boundedly rational agents in a network is identical to the condition for cooperation among rational agents in a PD with grim trigger strategies when P is normalised to zero.

First we have $\lim_{\bar{\eta}\to 1}$ so that player i could in principle establish in every round a link to any player $j \in N \setminus \{i\}$ with whom he is not currently linked. Further, links are almost always deleted immediately (players have one shot interactions) and we get $\lim_{\lambda,\bar{\eta}\to 1}\xi_1 = \frac{1}{1-\delta}$ and $\lim_{\lambda,\bar{\eta}\to 1}\xi_2 = 0$. Thus the overall expected utility of defection and cooperation from the perspective of a boundedly rational agent can be calculated as

$$\begin{aligned} u^d &= n^c_{i,t}T + n^d_{i,t}P + \delta\frac{n^d_{-i,t}}{1-\delta}P \\ u^c &= n^c_{i,t}R + n^d_{i,t}S + \delta\frac{n^c_{-i,t}}{1-\delta}R \end{aligned}$$

Now assume we are in a state in which all players cooperate, i.e. $n^c_t = n$ and

$n_t^d = 0$. Given that player i has the chance to revise his strategy, he has the highest incentive to defect if $n_{i,t}^c = n - 1$, i.e. he is currently linked to all other players. We get

$$u^c \gtreqless u^d \Leftrightarrow \frac{1}{1-\delta} R \gtreqless T$$

This is the same condition on δ for cooperation as in a 2-person PD with $P = 0$ in which the trigger strategies prescribe cooperation for both players on the intended path and mutual never ending defection after any deviation.

5 'Folk Theorem' and Dynamics

Next we can generalise this finding. As we defined earlier, we have $a_t = (c, \ldots, c)$ for any fully cooperative state $\omega_t \in \Omega^c$. Additionally, full cooperation is called stable iff Ω^c is a recurrent set of the process. When full cooperation is stable, nobody has an incentive to unilaterally deviate from unanimous cooperation regardless of the current linking situation. We can show that for any network configuration full cooperation is stable for δ high enough:

Theorem 5. *For any network configuration $\lambda \in (0,1)$, $\nu \in (0,1)$, and $\bar{\eta} \in (0,1)$ there is $\delta^* < 1$ such that full cooperation is stable (and Ω^c a recurrent set) when $\delta > \delta^*$.*

Proof. First we show that the incentive for unilateral deviation from full cooperation (i.e. the relative utility gain from defection) is strongest when a player i is linked to all other cooperative players. The incentive for unilateral deviation from full cooperation can be written by our former finding as follows (with $n_t^c = n$ and $n_t^d = n_{i,t}^d = n_{-i,t}^d = 0$):

$$\begin{aligned} u^d - u^c \;\; = \;\; & \frac{n_{i,t}^c}{1-\delta(1-\lambda)} T - \{ \frac{n_{i,t}^c}{1-\delta(1-\bar{\eta})(1-\lambda)} \\ & + \frac{\bar{\eta}\delta n_{-i,t}^c}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]} \} R \end{aligned}$$

Since $T > R$ and $\frac{1}{1-\delta(1-\lambda)} > \frac{1}{1-\delta(1-\bar{\eta})(1-\lambda)}$, the incentive for unilateral deviation increases in $n^c_{i,t}$. Hence a player i has the strongest incentive to deviate from full cooperation if $n^c_{i,t} = n^c_{-i,t} = n - 1$. There is always a positive probability that a (cooperating) player will form links to all other cooperating players through the dynamics. Hence as a necessary and sufficient condition for the stability of full cooperation, $u^d - u^c < 0$ has to hold if $n^c_{i,t} = n^c_{-i,t} = n-1$. In other words, we have to guarantee that a player would choose cooperation even if he is currently linked to all other (cooperating) players. The condition becomes:

$$\begin{aligned} 0 &> \frac{1}{1-\delta(1-\lambda)}T - \{\frac{1}{1-\delta(1-\bar{\eta})(1-\lambda)} \\ &\quad + \frac{\bar{\eta}\delta}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]}\}R \end{aligned}$$

Or:

$$\begin{aligned} \frac{T}{R} &< \frac{1-\delta(1-\lambda)}{1-\delta(1-\bar{\eta})(1-\lambda)} \\ &\quad + \frac{\bar{\eta}\delta(1-\delta(1-\lambda))}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]} \end{aligned} \tag{1}$$

For the first term of the right hand side, we know $\lim_{\delta\to 0}[\frac{1-\delta(1-\lambda)}{1-\delta(1-\bar{\eta})(1-\lambda)}] = 1$ and $\lim_{\delta\to 1}[\frac{1-\delta(1-\lambda)}{1-\delta(1-\bar{\eta})(1-\lambda)}] = \frac{1-(1-\lambda)}{1-(1-\bar{\eta})(1-\lambda)} < 1$. As a function of δ this term is strictly decreasing and concave in the relevant range of parameters. All higher order derivatives are strictly negative and assume finite values for $\delta = 1$. In contrast, the second term goes to infinity for $\lim_{\delta\to 1}$ and is zero for $\lim_{\delta\to 0}$. It is strictly increasing and convex. The condition is not fulfilled for $\lim_{\delta\to 0}$ since by definition $\frac{T}{R} > 1$ and the right hand side equals 1. Then there is a δ^* which solves the equation with equality such that all $\delta > \delta^*$ have the desired property. If $\delta > \delta^*$, the process never leaves the set of fully cooperative states Ω^c once some state $\omega_t \in \Omega^c$ is reached. Further, every state $\omega' \in \Omega^c$ is accessible from any other state $\omega \in \Omega^c$ and so the set Ω^c is

	$\frac{T}{R}$	1.1	1.5	2.0	5.0
$\lambda = 0.1$	$\bar{\eta} = 0.1$	0.78	0.93	0.96	0.99
$\lambda = 0.1$	$\bar{\eta} = 0.9$	0.26	0.51	0.64	0.85
$\lambda = 0.5$	$\bar{\eta} = 0.5$	0.27	0.63	0.76	0.93
$\lambda = 0.9$	$\bar{\eta} = 0.1$	0.62	0.90	0.95	0.99
$\lambda = 0.9$	$\bar{\eta} = 0.9$	0.11	0.38	0.55	0.83

Figure 3: Threshold values δ^* for stability of cooperation under different specifications

recurrent. $\square$

In the next paragraph, we address the practical implications of our findings so far. Though we have proven that cooperation is always stable for discount factors δ high enough in theorem 5, the natural question arises whether this result basically only holds when δ is almost equal to one. Furthermore, the dependence of threshold values δ^* on different network specifications can give some insight into the likelihood of cooperation in different real world settings. In equation 1, we implicitly defined $\delta^* = \delta^*(T, R, \lambda, \bar{\eta})$. The (local) stability of cooperation depends merely on the temptation payoff relative to the reward payoff in the PD and the network parameters. This fact holds since we do only consider the incentive of a single individual for unilateral defection. As will be shown later on, stochastic stability of cooperation in contrast also depends on the remaining payoff parameters of the PD. Since it is hard to solve equation 1 explicitly for δ, we provide a numerical analysis of threshold values for typical parameter values in figure 3. Keeping in mind that by definition $T > R$, we consider different typical ratios between the temptation and reward payoff with $\frac{T}{R} \in \{1.1, 1.5, 2.0, 5.0\}$. In other words, we consider cases when cheating in a one time interaction would give a player up to 5 times the payoff he gains from mutual cooperation. We also contrast different network types ranging from very stable networks ($\lambda = \bar{\eta} = 0.1$) to very volatile ($\lambda = \bar{\eta} = 0.9$) ones and low average degree ($\lambda = 0.9$, $\bar{\eta} = 0.1$) to high average degree ($\lambda = 0.1$, $\bar{\eta} = 0.9$). Let us first consider a network with medium configuration, i.e. $\lambda = \bar{\eta} = 0.5$. As apparent from equation 1, the higher the relative temptation $\frac{T}{R}$, the higher the threshold on δ. For the configurations under consideration, the threshold is moderate and ranges from 0.27 to 0.93 for different $\frac{T}{R}$. Given an annual interest rate of 4%, we might empirically expect an average annual discount factors of round about 0.96. Frederick/Loewenstein/O'Donoghue (2002) discuss time preferences in general and give an overview about different empirical estimates of the discount factor. Though different estimates vary quite a lot, our thresholds are fairly below 1 for most specifications and cooperation is stable for a reasonable range of discount factors. In general, it seems that the thresholds decrease in $\bar{\eta}$ and (to a lesser degree) in λ. Thus volatility and high average degree facilitate cooperation in these configurations.[8] Figure 4 shows the thresholds for a temptation to reward ratio of 1.5 graphically. Let us consider the case of balanced networks with common probability for link decay and link formation opportunities. For balanced networks, it is easy to show that increasing volatility always fosters cooperation. Let us define $\theta = \lambda = \bar{\eta}$. Plugging into the right hand side of equation 1, it reduces to the following function:

$$\Theta(\theta, \delta) \quad = \quad \frac{[1-\delta(1-\theta)]^2}{[1-\delta(1-\theta)^2][1-\delta]} \text{ with } \theta, \delta \in (0,1)$$

Check that similarly to above $\lim_{\delta \to 0} \Theta = 1$ and $\lim_{\delta \to \infty} \Theta = \infty$, hence cooperation is always stable for δ high enough. Since $\frac{T}{R} > 1$, we conclude

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[8] "Facilitation" refers to the fact that cooperation is already stable for lower discount factors. In particular, one could assume that there is a distribution of discount factors among individuals in the society—which we do not explicit model. Then cooperation is stable in our sense, if $\min\{\delta_i\} > \delta^*$. If agents' types are determined randomly, e.g. from a uniform distribution on $(0, 1)$, then stability of cooperation is more likely for lower δ^*.

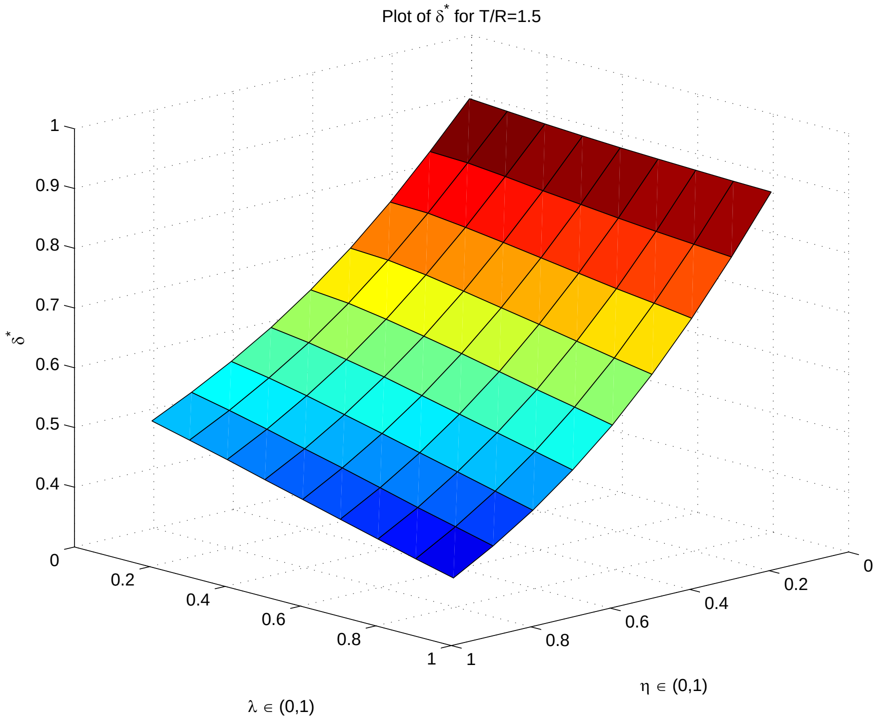

Figure 4: Threshold values δ^* for stability of cooperation for $\frac{T}{R} = 1.5$

$\delta^*(\theta, \frac{T}{R}) > 0$. Further, Θ strictly increases in both its arguments:

$$\begin{aligned}\frac{\partial\Theta}{\partial\delta} &= \theta^2\frac{\theta\delta^2(2-\theta)+(1-\delta)(1+\delta)}{[1-\delta(1-\theta)^2]^2[1-\delta]^2} > 0\\ \frac{\partial\Theta}{\partial\theta} &= \frac{2\theta\delta[1-\delta(1-\theta)]}{[1-\delta(1-\theta)^2]^2[1-\delta]} > 0\end{aligned}$$

Since Θ strictly increases in δ, there is a *unique* $\delta \equiv \delta^*(\theta, \frac{T}{R})$ such that $\Theta(\theta, \delta) = \frac{T}{R}$ for any given θ and $\frac{T}{R}$. If the temptation reward ratio is nearly one, cooperation is stable for almost any distribution of discount factors as $\lim_{T\to R}\delta^* = 0$. Additionally, since Θ also strictly increases in θ, the critical discount factor must be strictly decreasing in θ, i.e. $\frac{\partial\delta^*(\theta,\frac{T}{R})}{\partial\theta} < 0$. Increasing volatility in a balanced network therefore always lowers the threshold δ^* for stability of cooperation.[9] This gives us some insights for general networks: Graphically speaking, the threshold δ^* increases along the bisecting line of the λ-η-plane for any network and any PD parameters in a graphical representation such as figure 3. There is a neat interpretation for these findings: For higher λ, existing links decay faster so that the relative advantage of exploiting existing partners becomes smaller. Even more decisively, faster linking by high $\bar{\eta}$ increases the utility from the *attraction effect* and thus strengthens the implicit punishment of defectors. We conclude that our model potentially sheds light on the enforcement of cooperation in highly volatile environments in which explicit punishment cannot be carried out. For instance, think of the economy in a "failed state": Due to lack of institution, only short term contracts might be possible. Cooperation then can be enforced through the linking decisions and its implicit punishment of defection.

Let us turn to the dynamics of the unperturbed process. We can show that in any network, if the number of defectors in the network is not too high and the discount factor δ is high enough, stable full cooperation is reached with positive probability, i.e. the process converges to the recurrent set Ω^c

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[9] In example 4, we have already analysed the limit case for $\theta \to 1$.

with some probability. Furthermore, the set of fully cooperative states Ω^c and the set of fully defective states Ω^d are the only possible recurrent sets of the unperturbed process.

Theorem 6. *(i) The set of fully defective states* Ω^d *is always a basin of attraction (recurrent set) of the process and the set of fully cooperative states* Ω^c *for* $\delta > \delta^*$. *These are the only basins of attraction (recurrent sets) of the process. (ii) For* $n_t^d > \frac{(n-1)R}{R+P} + 1$ *in some round t, the process reaches full stable defection with probability one. If instead* $n_t^d \leq \frac{(n-1)R}{R+P} + 1$, *there is a positive probability to reach full cooperation.*

Proof. We will start by proofing (ii) and, thereafter, turn to (i). For the following arguments, we first consider a preliminary claim: Fix any number of defectors apart from player i ($n_{-i,t}^d$) in the network in round t (and, therefore, fix the number of cooperators apart from i as well). Then i's incentive to defect strictly increases in the number of defecting partners $n_{i,t}^d$ and in the number of cooperating partners $n_{i,t}^c$. The opposite holds true for the incentive to cooperate. The utility gain from defection $u^d - u^c$ can be written as:

$$\begin{aligned} u^d - u^c \;\; = \;\; & \frac{n_{i,t}^d}{1-\delta(1-\bar{\eta})(1-\lambda)} P + \frac{n_{i,t}^c}{1-\delta(1-\lambda)} T \\ & - \frac{n_{i,t}^c}{1-\delta(1-\bar{\eta})(1-\lambda)} R - \frac{n_{i,t}^d}{1-\delta(1-\lambda)} S \\ & + f(n_{-i,t}^d, n_{-i,t}^c) \end{aligned}$$

The claim follows similarly to above since $T > R$ and $\frac{1}{1-\delta(1-\lambda)} > \frac{1}{1-\delta(1-\bar{\eta})(1-\lambda)}$ so that the relative incentive to deviate increases in $n_{i,t}^c$. Furthermore, since $S < 0$ it also increases in $n_{i,t}^d$. Hence any player i has the strongest incentive to cooperate if he has no links at all and the strongest incentive for deviation if he is fully linked to all other players. For the first part of claim (ii), we have to find a sufficient condition so that no defector in the network will ever switch to cooperation. Hence we have to explore the incentive to defect for an unlinked defector with $n^d_{i,t} = n^c_{i,t} = 0$:

$$\begin{aligned} u^d - u^c &= \frac{\bar{\eta}\delta n^d_{-i,t}}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]}P \\ &\quad -\frac{\bar{\eta}\delta n^c_{-i,t}}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]}R \end{aligned} \tag{2}$$

We want to establish that $u^d - u^c > 0$ which is similar to $n^d_{-i,t}P - n^c_{-i,t}R > 0$. Since we explore the decision of a current defector, $n^d_t = n^d_{-i,t} + 1$ and $n^c_{-i,t} = n - n^d_t$. Inserting gives the desired condition $n^d_t > \frac{(n-1)R}{R+P} + 1$. Assume that this condition holds so that no defector switches to cooperation in round t. Since equation (2) increases in $n^d_{-i,t}$ and decreases in $n^c_{-i,t}$ no defector will ever switch back to cooperation in any round $\tau > t$ with $n^d_\tau \geq n^d_t$. Now turn to a current cooperator j who is unlinked as well. Then we get $n^d_{-j,t} = n^d_{-i,t}+1$ and $n^c_{-j,t} = n^c_{-i,t}-1$. Since equation (2) increases in $n^d_{-i,t}$ and decreases in $n^c_{-i,t}$, an unlinked cooperator j would switch to defection when $n^d_t > \frac{(n-1)R}{R+P} + 1$. In fact, since the incentive to defect increases in current links, all cooperators will switch to defection. Thus the process reaches full defection with probability 1 and this is stable. In other words, a state $\omega_t \in \Omega^d$ is reached with probability one and the process thereafter never leaves the set Ω^d. Since further any state $\omega' \in \Omega^d$ is accessible from any other state $\omega \in \Omega^d$, the set of defective states Ω^d is a recurrent set of the process. Assume instead $n^d_t \leq \frac{(n-1)R}{R+P} + 1$. Then unlinked defectors might switch to cooperation. Hence there is a positive probability that all defectors become unlinked and switch to cooperation, whereas no cooperator can revise his action in the meantime. So full cooperation is reached with positive probability. However, it might not be stable.

Finally, we turn to claim (i). From claim (ii) we know that the set of fully defective states Ω^d is recurrent and from theorem 5 we know that the set of fully cooperative states Ω^c is recurrent for $\delta > \delta^*$. It remains to show that these are the only two possible recurrent sets of the process. First assume

$\delta > \delta^*$ indeed. Then we have to show that from any state $\omega_t \in \Omega \setminus \{\Omega^d \cup \Omega^c\}$ a state in $\omega_t \in \Omega^d \cup \Omega^c$ can be reached with positive probability, i.e. all states $\omega_t \in \Omega \setminus \{\Omega^d \cup \Omega^c\}$ are transient. This follows directly from claim (ii): From any state $\omega_t \in \Omega \setminus \{\Omega^d \cup \Omega^c\}$, either full defection is reached with probability one or full cooperation is reached with positive probability depending on $n_t^d \gtreqqless \frac{(n-1)R}{R+P} + 1$. Hence assume $\delta \leq \delta^*$ so that S^c is not a recurrent set. Assume that the process is in some state $\omega_t \in \Omega \setminus \Omega^d$. We show that there is a positive probability to reach full defection even if $n_t^d \leq \frac{(n-1)R}{R+P} + 1$: Similar to above, full cooperation is reached with positive probability in this case. Given full cooperation is reached, there is a strictly positive probability that in the next round all possible link establishing opportunities arise. Since all players played cooperation in the last period, all links are established—i.e. a complete network of cooperators is formed. Then there is a positive probability that all players can revise their action at once; since $\delta \leq \delta^*$, all players can change to defection with positive probability (and with probability 1 for $\delta < \delta^*$). Therefore, full defection is reached with positive probability from any state $\omega \in \Omega \setminus \Omega^d$.

□

6 Stochastic Stability

Henceforth, whenever some player i has the chance to revise his action, he chooses his optimal action with probability $1 - \epsilon$ and with probability ϵ he chooses the opposite action. These so-called ϵ-errors mirror the possibility of mistakes or experimentation in the action revision process. Thus we define a Markov process with a finite number of states. We will show that the process has a unique recurrent class (and hence is irreducible): Every state can be reached from every other state with positive probability in finitely many periods. Additionally, the process is aperiodic. If these two properties are fulfilled, a process is called ergodic. The long run distribution over states does not depend on the initial state. We call the unique invariant long run probability distribution over states μ_ϵ. We are interested in the states in which the system is most often for low probabilities ϵ of mistakes. These states will be in recurrent sets of the unperturbed process. We assume that $\delta > \delta^*$ so that both Ω^c and Ω^d are recurrent sets of the unperturbed process. Then we can establish a selection criterion between them: Formally, we define $\lim_{\epsilon\to 0} \mu_\epsilon = \hat{\mu}$ as the limit distribution for small ϵ. A state ω is called stochastically stable if and only if $\hat{\mu}(\omega) > 0$. We will show that states in Ω^c are stochastically stable for δ high enough. As a first step, we will establish that the perturbed process is ergodic under this setting:

Proposition 7. *For $\epsilon \in (0,1)$, $\lambda \in (0,1)$, $\nu \in (0,1)$, and $\bar{\eta} \in (0,1)$ the process described above is ergodic.*

Proof. First, we show that the perturbed process has a unique recurrent class Ω. Hence it must be possible to access any state ω' from any state ω in countably many steps with positive probability. Assume the system is currently in state $\omega_t = \omega$. Then there is a positive probability that all players which currently play a particular action (either cooperation or defection) receive their action revision opportunity in round $t+1$ and—by mistake or optimisation—choose the opposite action. Hence all players play the same action in period $t+1$. In period $t+2$, there is a strictly positive probability that all possible link formation opportunities arise. Since all agents played the same action in the last period, all links are established. Hence the players form a complete network. Then there is a positive probability that in the beginning of round $t+3$ exactly those links decay that are not supported in state ω'. Further, nobody might get any link formation opportunity. Finally, all players whose current action differs from that prescribed in ω' get an action revision opportunity and change their action to the prescribed one in ω' by optimising or mistake. Hence with positive probability $\omega_{t+3} = \omega'$. Since ω and ω' were chosen arbitrarily, the process has a single recurrent class. Second, we have to show aperiodicity of the process. A process is aperiodic if all states are aperiodic. In fact, the process can return to any state ω with positive probability within any $z \in \mathbb{N}^+$ periods. This is a sufficient condition for each state to be aperiodic. Finally, the process is ergodic since it has a unique recurrent class and is aperiodic. $\square$

Given that the perturbed process is ergodic, we can examine the stochastic stability of cooperation. In other words, we will characterise conditions on the discount factor so that cooperation is the prevailing action in the network when there is a vanishingly small chance of mistakes or trembles:

Theorem 8. *For the process described above with $\delta > \delta^*$, fully cooperative states in Ω^c are stochastically stable for discount factors δ high enough with $\delta > \delta^{**} > \delta^*$ in societies not too small.*

Proof. For this proof, we are using methods developed by Kandori/Mailath/ Rob (1993) and Young (1993). Since there are only two recurrent classes in the unperturbed process, we do not have to construct least resistance trees. It suffices to count the number of ϵ errors mandatory to get from any state in Ω^d to a state in Ω^c, $\epsilon_{d\to c}$, and the number of errors to get from any state Ω^c to a state in Ω^d, $\epsilon_{c\to d}$. If $\epsilon_{c\to d} \leq \epsilon_{d\to c}$, then cooperation is indeed stochastically stable. In order to analyse stochastic stability, full cooperation must be a recurrent class of the unperturbed process. Hence we assume $\delta > \delta^*$. From claim (ii) of theorem 6, we know that the unperturbed process returns to full defection with probability 1 for $n_t^d > \frac{(n-1)R}{R+P} + 1$. If $n_t^d \leq \frac{(n-1)R}{R+P} + 1$ instead, full cooperation can be reached with positive probability. Since $n_t^d = n - n_t^c$, the last condition is equivalent to $n_t^c \geq n - \frac{(n-1)R}{R+P} - 1 = \frac{(n-1)P}{R+P}$. Assume that we start in a state $\omega_t \in \Omega^d$. If $\frac{(n-1)P}{R+P}$ defectors change their action into cooperation by mutation, then there is a positive probability to end up in full cooperation through the unperturbed dynamics. Since the number of errors has to be an integer value, we get the minimum number of necessary errors as $\epsilon_{d\to c} = \lceil \frac{(n-1)P}{R+P} \rceil$.[10] In fact, the minimal number of errors to move from

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[10] $\lceil z \rceil$ denotes the smallest integer greater than z.

full defection to full cooperation is independent of the network configuration and the discount factor. It merely depends on the relative payoffs in the PD and (obviously) the number of agents in the network. Specifically, since we assume $R > P$, $\frac{(n-1)P}{R+P} < \frac{n-1}{2}$ and for $R \gg P$ only a small fraction of the total population has to switch from defection to cooperation.

Next, we turn to the minimum number of mutations mandatory to move from a state in Ω^c to a state in Ω^d. Since the incentive to defect increases for any player i in the number of current links to defectors $n^d_{i,t}$ and cooperators $n^c_{i,t}$ and since we want to find the minimally necessary amount of errors $\epsilon_{c\to d}$, we have to consider the case when players are maximally linked. Hence we are starting from the complete network of cooperators and assume no links decay. Thus we get $n^c_{i,t} = n^c_{-i,t}$ and $n^d_{i,t} = n^d_{-i,t}$ for all t considered. $\epsilon_{c\to d}$ is then the number of defectors $n^d_{-i,t}$ minimally necessary so that a currently cooperating player would voluntarily choose defection. In other words, after $\epsilon_{c\to d}$ errors, the unperturbed dynamics can drive the system into a state $\omega \in \Omega^d$. For choosing defection voluntarily $u^d - u^c \geq 0$ so that we get the condition:

$$\begin{aligned} 0 \leq\ & \frac{n^d_{-i,t}}{1-\delta(1-\bar{\eta})(1-\lambda)}P + \frac{n-1-n^d_{-i,t}}{1-\delta(1-\lambda)}T \\ & -\frac{n-1-n^d_{-i,t}}{1-\delta(1-\bar{\eta})(1-\lambda)}R - \frac{n^d_{-i,t}}{1-\delta(1-\lambda)}S \\ & +\frac{\bar{\eta}\delta n^d_{-i,t}}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]}P \\ & -\frac{\bar{\eta}\delta(n-1-n^d_{-i,t})}{[1-\delta(1-\lambda)+(1-\bar{\eta})\lambda\delta][1-\delta]}R \end{aligned} \tag{3}$$

In particular, the right side increases in $n^d_{-i,t}$ since we assume $\delta > \delta^*$ which is similar to the following equation (cf. theorem 5):

$$\begin{aligned} 0 &> \frac{1}{1-\delta(1-\lambda)}T - \{\frac{1}{1-\delta(1-\overline{\eta})(1-\lambda)} \\ &+\frac{\overline{\eta}\delta}{[1-\delta(1-\lambda)+(1-\overline{\eta})\lambda\delta][1-\delta]}\}R \end{aligned}$$

Therefore, once some fully linked cooperator chooses defection voluntarily, the unperturbed process can indeed drive the system into full defection. We get $\epsilon_{c\to d} = \min\{n^d_{-i,t} \,|\, n^d_{-i,t}$ fulfills equation (3)$\}$. Since we want to establish that $\epsilon_{c\to d} > \epsilon_{d\to c}$, it suffices to establish that the above condition is not fulfilled for $\epsilon_{d\to c}$. Rewriting equation 3, we get the following condition for stoachstic stability of cooperation:

$$\begin{aligned} \frac{(n-1-\epsilon_{d\to c})T-\epsilon_{d\to c}S}{(n-1-\epsilon_{d\to c})R-\epsilon_{d\to c}P} &< \frac{1-\delta(1-\lambda)}{1-\delta(1-\overline{\eta})(1-\lambda)} \\ &+\frac{\overline{\eta}\delta(1-\delta(1-\lambda))}{[1-\delta(1-\lambda)+(1-\overline{\eta})\lambda\delta][1-\delta]} \end{aligned} \tag{4}$$

We have already found that $\epsilon_{d\to c} = \lceil\frac{(n-1)P}{R+P}\rceil$. By definition $\frac{P}{P+R} < \frac{1}{2}$ since $R > P$. For ease of exposition, we assume that n is not too small so that we get $\lceil\frac{(n-1)P}{R+P}\rceil < \frac{n-1}{2}$ and n is odd. Hence for proving the claim it suffices to show that equation 3 is not fulfilled for $\epsilon'_{d\to c} = \frac{n-1}{2}$ and some δ high enough. In other words, we show that more than $\frac{n-1}{2}$ errors would be necessary to get from a state in Ω^c to some state in Ω^d. Using the condition for $u^d - u^c < 0$ and plugging in we get:

$$\begin{aligned} \frac{T-S}{R-P} &< \frac{1-\delta(1-\lambda)}{1-\delta(1-\overline{\eta})(1-\lambda)} \\ &+\frac{\overline{\eta}\delta(1-\delta(1-\lambda))}{[1-\delta(1-\lambda)+(1-\overline{\eta})\lambda\delta][1-\delta]} \end{aligned}$$

From the definition of the PD payoffs, $\frac{T-S}{R-P} > 1$. By the same argument used several times above, the right hand side of the equation equals 1 for

		δ^*	δ^{**}
$\lambda = 0.1$	$\bar{\eta} = 0.1$	0.93	0.95
$\lambda = 0.1$	$\bar{\eta} = 0.9$	0.51	0.60
$\lambda = 0.5$	$\bar{\eta} = 0.5$	0.63	0.73
$\lambda = 0.9$	$\bar{\eta} = 0.1$	0.90	0.94
$\lambda = 0.9$	$\bar{\eta} = 0.9$	0.38	0.50

Figure 5: Threshold values of δ for stability and stochastic stability of cooperation in a typical specification

$\lim_{\delta \to 0}$, increases unboundedly for $\lim_{\delta \to 1}$. Thus there is a δ^{**} such that for any $\delta > \delta^{**}$, more than $\frac{n-1}{2}$ errors are necessary to get from Ω^c to Ω^d Therefore, cooperation is stochastically stable. Finally, we want to establish that $\delta^{**} > \delta^*$. For societies not too small, we have $\lceil \frac{(n-1)P}{R+P} \rceil < \frac{n-1}{2}$ so that $(n-1-\epsilon_{d\to c})R - \epsilon_{d \to c}P > 0$. It follows that $\frac{(n-1-\epsilon_{d\to c})T - \epsilon_{d\to c}S}{(n-1-\epsilon_{d\to c})R-\epsilon_{d\to c}P} > \frac{T}{R}$. Comparing the implicit definition of δ^{**} in equation 4 and of δ^* in equation 1, one can see that $\delta^{**} > \delta^*$. $\square$

The concept of stochastic stability is obviously much stronger than local stability. If cooperation is also stochastically stable, it is observed most of the time for a vanishing probability of mistakes in the long run. Thus it becomes interesting, whether thresholds for stochastic stability of cooperation also allow for a wide range of discount factors as we have seen for local stability. Ignoring slight rounding effects, one can see that δ^{**} is independent of the total number of people in the society n.[11] Hence we have $\delta^{**} = \delta^{**}(T, R, P, S, \lambda, \bar{\eta})$. In figure 5, we present thresholds for stability and stochastic stability for a typical setting of parameters. In particular, we assume that $T = 4.5$, $R = 3$, $P = 1$, and $S = -1$. Thus we have a temptation to reward ratio $\frac{T}{R} = 1.5$, corresponding to the second data column in figure 3. Further, we get $\frac{P}{P+R} = \frac{1}{4}$ and for any $n-1$ which is a multiple of 4

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[11] Strictly speaking, this holds only when $n-1$ is a multiple of $R+P$ but is approximately true for large n.

$\frac{(n-1-n^d_{-i,t})T-n^d_{-i,t}S}{(n-1-n^d_{-i,t})R-n^d_{-i,t}P} = 1.8125$.[12] The thresholds for stochastic stability in this particular setting do not differ much from the thresholds for local stability. They still allow for a reasonable range of discount factors for some specification. Since the implicit condition for stochastic stability in equation 4 is similar to the condition for local stability in equation 1, the same network characteristics foster stochastic stability: high volatility and high average degree.

7 Variations of the Model

In this section, we will consider two variations of the basic model. First, we will address the effect of changing the reference point of no connection in the PD-payoffs. In our discussion so far, we have assumed that being exploited in the PD by another agent j is worse than having no link to him at all, i.e. $P > 0 > S$. We discuss different variations of this assumption: If $S > 0$, it would be weakly dominant for players to consent to any link formation independent of their current action in the PD. Hence there is no scope for assortative linking. Without assortative linking, defection will not be punished and—as in the standard PD case—any agent who can revise his action will choose defection. In other words, given any initial setup, the system converges to the recurrent set Ω^d. Next assume $0 > T$: Then obviously it becomes weakly dominant to veto any link formation regardless of the current action a player performs in PDs. We are left with the more interesting setting $R > 0 > P$: It is still weakly dominant for a cooperator to agree to a link formation iff the opponent is a cooperator as before. However, for a current defector, it becomes weakly dominant to veto link establishment with other defectors. Both facts holds without any condition on ν. New links are therefore only established between cooperators. Since defectors

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[12] The numbers presented for δ^{**} actually assume that $n-1$ is a multiple of four. Otherwise there might be slight rounding effects.

cannot establish any new links, the utility from defection consists only of the defection effect, i.e. $u^d = \frac{n^c_{i,t}T+n^d_{i,t}P}{1-\delta(1-\lambda)}$. In contrast, the utility from cooperation is unchanged. The set of fully defective states Ω^d is no longer a recurrent set of the unperturbed process: Assume the system is in state $\omega_t \in \Omega^d$. Then there is some probability that all existing links between players decay. We have $u^d = u^c = 0$ in such a situation and any number of players who simultaneously get the chance to revise their action might choose cooperation. If $\delta > \delta^*$, Ω^c is a recurrent set of the process and the unperturbed system will converge to Ω^c with probability 1. In contrast, if $\delta \leq \delta^*$, Ω is a recurrent set and the process never settles down.

Next, we consider a different assumption about the bounded rationality of agents: Assume that agents look r rounds in the future and evaluate their total sum of utility during that time. Hence agents myopically maximise their undiscounted sum of payoffs for the coming r rounds assuming everyone sticks to his last period action. Players do no longer believe that others (or themselves) stick to their action *forever*, but ignore all effects which lie further than r rounds in the future. Similarly to above, we can find the utility from defection and cooperation as follows:

$$\begin{aligned} u^d &= \sum_{\tau=1}^{r} \bar{\eta}(n^d_{-i,t} - (1-\lambda)E[n^d_{i,t+\tau-1}|n^d_{i,t}])(\sum_{s=\tau}^{r}(1-\lambda)^{s-\tau}P) \\ &\quad + \sum_{\tau=0}^{r}(1-\lambda)^{\tau}(n^c_{i,t}T + n^d_{i,t}P) \\ u^c &= \sum_{\tau=1}^{r} \bar{\eta}(n^c_{-i,t} - (1-\lambda)E[n^c_{i,t+\tau-1}|n^c_{i,t}])(\sum_{s=\tau}^{r}(1-\lambda)^{s-\tau}R) \\ &\quad + \sum_{\tau=0}^{r}(1-\lambda)^{\tau}(n^c_{i,t}R + n^d_{i,t}S) \end{aligned}$$

For the stability of cooperation, one must again establish $u^d - u^c < 0$ for the case $n^d_{i,t} = n^d_{-i,t} = 0$. Since the incentive to defect still increases in $n^c_{i,t}$, we have to show that even a fully linked cooperator sticks to cooperation. Hence the condition for stability of cooperation $0 > u^d - u^c$ can be written as:

$$\begin{aligned} 0 &> \sum_{\tau=0}^{r}(1-\lambda)^{\tau} n_{-i,t}^{c}(T-R) \\ &\quad - \sum_{\tau=1}^{r} \bar{\eta}(n_{-i,t}^{c} - (1-\lambda)E[n_{i,t+\tau-1}^{c} | n_{i,t}^{c} = n_{-i,t}^{c}])(\sum_{s=\tau}^{r}(1-\lambda)^{s-\tau}R) \end{aligned}$$

From theorem 5 and the implicit definition of δ^* in equation 1, we know that $u^d - u^c \to -\infty$ for $r \to \infty$ and $u^d - u^c > 0$ for $r = 0$ by inspection. Thus cooperation is stable if r is high enough, i.e. the time period considered by the agents must be long enough. The condition on δ translates into a condition on r and the model can be re-interpreted as a model of myopic agents. Specifically, the assumption about agents optimisation are then similar to the model presented in Ule (2008), chapter six.

8 Conclusion

In the model presented in this paper, we inquire the possibility of cooperation in a randomly evolving network. There are stochastic link formation opportunities, action revision opportunities, and link decay. Every agents plays a PD against all his current neighbours in every round. Agents are boundedly rational: they believe everyone sticks to the same action as performed in the last period. When they decide about action revision, they either maximise their discounted future payoff assuming that they will stick to their action or maximise the total sum of payoffs for the next r rounds assuming that everyone sticks to their action for these r rounds. Assortative linking results when agents follow their weakly dominant linking strategies. Defectors hence incur additional costs since they forgo future links to cooperators. Following this line, we split the expected utility from different action choices in two parts: The utility form the *defection effect* describes the gain from current expiring partnerships. If it was only for this part of the total expected utility, defection would clearly be the dominant action. On the other hand, the utility from the *attraction effect* describes the gain from establishing new links in the future. We discussed the relation of this model to one with totally rational agents using trigger strategies. For our analysis of the dynamics, we use the fact that the incentive to defect increases in the number of current neighbours at several occasions. We proof that there is always a critical discount factor δ^* such that cooperation is locally stable for δ higher than that threshold. A similar (but higher) threshold exists for stochastic stability. Numerical analysis for reasonable ranges of parameters suggest that cooperation is stable or even stochastically stable for commonly used discount factors. Volatility fosters cooperation in our model: If links decay faster, the incentive to exploit current neighbours shrinks. Similarly, if there is a higher probability of establishing links in the future, the relative importance of attracting cooperators in the future increases. Thus our model shows how cooperation can sustain in highly volatile environments where there is no possibility of explicit punishment.

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