

Theory of the firm: Bargaining and competitive equilibrium

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Abstract

Suppose that a firm has several owners and that the future is uncertain in the sense that one out of many different states of nature will realize tomorrow. An owner's time preference and risk attitude will determine the importance she places on payoffs in the different states. It is a well-known problem in the literature that under incomplete asset markets, a conflict about the firm's objective function tends to arise among its owners. In this paper, we take a new approach to this problem, which is based on non-cooperative bargaining. The owners of the firm play a bargaining game in order to choose the firm's production plan and a scheme of transfers which are payable before the uncertainty about the future state of nature is resolved. We analyze the resulting firm decision in the limit of subgame-perfect equilibria in stationary strategies. Indeed, given the distribution of bargaining power, we obtain a unique prediction for a production plan and a transfer scheme. We interpret this outcome in light of the existing literature and show that it is almost always different from the outcome implied by the well-known Drèze criterion.

1 Introduction

Suppose that a firm has several owners and that the future is uncertain in the sense that one out of many different states of nature will realize tomorrow. An owner's time preference and risk attitude will determine the importance she places on payoffs in the different states. It is a well-known problem in the literature that under incomplete asset markets, a conflict about the firm's objective function tends to arise among its owners.

We take a non-cooperative bargaining approach to this conflict, which is new to the literature. We present a model in which the internal decision-making of the firm is formalized explicitly as a strategic bargaining game. In contrast to standard theory, we thus consider the firm as a coalition of owners who use strategic power in order to influence the firm's production decision and thereby maximize their own payoffs.

The Drèze criterion is a standard approach in the existing literature on incomplete markets with production. It says that one should choose a production plan which is optimal when evaluated against a weighted average of the shareholders' utility gradients, where the weights are given by the shares of ownership. Such a production plan will also have the following feature: It is not possible to achieve a Pareto-improvement by adopting any other production plan together with a scheme of transfers payable before the uncertainty is resolved. Drèze (1974) introduces a notion of general equilibrium in which each firm chooses its production plan in accordance with the shareholder-weighted criterion and the shares of each firm are traded in a stock market where prices are formed through market-clearing. It is shown that such an equilibrium exists. However, Dierker, Dierker and Grodal (2002) provide an example of an economy where all such market equilibria are constrained inefficient. Kelsey and Milne (1996) give a proof for equilibrium existence in a more general model which emphasizes externalities between firms and shareholders. Bisin, Gottardi and Ruta (2009) present a general equilibrium model with an emphasis on the choice of a firm's optimal corporate finance policy when markets are incomplete.

Recognizing that the Drèze criterion is purely normative in nature, some authors have tried to link it to outcomes of majority voting. A typical approach in this stream of literature is to ask: If a certain production plan was given as a default option, would there exist an alternative plan preferred by more than at least a certain (super-)majority of the shareholders? If no winning alternative can be found, the default plan is considered "stable". Tvede and Crès (2005) discuss the relationship between the Drèze criterion and such a voting approach and find conditions under which both lead to compatible predictions. Another voting analysis is given by DeMarzo (1993) who emphasizes the importance of the largest shareholder. However, both the pure application of the Drèze criterion and the majority voting seem to suffer from a common problem. Both approaches ask only which production plans are stable to being replaced by other plans through a certain mechanism. However, there is no explanation why a particular plan would serve as a default setting or how any one particular plan is to be chosen in case there are several plans satisfying the criterion used.

Few papers have taken a truly positive approach to decision-making within the firm. DeMichelis and Ritzberger (2008) is an example with an explicit game of voting in order

to define the firm's objective. However, the problem addressed in this paper is substantially different from the issue under consideration here. More specifically, they discuss a situation where a conflict arises from imperfect competition rather than from missing assets. A monopolistic firm produces a good which is also consumed by the firm's owners. Consequently, the agents benefit from competitive pricing in their capacity as consumers, but from monopolistic pricing in their capacity as shareholders.

In our setup, there is a single firm which exists in an environment of competitive but potentially incomplete markets. We take the ownership structure of the firm as exogenously fixed. The firm will be active in two time periods. A production plan has to be chosen knowing the state of the world in the first period, but under uncertainty about the state of the world in the second period. There are assets which the agents can use to shift consumption across time periods and states. Each agent is a price-taker in each asset market.

The owners of the firm use a natural bargaining procedure in order to determine a production plan for the firm. In each round of bargaining, an agent is chosen from a fixed probability distribution to be the proposer. This agent can then offer a production plan as well as a scheme of side-payments in terms of first-period consumption. If agents unanimously agree, the proposal is implemented. Otherwise, the negotiation breaks down with some probability and the firm does not become active. An interpretation is that when the owners do not readily exploit the investment opportunity, it will be lost with some probability. With the complementary continuation probability, the bargaining continues to another round. In the case of perpetual disagreement, the firm will not produce.

Our main results are the following: We show that the outcome of the bargaining process corresponds to a weighted Nash Bargaining Solution, where the weights are given by the recognition probabilities; a unique prediction for a production plan as well as a system of transfer payments in terms of good 0 is derived.

In the special case where markets are complete, the bargaining approach selects the profit-maximizing production plan and is therefore in line with the predictions of the standard Arrow-Debreu model as far as the firm's production decision is concerned. However, contrary to the Arrow-Debreu model, owners make use of their bargaining power to redistribute the profits among themselves. Hence, owners obtain payoffs which are generically different from those in standard economic theory, even if markets are complete.

In the case of incomplete markets, we find that the production plan which the firm adopts under the bargaining mechanism is almost always different from the production plan which would be predicted by the well-known Drèze criterion. Moreover, non-zero transfers are made in general.

In section 2, we present the model and the most important assumptions formally. In section 3, we establish the relationship between the explicit bargaining about a production plan and a transfer scheme on the one hand and the implicit bargaining problem in the payoff space on the other hand. In section 4, we characterize the relevant bargaining outcome. In section 5, this bargaining outcome is interpreted in light of the existing literature and, in particular, compared to the Drèze criterion. Section 6 concludes.

2 Model Description

The firm is owned by a finite number of owners. Each owner $i = 1, \dots, I$ holds a share θ^i of the firm so that $\sum_{i=1}^I \theta^i = 1$. The set of agents $\{1, \dots, I\}$ will be denoted by \mathcal{I} .

The firm can carry out some productive activity which extends over two periods. In the first period, the state of the world is known to be $s = 0$. In the second period, any one of the states of the world $s = 1, \dots, S$ may be realized; we will write $\mathcal{S} = \{0, 1, \dots, S\}$. A particular productive activity of the firm is described by a production plan $y \in \mathbb{R}^{S+1}$. If $y_s < 0$ for some $s = 0, 1, \dots, S$, then each agent i has to provide the firm with an input of $|\theta^i y_s|$ in state s . Similarly, if $y_s > 0$, then this output will be distributed to the agents in the proportion in which they own the firm. The set of all production plans which are possible for the firm is called its production set and is denoted by Y .

Each agent i has initial endowments $\omega^i \in \mathbb{R}_{++}^{S+1}$, which can be used to finance the provision of inputs. Also, consumption in state $s = 0$ can be transferred across agents and assets can be used to shift consumption across states. Transfers and assets will be introduced in detail in the sequel.

A production plan can be chosen only by unanimous consent of the owners. In order to reach agreement, the following bargaining procedure is used. Bargaining takes place in a potentially infinite number of rounds $r = 0, 1, \dots$. In the beginning of any round r , a draw from a given probability distribution μ on the agents determines the proposer. The proposer then makes an offer $(y, \tau) \in Y \times T$, where $T = \{\tau \in \mathbb{R}^I \mid \sum_{i \in \mathcal{I}} \tau^i \leq 0\}$. We interpret τ^i as a (net) transfer payment which agent i receives, and which is made in terms of consumption in state $s = 0$. The agents then either accept or reject the offer in some given order. It is assumed that an agent cannot accept a proposal which leads to his insolvency irrespective of his choice of an asset portfolio. This condition will be stated more formally later.

If unanimous agreement is reached, the bargaining stage ends and all owners individually choose their asset portfolios. Then, the firm becomes active in the aforementioned states $s = 0, 1, \dots, S$ according to the agreed-upon production plan and transfer scheme. If, however, an agent rejects a proposal, the bargaining moves to round $r + 1$ with probability δ . With probability $1 - \delta$, an exogenous breakdown of the bargaining occurs. In that case, no production will take place (recall that $0 \in Y$) and no transfers will be made. Each individual owner is merely left to choose his asset portfolio. Likewise, perpetual disagreement means that no production takes place. One interpretation of the breakdown probability is that the investment opportunities implicit in the production set tend to slip away if one waits too long to exploit them.

Once the bargaining has ended, each agent individually decides on an asset portfolio. The agent can purchase assets $j = 1, \dots, J$ at prices q_1, \dots, q_J in state $s = 0$. These prices are taken as given. In state $s = 1, \dots, S$, each unit of asset j will give a payoff of a_s^j . We summarize the asset structure in the $(S \times J)$ -matrix A , which we assume to be of full column rank. Duplicate assets are ignored without loss of generality. It will sometimes be convenient to use the notation $W = \begin{pmatrix} -q \\ A \end{pmatrix}$. The assets are perfectly divisible and may

be sold short. We write z^{ij} for agent i 's holdings of asset j .

Finally, the production plan and transfers are implemented as agreed and the assets pay off according to A . Once the bargaining has led to an agreement (or broken down), the sequel of the model does not incorporate any strategic interaction anymore. Production and consumption takes place and payoffs realize. Agent i has preferences over consumption plans $x^i \in \mathbb{R}_+^{S+1}$, which are represented by a utility function $u^i : \mathbb{R}_+^{S+1} \rightarrow \mathbb{R}_+$. Given the bargaining outcome $(y, \tau) \in Y \times T$, agent i solves

$$\begin{aligned} \xi^i(y, \tau) &= \arg \max_{x^i} u^i(x^i) \\ \text{subject to } x^i &= \begin{pmatrix} \omega_0^i + \theta^i y_0 + \tau^i - qz^i \\ \omega_1^i + \theta^i y_1 + Az^i \end{pmatrix}, z^i \in \mathbb{R}^J \end{aligned}$$

to determine his optimal consumption plan $\xi^i(y, \tau)$, and the unique asset portfolio needed to achieve it.

Let $e(0)$ denote the $(S+1)$ -vector $(1, 0, \dots, 0)^\top$. Define $B^i = \{(y, \tau) | \exists z \in \mathbb{R}_+^{S+1} : \omega^i + \theta^i y + e(0)\tau^i + Wz^i \geq 0\}$ and $B = \cap_{i \in \mathcal{I}} B^i$. The set B now contains all bargaining outcomes which allow each agent to remain solvent by an appropriate portfolio choice. We assume that agent i can only accept a proposal if it lies in B^i . Since u^i has been defined on \mathbb{R}_+^{S+1} , this technical assumption is needed to ensure that the optimal consumption plan $\xi(y, \tau)$ is well-defined. We note that B is guaranteed to be non-empty since $\omega^i \gg 0$ for all i .

For any $(y, \tau) \in B$, we can now define i 's indirect utility function $\varphi^i(y, \tau)$ as

$$\varphi^i(y, \tau) = u^i[\xi^i(y, \tau)]$$

We write $\varphi(y, \tau)$ for $(\varphi^1(y, \tau), \dots, \varphi^I(y, \tau))$. We can then derive the following set of feasible payoffs:

$$V = \{v \in \mathbb{R}_+^I | v = \varphi(y, \tau) \text{ for some } (y, \tau) \in B\}$$

The results in this paper are derived under a number of assumptions on the utility functions, the production plan, and the asset structure. These assumptions are now introduced.

Assumption 2.1 (*Production Set*)

1. Y is closed and strictly convex.
2. $Y \supset \mathbb{R}_-^{S+1}$: Output can be freely disposed of and inaction is possible.
3. $Y \cap \mathbb{R}_+^{S+1} \subset \{0\}$: One cannot produce a positive output without inputs.
4. The interior of Y is non-empty.

5. The boundary ∂Y is differentiable: There is a unique outward normal vector at any point of ∂Y .

Assumption 2.2 (*Utility functions*)

For all $i \in \mathcal{I}$:

1. u^i is continuous on \mathbb{R}_+^{S+1} and twice continuously differentiable on \mathbb{R}_{++}^{S+1} .
2. For any $x^i \in \mathbb{R}_{++}^{S+1}$, it holds that $\nabla u_i(x^i) \gg 0$: Utility is strictly increasing in the consumption of each state.
3. u^i is strictly concave on \mathbb{R}_{++}^{S+1} ; i.e., the Hessian matrix of u^i is negative-definite.
4. If $x_s^i = 0$ for some $s = 0, 1, \dots, S$, then $u^i(x^i) = 0$.

We define a set of (normalized) state prices

$$\Pi = \{\pi \in \mathbb{R}_{++}^S \mid (1, \pi)W = (0, \dots, 0)^\top\}$$

In order to rule out arbitrage opportunities, we assume that Π is non-empty. Moreover, it is assumed that there exists $\pi \in \Pi$ and $b \in \mathbb{R}$ such that for all $y \in Y$, it holds that $(1, \pi)y \leq b$. Hence, profits are bounded. This assumption can be seen as a restriction on the asset prices q . It would be satisfied in general equilibrium, but has to be imposed here since we conduct our analysis in a partial equilibrium context.

Asset markets are said to be complete if Π is single-valued. If Π is not single-valued, markets are incomplete. We also note that Π is a convex set.

3 Reduced Form Bargaining Game

When the owners bargain about a production plan and a transfer scheme, they implicitly bargain about the associated payoffs. In this section, we will analyze the bargaining problem in the payoff space. In order to motivate this approach, we will first show that a point on the Pareto frontier

$$\partial V = \{v \in V \mid \nexists v' \in V \setminus \{v\} : v' \geq v\}$$

of V corresponds to a unique production plan and transfer scheme. Put another way, any efficient outcome of the implicit bargaining about payoffs corresponds to a unique outcome of the explicit bargaining about a production plan and transfer scheme. This result is driven by the assumption that the convexity of the production set and the concavity of the utility functions are strict.

Theorem 3.1 *Let $\bar{v} \in \partial V$. There is a unique $(\bar{y}, \bar{\tau}, \bar{z})$ such that $u^i(\omega^i + e(0)\bar{\tau}^i + \theta^i\bar{y} + W\bar{z}^i) = \bar{v}^i$ for all $i \in \mathcal{I}$.*

Proof:

Suppose that $(\hat{x}^i, \hat{y}, \hat{\tau}^i, \hat{z}^i)$ and $(\check{x}^i, \check{y}, \check{\tau}^i, \check{z}^i)$ are such that

$$\begin{aligned}\bar{v}^i &= u^i(\hat{x}^i) = u^i(\omega^i + e(0)\hat{\tau}^i + \theta^i\hat{y} + W\hat{z}^i) \\ \bar{v}^i &= u^i(\check{x}^i) = u^i(\omega^i + e(0)\check{\tau}^i + \theta^i\check{y} + W\check{z}^i)\end{aligned}$$

Furthermore, let $\tilde{\tau} = \alpha\hat{\tau} + (1-\alpha)\check{\tau}$, $\tilde{z} = \alpha\hat{z} + (1-\alpha)\check{z}$, and $\tilde{y} = \alpha\hat{y} + (1-\alpha)\check{y}$ for some $\alpha \in (0, 1)$, and let $\tilde{x}^i = \omega^i + e(0)\tilde{\tau}^i + \theta^i\tilde{y} + W\tilde{z}^i$. We observe that $\tilde{x}^i = \alpha\hat{x}^i + (1-\alpha)\check{x}^i$.

Let us suppose first that $\hat{x}^i \neq \check{x}^i$. By strict concavity of u^i on \mathbb{R}_{++}^{S+1} , it holds that $u^i(\tilde{x}^i) > \bar{v}^i$ for all $i \in \mathcal{I}$. Hence, $\bar{v} \notin \partial V$, a contradiction. We have shown that $\hat{x}^i = \check{x}^i$.

Now suppose that $\hat{y} \neq \check{y}$. By strict convexity of Y , there is $\bar{y} \in Y$ such that $\bar{y} \gg \check{y}$. Therefore, $u^i(e(0)\check{\tau}^i + \theta^i\bar{y} + W\check{z}^i) > \bar{v}^i$ for all $i \in \mathcal{I}$. Again, we see that $\bar{v} \notin \partial V$, a contradiction.

We know that $v^i(y, \tau)$ is strictly increasing in τ^i for any given y . It follows that $\hat{\tau}^i = \check{\tau}^i$.

Now, we find that $W(\hat{z}^i - \check{z}^i) = 0$. Since the column rank of W is J by definition, it holds that $\hat{z}^i = \check{z}^i$, as desired. \square

We will say that the payoff set V is comprehensive if the following holds: If $\bar{v} \in V$, then any payoff vector \hat{v} such that $u(0) \leq \hat{v} \leq \bar{v}$ is in V as well.

Lemma 3.2 *V is comprehensive.*

Proof: Suppose that $\bar{v}^i = u^i(x^i) = u^i(\omega^i + \theta^i\bar{y} + W\bar{z}^i + e(0)\bar{\tau}^i)$. Define

$$\begin{aligned}\hat{z}^i &= \arg \max_{z^i \in \mathbb{R}^J} -qz^i \\ &\text{subject to } \omega^i + \theta^i\bar{y} + W\bar{z}^i + e(0)\bar{\tau}^i \geq 0\end{aligned}$$

and

$$t^i = \omega^i + \theta^i\bar{y}_0 - q\hat{z}^i + \bar{\tau}^i$$

Since $t \geq 0$, we have that $(\bar{y}, \bar{\tau} - t) \in Y \times T$. But $\varphi^i(\bar{y}, \bar{\tau} - t) = 0$. Since φ^i is a continuous function of agent i 's transfer receipts, it can also assume all values in the interval $[0, \bar{v}^i]$. \square

The comprehensiveness of V also implies that

$$\begin{aligned}V &= \{v \in \mathbb{R}^I \mid \exists (y, \tau, z) \in Y \times T \times \mathbb{R}^{IJ} : v^i = u^i(\omega^i + \theta^iy + e(0)\tau^i + Wz^i), \forall i \in \mathcal{I} \\ &\quad 0 \leq \omega^i + \theta^iy + e(0)\tau^i + Wz^i, \forall i \in \mathcal{I}\}\end{aligned}$$

The right hand side of the above equality is the set of all payoff vectors which would result from some choice of (y, τ, z) . The set V , however, is the set of all payoff vectors which result from a tuple (y, τ, z) such that z is the optimal set of asset portfolios given (y, τ) .

Lemma 3.3 *V is strictly convex.*

Proof: Let $\check{v}, \hat{v} \in V$ and $\tilde{v} = \alpha\check{v} + (1 - \alpha)\hat{v}$ for some $\alpha \in (0, 1)$. There exist $(\check{y}, \check{z}, \check{\tau})$ and $(\hat{y}, \hat{z}, \hat{\tau})$ in $Y \times \mathbb{R}^{IJ} \times \mathbb{R}^I$ and consumption plans $\check{x}^i, \hat{x}^i \in \mathbb{R}_+^{S+1}$ such that for all $i \in I$:

$$\begin{aligned}\omega^i + \theta^i \check{y} + W \check{z}^i + e(0) \check{\tau}^i &= \check{x}^i \\ \omega^i + \theta^i \hat{y} + W \hat{z}^i + e(0) \hat{\tau}^i &= \hat{x}^i \\ u^i(\check{x}^i) &= \check{v}^i \\ u^i(\hat{x}^i) &= \hat{v}^i\end{aligned}$$

Multiplying the first two equations by α and $1 - \alpha$ respectively and then summing them up yields

$$\omega^i + \theta^i \tilde{y} + W \tilde{z}^i + e(0) \tilde{\tau}^i = \alpha \check{x}^i + (1 - \alpha) \hat{x}^i$$

for all $i \in I$, where $(\tilde{y}, \tilde{z}, \tilde{\tau}) = \alpha(\check{y}, \check{z}, \check{\tau}) + (1 - \alpha)(\hat{y}, \hat{z}, \hat{\tau})$. By assumption, u is concave, so that

$$\alpha \check{v} + (1 - \alpha) \hat{v} < u(\tilde{x})$$

With comprehensiveness, this implies strict convexity of V . \square

Lemma 3.4 *All points in the boundary of V are strongly Pareto-efficient.*

Proof: Let $\check{v}, \hat{v} \in V$. Suppose that

$$\begin{aligned}\hat{v}^i &> \check{v}^i \\ \hat{v}_j &= \check{v}_j, \forall j \in I \setminus \{i\}\end{aligned}$$

We want to show that there exists $\tilde{v} \in V$ such that $\tilde{v} \gg \check{v}$.

Let $\hat{v}^i = u^i(\hat{x}^i)$ and $\check{v}^i = u^i(\check{x}^i)$. Since $u^i(\hat{x}^i) > u^i(\check{x}^i) \geq u^i(0)$, we know that $\hat{x}_0^i > 0$. Construct the consumption plan $\tilde{x}^i = \hat{x}^i - e(0)\varepsilon$ for agent i and consumption plans $\tilde{x}^j = \hat{x}^j + e(0)\frac{\varepsilon}{I-1}$ for agents $j \neq i$. Since u^i is continuous, $u^i(\tilde{x}^i) > u^i(\check{x}^i)$ for $\varepsilon > 0$ sufficiently small. Furthermore, since u^j is strictly increasing in all arguments, $u^j(\tilde{x}^j) > u^j(\hat{x}^j)$. Now, $u(\tilde{x}) \gg \check{v}$, as desired. \square

Lemma 3.5 *The set V is compact.*

Proof: Define the consumption set

$$X = \{x \in \mathbb{R}_+^{(S+1)I} \mid \exists (y, \tau, z) \in Y \times T \times \mathbb{R}^{IJ} : \begin{aligned} x^i &= \omega^i + \theta^i y + W z^i + e(0) \tau^i, \forall i \in I \\ \sum_{i \in I} \tau^i &= 0 \end{aligned}\}$$

Since $u : X \rightarrow V$ is a continuous function, V is compact if X is compact. X is clearly closed, we want to show that it is bounded. To this end, take any $x \in X$ and compute

$\sum_{i \in I} (x^i - \omega^i) = W \sum_{i \in I} z^i + y$. By assumption, $W \sum_{i \in I} z^i + y \subset \{x \in \mathbb{R}^{S+1} | (1, \pi)^\top x \leq b\}$. Furthermore, we have assumed that $x^i \in \mathbb{R}_+^{S+1}$, implying that $\sum_{i \in I} (x^i - \omega^i) \geq -\sum_{i \in I} \omega^i$. Defining $\sum_{i \in I} \omega^i = \omega$, we can conclude

$$\sum_{i \in I} (x^i - \omega^i) \in \{x \in \mathbb{R}^{S+1} | (1, \pi)x \leq b\} \cap \{x \in \mathbb{R}^{S+1} | x \geq -\omega\}$$

Clearly, the above intersection is bounded for any given endowment schedule. \square

Lemma 3.6 *The boundary ∂V of V is differentiable, that is, at any point $\bar{v} \in \partial V$, there is a unique normal to V .*

Proof: Take any $\bar{v} \in \partial V$.

Suppose that $\bar{v}^i = u^i(\bar{x}^i)$ for all i .

Consider the following set,

$$K(\bar{v}) = \left\{ \begin{array}{l} v \in \mathbb{R}^I | \exists t \in \mathbb{R}^I : \\ v^i = u^i(\bar{x}_0^i + t^i, \bar{x}_1^i, \dots, \bar{x}_S^i), \forall i \in I \\ \bar{x}_0^i + t^i \geq 0, \forall i \in I \\ \sum_{i=1}^I t^i \leq 0 \end{array} \right\}$$

We see that $K(\bar{v}) \subset V$. Informally, we consider the agreement that yields the payoff vector \bar{v} . Holding asset portfolios and production plan fixed, we reshuffle first-period consumption in order to obtain a new payoff vector. $K(\bar{v})$ is the set of all payoff vectors to be obtained in this way.

Trivially, it holds that $\bar{v} \in \partial K(\bar{v})$.

Since the direct utility functions are infinitely many times differentiable with respect to consumption in state $s = 0$, the boundary of $K(\bar{v})$ is differentiable at \bar{v} . In particular, there is a unique outward normal vector to $K(\bar{v})$ at \bar{v} , say n . Furthermore, $K(\bar{v})$ is a convex set because the direct utility functions are concave by assumption.

We want to show that n is also the unique normal to V at \bar{v} . Suppose to the contrary that there is a normal vector $n' \neq n$ to V at \bar{v} . Since n' cannot be normal to $K(\bar{v})$, there is $v \in K(\bar{v})$ such that $(v - \bar{v})^\top n' > 0$. But $K(\bar{v}) \subset V$, so that $v \in V$ as well, contradicting that n' is normal to V at \bar{v} . \square

It may be interesting to note that the differentiability of ∂V does *not* require differentiability of ∂Y .

In the current section, we have established a number of properties of the set V , which correspond to standard assumptions on the payoff set in the non-cooperative bargaining literature. The following theorem is therefore implied by the results of Miyakawa (2006) as well as Laruelle and Valenciano (2007). It can also be seen as a special case of Hart and MasColell (2006). Britz, Herings, and Predtetchinski (2008) have generalized the convergence result to proposer selection by a Markov process.

Definition 3.7 The μ -weighted Nash Bargaining Solution (μ -NBS) is the payoff allocation $v^* \in V$ which solves

$$\max_{v \in V} \prod_{i=1}^I [v^i - \varphi^i(0, 0)]^{\mu^i}$$

Theorem 3.8 In the limit as $\delta \rightarrow 1$, the payoffs of all subgame perfect bargaining equilibria in stationary strategies converge to the μ -NBS.

We will now state the relationship between the set of efficient and individually rational payoff allocations and the recognition probabilities. Define the *individually rational boundary* of V as

$$\partial V^+ = \{v \in \partial V \mid v \geq \varphi(0, 0)\}$$

and the set of possible vectors of bargaining weights as

$$\Delta_I = \{\mu \in \mathbb{R}_+^I \mid \sum_{i=1}^I \mu^i = 1\}$$

Lemma 3.9 The payoff vector v belongs to ∂V^+ if and only if there is exactly one $\mu \in \Delta_I$ such that v is the μ -weighted NBS.

4 Weighted Nash Bargaining Solution

In the previous section, we have obtained results on the equilibrium payoffs of the bargaining procedure. Now, we will analyze the production and transfer decisions which lead to these payoffs. We have already shown that any efficient and individually rational payoff allocation (and thus any μ -weighted NBS) is supported by a unique tuple (y, τ, z) , which we will now derive. To this end, we introduce the transformation function $f(y) = -\max\{t \mid y + \mathbf{1}^T t \in Y\}$. Clearly, $f(y) \leq 0$ if $y \in Y$ and $f(y) = 0$ if $y \in \partial Y$. If $y \in \partial Y$, the gradient of $f(y)$ corresponds to the outward normal vector to Y at y . The strict convexity of Y implies that f is strictly quasi-concave.

We will now derive the first-order conditions for maximizing the μ -weighted Nash product. Suppose indeed that (y, τ, z) lead to the μ -weighted NBS; then they must solve

$$\mathcal{L}(y, \tau, z^1, \dots, z^I, \lambda, \nu) = \prod_{i=1}^I [u^i(\omega^i + \theta^i y + e(0)\tau^i + Wz^i) - \varphi^i(0, 0)]^{\mu^i} - \lambda f(y) - \nu \sum_{i=1}^I \tau^i$$

leading to the first order conditions:

$$\begin{aligned}
\sum_{i \in I} \frac{\theta^i \mu^i}{u^i(x^i) - \varphi^i(0,0)} \partial_{x_s^i} u^i(x^i) &= \lambda \partial_{y_s} f(y), \quad s \in \mathcal{S} \\
\frac{\mu^i}{u^i(x^i) - \varphi^i(0,0)} \partial_{x_0^i} u^i(x^i) &= \nu, \quad i \in \mathcal{I} \\
(\partial_{x_0^i} u^i(x^i), \partial_{x_1^i} u^i(x^i), \dots, \partial_{x_S^i} u^i(x^i))^\top W &= 0 \\
f(y) &= 0 \\
\sum_{i \in I} \tau^i &= 0
\end{aligned}$$

We remark that these conditions are both necessary and sufficient.

We will write $\nabla u^i(x^i)$ for the S -vector with component $\nabla_s u^i(x^i) = \frac{\partial_{x_s^i} u^i(x^i)}{\partial_{x_0^i} u^i(x^i)}$ for $s = 1, \dots, S$. Similarly, we also denote by $\nabla f(y)$ the S -vector with component $\nabla_s f(y) = \frac{\partial_{y_s} f(y)}{\partial_{y_0} f(y)}$ for $s = 1, \dots, S$. Since $\partial_{x_0^i} u^i(x^i) = \nu \frac{u^i(x^i) - \varphi^i(0,0)}{\mu^i}$ for all $i \in \mathcal{I}$, the first set of first-order conditions is equivalent to

$$\begin{aligned}
\sum_{i \in \mathcal{I}} \theta^i \nabla u^i(x^i) - \nabla f(y) &= 0 \\
\nu (\partial_{y_0} f(y))^{-1} &= \lambda
\end{aligned}$$

The condition says that the (normalized) gradient of the transformation function is the θ -weighted average of the normalized gradients of all owners' utility functions.

For any $i \in \mathcal{I}$, define $\eta^i(x^i) = \frac{(\partial_{x_0^i} u^i(x^i))}{u^i(x^i) - \varphi^i(0,0)}$. Then, the second set of first-order conditions is equivalent to

$$\begin{aligned}
\mu^i \eta^i(x^i) - \mu^I \eta^I(x^I) &= 0, \quad i \in \mathcal{I} \setminus \{I\} \\
\mu^I \eta^I(x^I) &= \nu
\end{aligned}$$

In Aumann and Kurz (1977), η^i is considered a measure of the agent's *boldness*. Aumann and Kurz identify the point where boldness is equal across all agents as the Nash Bargaining Solution. The above condition says that a weighted Nash Bargaining Solution is the point where the product of boldness and proposal power is equal for all agents. We remark that only the marginal utility of consumption in state $s = 0$ enters η^i , whereas Aumann and Kurz define boldness in a context with a single good.

By definition of $\nabla u^i(x^i)$, we can write the third set of first-order conditions as

$$A^\top \nabla u^i(x^i) - q = 0, \quad i \in \mathcal{I}$$

These are simply the portfolio conditions for each agent. In view of the condition $\sum_{i \in \mathcal{I}} \theta^i \nabla u^i(x^i) - \nabla f(y) = 0$, they are equivalent to

$$\begin{aligned} A^\top \nabla u^i(x^i) - q &= 0, \quad i \in \mathcal{I} \setminus \{I\} \\ A^\top \nabla f(y) - q &= 0 \end{aligned}$$

The aforementioned first-order conditions for the μ -weighted Nash product and Lemma 3.9 imply the following result.

Theorem 4.1 *If and only if the tuple (y, τ, z) supports a payoff vector v belonging to ∂V^+ , then (y, τ, z) satisfy the following conditions,*

$$\begin{aligned} \sum_{i \in \mathcal{I}} \theta^i \nabla u^i(\omega^i + \theta^i y + e(0)\tau^i + Wz^i) - \nabla f(y) &= 0 \\ A^\top \nabla u^i(\omega^i + \theta^i y + e(0)\tau^i + Wz^i) - q &= 0 \quad i \in \mathcal{I} \setminus \{I\} \\ A^\top \nabla f(y) - q &= 0 \\ f(y) &= 0 \\ \sum_{i \in \mathcal{I}} \tau^i &= 0 \end{aligned}$$

Moreover, if (y, τ, z) support $v \in \partial V^+$, then v is the μ -weighted Nash Bargaining Solution for the vector $\mu \in \Delta_I$ given by

$$\mu^i \eta^i(y, \tau, z) - \mu^I \eta^I(y, \tau, z) = 0, \quad i \in \mathcal{I} \setminus \{I\}$$

We have shown that in the limit of subgame perfect equilibria in stationary strategies, the owners' bargaining procedure leads to payoffs corresponding to the μ -weighted Nash Bargaining Solution. Moreover, in the previous theorem, we have translated this result on the equilibrium payoffs into a result on the production plan and transfer scheme which support the μ -weighted NBS. We will now give a formal definition of the Drèze criterion and begin contrasting it with our findings.

Definition 4.2 *A production plan $y \in Y$ satisfies the Drèze criterion if there is $z \in \mathbb{R}^{IJ}$ such that*

$$\begin{aligned} \sum_{i \in \mathcal{I}} \theta^i \nabla u^i(\omega^i + \theta^i y + Wz^i) - \nabla f(y) &= 0 \\ A^\top \nabla u^i(\omega^i + \theta^i y + Wz^i) - q &= 0 \quad i \in \mathcal{I} \setminus \{I\} \\ A^\top \nabla f(y) - q &= 0 \\ f(y) &= 0 \end{aligned}$$

We will say that a payoff vector $\bar{v} \in \partial V$ is consistent with the Drèze criterion if there are a production plan $y \in Y$ and asset portfolios z^1, \dots, z^I which satisfy the above conditions and $u^i(\omega^i + \theta^i y + W z^i) = \bar{v}^i$ for all $i \in \mathcal{I}$.

We see that both the μ -NBS and the Drèze criterion require constrained Pareto-optimality, optimal choice of portfolios by all agents, and efficient production. The difference between both approaches is the selection which is made from the outcomes which satisfy these common conditions. Under the Drèze criterion, one chooses an allocation which does not require transfers. Bargaining power or the disagreement point play no role in this selection. Under the μ -NBS, one chooses the unique allocation which can be reached by non-wasteful transfers and at which the μ -weighted boldness of each agent is equal.

5 Producer Choice

We have studied the equilibrium production and transfer decision of the firm resulting from the bargaining procedure. In the last section, we have characterized the outcome of the bargaining procedure and given a first comparison to the Drèze criterion. In this section, we make use of the characterization given in Theorem 4.1 in order to study the bargaining outcome in much more detail and explore its relation to important concepts well-established in the literature, such as value-maximization and the Drèze criterion.

Any matrix of security payoffs $W = [-q, A]^\top$ implies a set Π of (normalized) state prices. A production plan is said to be *value-maximizing* if it is optimal with regard to some element of Π (De Marzo 1993). It turns out that the set of value-maximizing production plans is closely related to the boundary of V , parameterized by the weights μ of the Nash Bargaining Solution, which are in turn given by the recognition probabilities of the bargaining procedure.

Definition 5.1 *A production plan $\bar{y} \in Y$ is value-maximizing if there is a state price vector $\pi \in \Pi$ such that*

$$\pi^\top \bar{y} \geq \pi^\top y, \quad \forall y \in Y$$

We will say that a payoff vector $v \in \partial V^+$ is *supported by a production plan $y \in Y$* if there is a transfer scheme τ and asset portfolios z^1, \dots, z^I such that $u^i(\omega^i + \theta^i y + e(0)\tau^i + W z^i) = v^i$ for all $i \in \mathcal{I}$.

In what follows, we show how our previous characterization of the μ -weighted NBS relates to the value-maximization concept:

Lemma 5.2 *A production plan $\bar{y} \in Y$ is value-maximizing if and only if it satisfies*

$$\begin{aligned} A^\top \nabla f(\bar{y}) - q &= 0 \\ f(\bar{y}) &= 0 \end{aligned}$$

Proof: *If:* We have that $\nabla f(\bar{y}) \in \Pi$. We will show that \bar{y} is optimal against $\nabla f(\bar{y})$. Consider the problem $\arg \max_{y \in Y} \nabla f(\bar{y})^\top y$ s.t. $f(y) \leq 0$. The concomitant first-order conditions are $\nabla f(\bar{y}) = \kappa \nabla f(y)$ and $f(y) = 0$. Clearly, the only possible solution is $y = \bar{y}$. The necessary conditions are also sufficient because of the quasi-concavity of f .
Only If: If $\bar{y} \in Y$ is value-maximizing, then there is some $\pi \in \Pi$ against which \bar{y} is optimal. Hence, \bar{y} must satisfy the first-order conditions of the problem $\max \pi^\top y$ s.t. $f(y) \leq 0$. Hence, we have $f(\bar{y}) = 0$ and $\pi = \nabla f(\bar{y})$. Since $A^\top \pi - q = 0$, the statement holds. \square

Theorem 4.1 and Lemma 5.2 imply that ∂V^+ is supported only by production plans which are value-maximizing. But in the special case with complete markets, Π is single-valued and value-maximization reduces to the usual profit-maximization.

Corollary 5.3 *If $S = J$, then every $v \in \partial V^+$ is supported by the profit-maximizing production plan.*

If markets are complete, the bargaining outcome is thus consistent with the usual profit-maximizing predictions of general equilibrium theory. However, the distribution of bargaining power will determine how the firm's profits are to be divided among its owners.

For the remainder of the section, we need the following notation.

We will denote $\partial_{x_s^i x_{s''}^i}^2 u^i(x^i)$ by $u_{s's''}^i$ and $\partial_{x_{s'}^i} \left[\frac{\partial_{x_s^i} u^i(x^i)}{\partial_{x_0^i} u^i(x^i)} \right]$ by $\hat{u}_{s's''}^i$. We will summarize the second-order derivatives in matrices

$$U^i = \begin{pmatrix} u_{00}^i & u_{01}^i & \cdots & u_{0S}^i \\ u_{10}^i & u_{11}^i & \cdots & u_{1S}^i \\ \vdots & \vdots & \ddots & \vdots \\ u_{S0}^i & u_{S1}^i & \cdots & u_{SS}^i \end{pmatrix} \text{ and } \hat{U}^i = \begin{pmatrix} \hat{u}_{10}^i & \hat{u}_{11}^i & \cdots & \hat{u}_{1S}^i \\ \vdots & \vdots & \ddots & \vdots \\ \hat{u}_{S0}^i & \hat{u}_{S1}^i & \cdots & \hat{u}_{SS}^i \end{pmatrix}.$$

Similarly, we will write $f_{s's''} = \partial_{y_{s'} y_{s''}}^2 f(y)$ and $\hat{f}_{s's''} = \partial_{y_{s''}} \left[\frac{\partial_{y_s} f(y)}{\partial_{y_0} f(y)} \right]$, and use the matrices

$$F = \begin{pmatrix} f_{00} & f_{01} & \cdots & f_{0S} \\ f_{10} & f_{11} & \cdots & f_{1S} \\ \vdots & \vdots & \ddots & \vdots \\ f_{S0} & f_{S1} & \cdots & f_{SS} \end{pmatrix} \text{ and } \hat{F} = \begin{pmatrix} \hat{f}_{10} & \hat{f}_{11} & \cdots & \hat{f}_{1S} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{f}_{S0} & \hat{f}_{S1} & \cdots & \hat{f}_{SS} \end{pmatrix}.$$

Theorem 5.4 *The set of value-maximizing production plans is an $(S - J)$ -dimensional manifold.*

Proof: We want to proof that the following system contains J independent equations.

$$\begin{aligned} A^\top \nabla f(y) - q &= (0, \dots, 0)^\top \\ f(y) &= 0 \end{aligned}$$

To this end, we have to show that the rows of the matrix $\begin{pmatrix} A^\top \hat{F} \\ df(y)^\top \end{pmatrix}$ are linearly independent. We first rewrite the entry in row j and column s' of $A^\top \hat{F}$ as follows:

$$\begin{aligned} [A^\top \hat{F}]_j^{s'} &= \sum_{s=1}^S A_s^j \hat{f}_{ss'} \\ &= \sum_{s=1}^S A_s^j \left(\frac{f_{ss'} - f_s f_{0s'}}{f_0} \right) \\ &= \frac{1}{f_0} \left[\sum_{s=1}^S A_s^j f_{ss'} - \sum_{s=1}^S A_s^j f_s f_{0s'} \right] \\ &= \frac{1}{f_0} \left[\sum_{s=1}^S A_s^j f_{ss'} - q^j f_{0s'} \right] \\ &= \left[\frac{1}{f_0} W^\top F \right]_j^{s'} \end{aligned}$$

The second line follows by applying the quotient rule, and the fourth line results from $A^\top df(y) - q = 0$. Hence, we have that $A^\top \hat{F} = \frac{1}{f_0} W^\top F$. Now suppose by way of contradiction that there is $t \in \mathbb{R}^J \setminus \{0\}$ and $k \in \mathbb{R} \setminus \{0\}$ such that

$$tA^\top \hat{F} + kdf(y)^\top = 0$$

We obtain the following equalities.

$$\begin{aligned} tA^\top \hat{F}W &= 0 \\ tA^\top \hat{F}Wt^\top &= 0 \\ tW^\top FWt^\top &= 0 \end{aligned}$$

The first line results from right-multiplication by W and, again, $df(y)^\top W = 0$. The second line follows by right-multiplication with t^\top , and the last line from substituting the previously derived expression for $A^\top \hat{F}$. Since f is differentially quasi-concave, the last equation implies

$$df(y)^\top Wt^\top \neq 0$$

contradicting the fact that $df(y)^\top W = 0$. \square

We have assumed that U^i is negative-definite, implying that it is of full rank. In order to prove the next theorem, an important auxiliary result is that also the matrix of normalized second-order derivatives \hat{U}^i has linearly independent rows. This is shown in the following lemma.

Lemma 5.5 *The matrix \hat{U}^i has linearly independent rows.*

Proof: Suppose by way of contradiction that there is $t \in \mathbb{R}^S \setminus \{0\}$ such that $\sum_{s=1}^S t_s \hat{u}_{s\bar{s}}^i = 0$ for all $\bar{s} = 0, 1, \dots, S$. Then,

$$\begin{aligned} \sum_{s=1}^S t_s \left[\frac{u_{s\bar{s}}^i}{u_0^i} - \frac{u_{0\bar{s}}^i u_s^i}{(u_0^i)^2} \right] &= 0 \\ \sum_{s=1}^S t_s u_{s\bar{s}}^i - \sum_{s=1}^S t_s \frac{u_{0\bar{s}}^i u_s^i}{u_0^i} &= 0 \end{aligned}$$

The first equation follows from the definition of \hat{u}^i by the quotient rule. Now define a vector $t' = (-\sum_{s=1}^S t_s \frac{u_s^i}{u_0^i}, t_1, \dots, t_S)$. Then, $\sum_{s=0}^S t'_s u_{s\bar{s}}^i = 0$, contradicting the assumption that U^i is negative-definite. \square

Consider a particular $\bar{v} \in \partial V^+$. If there exist a production plan $\bar{y} \in Y$ and asset portfolios $\bar{z}^1, \dots, \bar{z}^I$ such that $u^i(\omega^i + \theta^i \bar{y} + W \bar{z}^i) = \bar{v}^i$ for all $i \in \mathcal{I}$, then we say that the point \bar{v} is *supported without transfers*.

In what follows, it will be important that the Pareto boundary itself is dependent on the endowments; we will henceforth denote it by ∂V_ω^+ .

Theorem 5.6 *For almost all endowments $\omega \in \mathbb{R}_{++}^{(S+1)I}$, at most finitely many elements of ∂V_ω^+ are supported without transfers.*

Proof: We consider the following system of equations.

$$\begin{aligned} \sum_i \theta^i \nabla u^i(\omega^i + \theta^i y + W z^i) - \nabla f(y) &= 0 \\ A^\top \nabla u^i(\omega^i + \theta^i y + W z^i) - q &= (0, \dots, 0)^\top, \quad i \in \mathcal{I} \setminus \{I\} \\ A^\top \nabla f(y) - q &= (0, \dots, 0)^\top \\ f(y) &= 0 \end{aligned}$$

We want to show first that the set of $y \in Y$ and $z \in \mathbb{R}^{IJ}$ which solve this system has at most finitely many elements, or, equivalently, that the above system contains $S + 1 + IJ$ independent equations for generically chosen $\omega \in \mathbb{R}_{++}^{S+1}$.

Taking derivatives of the left-hand sides of each equation with respect to $\omega^I, \omega^1, \dots, \omega^{I-1}, y$ we obtain the following matrix.

$$N = \begin{pmatrix} \hat{U}^I & \hat{U}^1 & \dots & \dots & \hat{U}^{I-1} & * \\ 0 & A^\top \hat{U}^1 & 0 & \dots & 0 & * \\ 0 & 0 & \ddots & 0 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 0 & A^\top \hat{U}^{I-1} & * \\ 0 & 0 & \dots & \dots & 0 & A^\top \hat{F} \\ 0 & 0 & \dots & \dots & 0 & df(y)^\top \end{pmatrix}$$

Lemma 5.5 and the full column rank of A imply that the diagonal blocks $A^\top \hat{U}^i$, $i = 1, \dots, I-1$ have linearly independent rows. Moreover, the linear independence of the rows in the block $\begin{pmatrix} A^\top \hat{F} \\ df(y)^\top \end{pmatrix}$ is implied by the earlier result that the set of value-maximizing production plans is of dimension $(S - J)$. \square

Theorem 5.7 *For almost all endowments $\omega \in \mathbb{R}_{++}^{(S+1)I}$ and almost all bargaining weights $\mu \in \Delta_I$, the tuple (y, τ, z) supporting the μ -weighted NBS satisfies $\tau \neq (0, \dots, 0)$.*

Proof: Add to the system of equations in the proof of Theorem 5.6 the additional restrictions

$$\mu^i \eta^i(y, 0, z) - \mu^I \eta^I(y, 0, z) = 0, \quad i \in \mathcal{I} \setminus \{I\}$$

We know from the proof of Theorem 5.6 that without the additional restrictions the system admits at most finitely many solutions. Now we want to show that, generically in ω and μ , it is over-determined. To this end, consider the derivatives of $\mu^i \eta^i(y, 0, z) - \mu^I \eta^I(y, 0, z)$ for all $i \in \mathcal{I} \setminus \{I\}$ with respect to $\omega^I, \omega^1, \dots, \omega^{I-1}, y, \mu^1, \dots, \mu^{I-1}$. We obtain the matrix

$$N' = \begin{pmatrix} N & 0 & \dots & \dots & \dots & 0 \\ * & * & 0 & \dots & \dots & 0 \\ * & 0 & * & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & * & 0 \\ * & 0 & \dots & \dots & 0 & * \end{pmatrix}$$

For any $i \in \mathcal{I} \setminus \{I\}$, the expression $\mu^i \eta^i(y, 0, z) - \mu^I \eta^I(y, 0, z)$ is independent of any μ^j , $j \in \mathcal{I} \setminus \{i, I\}$. This explains the block-diagonal structure of the last $I-1$ rows and columns of N' , and thus ensures linear independence among the last $I-1$ rows of N' . Moreover, none of those rows can depend on previous rows of N' because all the previous rows have only zeroes in their last $I-1$ columns. This is due to the fact that the system of equations which gave rise to the earlier matrix N is completely independent of μ . Since the rows of N' are linearly independent, the theorem holds. \square

Theorem 5.7 says that generically in endowments and bargaining weights, the bargaining procedure will lead to transfers. Since the Drèze criterion selects exactly those points in ∂V which do not require transfers, the theorem also implies that the payoff allocations resulting from the bargaining procedure and from the Drèze criterion are different for almost all endowments and bargaining weights. This difference in payoff allocation holds for both complete and incomplete markets. In the case of complete markets, we have previously shown that the profit-maximizing production plan is selected by the bargaining procedure. Thus, with complete markets, any difference in payoff allocation must be due to the transfers. With regard to incomplete markets, however, we will show in the sequel of this section that the difference in payoff allocation is not only the result of transfers, but that the chosen production plan is different as well.

The individually rational boundary ∂V_ω^+ is $(I - 1)$ -dimensional. We have shown previously that each of its elements is supported by a unique value-maximizing production plan. The set of such production plans is $(S - J)$ -dimensional by the previous corollary. In the following theorem, we complement these findings, showing that any given value-maximizing production plan supports an $(I - 1 - S + J)$ -dimensional subset of ∂V_ω^+ for almost all endowments.

Theorem 5.8 *Let \bar{y} be some value-maximizing production plan. Then, for almost all endowments $\omega \in \mathbb{R}_{++}^{(S+1)I}$, the subset of ∂V_ω^+ which is supported by \bar{y} is an $(I - 1 - S + J)$ -dimensional manifold.*

Proof: Consider the following system of equations.

$$\begin{aligned} \sum_i \theta^i \nabla u^i(\omega^i + \theta^i \bar{y} + e(0)\tau^i) - \nabla f(\bar{y}) &= 0 \\ A^T \nabla u^i(\omega^i + \theta^i \bar{y} + e(0)\tau^i) - q &= (0, \dots, 0)^\top, \quad i \in \mathcal{I} \setminus \{I\} \\ \sum_i \tau^i &= 0 \end{aligned}$$

For generically chosen parameters $\omega \in \mathbb{R}_{++}^{S+1}$, we are interested in the dimension of the set of $\tau \in \mathbb{R}^I$ and $z \in \mathbb{R}^{IJ}$ which solve this system.

Since \bar{y} is value-maximizing, it holds that $A^\top \nabla f(\bar{y}) - q = (0, \dots, 0)$. Multiplying the first S equations of the above system from the left by A^\top reveals that the J equations

$$A^T \nabla u^I(\omega^I + \theta^I \bar{y} + e(0)\tau^I) - q = (0, \dots, 0)^\top$$

are implied. This explains why they have been excluded from the system above. Consider a matrix M of derivatives with the following structure. The matrix has $S + (I - 1)J + 1$ rows, each corresponding to one of the above equations. Furthermore, there are $(S + 1)I + 1$ columns for the derivatives with respect to $\omega^I, \omega^1, \dots, \omega^{I-1}, \tau^I$. We will show that the rows

of M are linearly independent.

$$M = \begin{pmatrix} \hat{U}^I & \hat{U}^1 & \dots & \dots & \hat{U}^{I-1} & [\hat{u}^I]^0 \\ 0 & A^\top \hat{U}^1 & 0 & \dots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 0 & A^\top \hat{U}^{I-1} & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

As in the proof of the previous theorem, Lemma 5.5 and the full column rank of A mean that the diagonal blocks $A^\top \hat{U}^i$, $i = 1, \dots, I-1$ have linearly independent rows. We remark that the columns corresponding to the derivatives with respect to ω_0^I and τ^I must be identical except for the last row. We conclude that the rows of M are linearly independent.

The pre-image theorem implies that the set of solutions to the system of equations is of dimension $I-1-S+J$ for almost all $\omega \in \mathbb{R}_{++}^{S+1}$. \square

One implication of Theorem 5.8 is that for a small degree of market incompleteness ($S-J \leq I-1$) any particular value-maximizing production plan is needed to support the boundary for (almost) all choices of endowments. If, however, the degree of market incompleteness is high, the set of production plans which support ∂V_ω^+ depends strongly on the choice of ω .

Theorem 5.9 *For almost all endowments $\omega \in \mathbb{R}_{++}^{S+1}$, the subset of ∂V_ω^+ which is supported by a production plan satisfying the Drèze criterion is a manifold of dimension $I-1-S+J$.*

Proof: Consider the problem of finding a tuple $(y, \tau, \hat{z}, \check{z}) \in Y \times T \times \mathbb{R}^{IJ} \times \mathbb{R}^{IJ}$ which solves the following system of equations.

$$\begin{aligned} \sum_{i \in \mathcal{I}} \theta^i \nabla u^i(\omega^i + \theta^i y + W \hat{z}^i) - \nabla f(y) &= 0 \\ A^\top \nabla u^i(\omega^i + \theta^i y + W \hat{z}^i) - q &= 0, \quad i \in \mathcal{I} \setminus \{I\} \\ A^\top \nabla f(y) - q &= 0 \\ f(y) &= 0 \\ \sum_{i \in \mathcal{I}} \theta^i \nabla u^i(\omega^i + \theta^i y + e(0)\tau^i + W \check{z}^i) - \nabla f(y) &= 0 \\ A^\top \nabla u^i(\omega^i + \theta^i y + e(0)\tau^i + W \check{z}^i) - q &= 0, \quad i \in \mathcal{I} \setminus \{I\} \\ \sum_{i \in \mathcal{I}} \tau^i &= 0 \end{aligned}$$

In words, the first four conditions above mean that the tuple (y, τ, \hat{z}) supports a payoff vector in ∂V_ω^+ , and the last three conditions say that $(y, 0, \hat{z})$ supports a payoff vector

which is consistent with the Drèze criterion given ω . We have to show that generically in ω , the above equations are linearly independent.

To this end, we consider the derivatives of the left-hand sides of the above equations with respect to $\omega^I, \omega^1, \dots, \omega^{I-1}, y, \tau^I, \tilde{z}^{IJ}, \tilde{z}^{1J}, \dots, \tilde{z}^{(I-1)J}$.

We write the resulting matrix of derivatives in the following form,

$$Q = \begin{pmatrix} N^* & 0 \\ M^* & P \end{pmatrix}$$

and argue that the blocks N^* and P each have linearly independent rows: The first four lines in the system of equations correspond to the system used in the proof of Theorem 5.6. Since they are independent of τ , we have $N^* = [N, 0]$. But N has linearly independent rows. The asset holding \tilde{z}^{IJ} is present only in the fifth line of the system, and the terms $\tilde{z}^{1J}, \dots, \tilde{z}^{(I-1)J}$ each show up in the fifth line and in exactly one of the conditions summarized in the sixth line. Hence, we know that P is of the following form;

$$P = \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & 0 & \dots & \dots & 0 \\ \vdots & 0 & * & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & * & 0 \\ 0 & 0 & \dots & \dots & 0 & * \end{pmatrix}$$

its rows are linearly independent. Indeed, the set of solutions is an $(I - 1 - S + J)$ -dimensional manifold. \square

It is important to note the difference between Theorems 5.8 and 5.9 above. The set of value-maximizing production plans depends only on the production set and on the prices and payoffs of the assets. In particular, whether a production plan is or is not value-maximizing is completely independent of the endowments. In Theorem 5.8, one particular value-maximizing production plan is fixed, and then for each choice of ω , the subset of ∂V_ω^+ supported by that plan is considered. For almost all ω , it is a manifold of dimension $I - 1 - S + J$. However, this does *not* imply that the subset of ∂V_ω^+ supported by a Drèze production plan is a manifold of this dimension. While it is true that any Drèze production plan is always value-maximizing, a given value-maximizing production plan may satisfy the Drèze criterion for some choices of endowments and fail it for others. Hence, the statement of Theorem 5.9 does not fix any particular production plan.

Theorem 5.10 *For almost all endowments $\omega \in \mathbb{R}_{++}^{S+1}$ and almost all bargaining weights $\mu \in \Delta_I$, the production plan corresponding to the μ -NBS fails the Drèze criterion, unless markets are complete.*

Proof: The proof is analogous to Theorem 5.7. Consider the system of equations used in the proof of Theorem 5.9, and add the restrictions

$$\mu^i \eta^i(y, \tau, \tilde{z}) - \mu^I \eta^I(y, \tau, \tilde{z}) = 0, \quad i \in \mathcal{I} \setminus \{I\}$$

for some $\mu \in \Delta_I$. Then let us take the derivatives with respect to $\omega^I, \omega^1, \dots, \omega^{I-1}, y, \tau^I, \tilde{z}^{IJ}, \tilde{z}^{1J}, \dots, \tilde{z}^{(I-1)J}, \mu^I, \mu^1, \dots, \mu^{I-1}$, which we can summarize in the matrix

$$Q' = \begin{pmatrix} Q & 0 & \cdots & \cdots & \cdots & 0 \\ * & * & 0 & \cdots & \cdots & 0 \\ * & 0 & * & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & * & 0 \\ * & 0 & \cdots & \cdots & 0 & * \end{pmatrix}$$

We see that the rows of Q' are linearly independent. The set of solutions to the new system is a manifold of dimension $J - S$ for almost all endowments $\omega \in \mathbb{R}_{++}^{S+1}$ and almost all bargaining weights $\mu \in \Delta_I$. Indeed, if markets are incomplete, the production plan corresponding to the μ -NBS generically fails the Drèze criterion. \square

We had previously shown that generically in endowments and bargaining weights the payoff allocation resulting from our bargaining approach and that resulting from the use of the Drèze criterion are different. We have seen that in the case of complete markets, the production plan chosen under both approaches is the same, so that the different payoff allocation is merely a result of redistribution via transfers. The last theorem complements these findings by saying that for the case of incomplete markets, the two approaches lead to different production plans for almost all choices of endowments and bargaining weights.

6 Conclusion

We have introduced a non-cooperative bargaining procedure to resolve the conflict among shareholders of a firm when markets are incomplete. In contrast to many existing models, we obtain a unique prediction for the production plan as well as for the resulting payoff allocation. This solution is parameterized by the distribution of bargaining power across the different owners. This distribution may, but does not need to, be given by the shares of ownership. An important feature of the model is that transfers are possible in equilibrium. The well-known Drèze criterion rules out a production plan which can be Pareto-improved upon by an alternative plan and transfers. However, the chosen production plan itself should be implemented without transfers under that criterion. In our model, however, it turns out that when transfers are allowed, they will almost always be used. If the degree of market incompleteness is small, any given value-maximizing production plan will be chosen for a set of possible distributions of bargaining power. Generically in endowments,

the dimension of this set is $I - 1 - S + J$. If the degree of market incompleteness is high ($S - J > I - 1$), then a perturbation of endowments changes the set of production plans which may result from the bargaining procedure. The outcome of the bargaining procedure proposed in this paper is different from the predictions of standard economic theory. If markets are complete, the production decision of the firm is driven by profit-maximization as in the Arrow-Debreu model. However, the profits are redistributed among the owners of the firm in accordance with their bargaining power, which derives from the ability to make a proposal and from the disagreement payoff. In the case of incomplete markets, the production plan adopted under the bargaining procedure almost always fails the Drèze criterion. Non-zero transfers are almost always made.

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