The INDIRECT FUNCTION of COMPROMISE STABLE TU GAMES and CLAN TU GAMES as a tool for the determination of its NUCLEOLUS and PREKERNEL

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Abstract

The main goal is to illustrate that the so-called indirect function of a cooperative game in characteristic function form is applicable to determine the nucleolus for a subclass of coalitional games called compromise stable TU games. In accordance with the Fenchel-Moreau theory on conjugate functions, the indirect function is known as the dual representation of the characteristic function of the coalitional game. The key feature of compromise stable TU games is the coincidence of its core with a box prescribed by certain upper and lower core bounds. For the purpose of the determination of the nucleolus, we benefit from the interrelationship between the indirect function and the prekernel of coalitional TU games. The class of compromise stable TU games contains the subclasses of clan games, big boss games, 1- and 2-convex n-person TU games. As an adjunct, this paper reports the indirect function of clan games for the purpose to determine its nucleolus.

Keywords: cooperative game in characteristic function form, dual representation, indirect function, compromise stable TU game, clan game, core, prekernel

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1 Compromise stable TU Games

Fix the finite player set N and its power set $\mathcal{P}(N) = \{S | S \subseteq N\}$ consisting of all the subsets of N (including the empty set $\emptyset$). A *cooperative transferable utility game*, or TU game for

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short, is given by the so-called *characteristic function* $v : \mathcal{P}(N) \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. That is, the TU game v assigns to each coalition $S \subseteq N$ its *worth* $v(S)$ amounting the monetary benefits achieved by cooperation among the members of S.
In the framework of set-valued solution concepts for TU games, we aim to determine the prekernel for a special subclass of TU games called compromise stable TU games ([11]) using a new mathematical tool called the indirect function ([7]). The economic interpretation of this function is the following. An employer has to select among the players those who will produce the maximum profit to him. In case the non-empty coalition $S \subseteq N$ is selected, then its members will produce, using the resources that are available to the employer, a total amount of output the monetary utility of which is represented by the worth $v(S)$. The expression $e^v(S, \vec{y}) = v(S) - \sum_{k \in S} y_k$, called the *excess* of coalition S at the payoff vector $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ in the TU game v, is thus the net profit the employer would obtain from the coalition S if the (possibly negative) salary required by the player i amounts y_i, $i \in N$. Write $e^v(\emptyset, \vec{y}) = 0$. In accordance with the Fenchel-Moreau theory on conjugate functions, the indirect function provides a dual representation to TU games in the sense that indirect functions provide the same information as characteristic functions because a simple formula permits to recover any characteristic function from its associated indirect function.

Definition 1.1. ([7], page 292) With every TU game $v : \mathcal{P}(N) \to \mathbb{R}$, there is associated the *indirect function* $\pi^v : \mathbb{R}^N \to \mathbb{R}$, given by

$$\pi^v(\vec{y}) = \max_{S \subseteq N} e^v(S, \vec{y}) = \max_{S \subseteq N} \left[v(S) - \sum_{k \in S} y_k \right] \qquad \text{for all } \vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N. \tag{1.1}$$

Definition 1.2. The *Core(v)* of the TU game $v : \mathcal{P}(N) \to \mathbb{R}$ consists of *efficient* salary vectors of which all the excesses are non-positive, that is

$$Core(v) = \{\vec{y} \in \mathbb{R}^N | e^v(N, \vec{y}) = 0 \quad \text{and} \quad e^v(S, \vec{y}) \leq 0 \quad \text{for all } S \subsetneq N,\, S \neq \emptyset\}. \tag{1.2}$$

Equivalently, $\vec{y} \in Core(v)$ if and only if $e^v(N, \vec{y}) = 0$ and $\pi^v(\vec{y}) = 0$.

Concerning the definition of compromise stable TU games, we follow the notation as used in [11].

Definition 1.3. Let $v : \mathcal{P}(N) \to \mathbb{R}$ be a TU game.

(i) The *utopia demand* vector $\vec{M}^v = (M^v_k)_{k \in N} \in \mathbb{R}^N$ is given by $M^v_i = v(N) - v(N \backslash \{i\})$ for all $i \in N$.

(ii) The *minimum right* vector $\vec{m}^v = (m^v_k)_{k \in N} \in \mathbb{R}^N$ is given by

$$m^v_i = \max \left[v(S) - \sum_{k \in S \backslash \{i\}} M^v_k \mid \quad S \subseteq N, \quad i \in S \right] \quad \text{for all } i \in N. \tag{1.3}$$

(iii) The *core cover* $CC(v) \subseteq R^N$ consist of efficient payoff vectors representing compromises between utopia demands as well as minimum rights, that is

$$CC(v) = \{\vec{y} \in \mathbb{R}^N | e^v(N, \vec{y}) = 0 \quad \text{and} \quad m^v_i \leq y_i \leq M^v_i \quad \text{for all } i \in N\}. \tag{1.4}$$

(iv) The TU game v is called *compromise stable* if $CC(v) = Core(v)$.

We remark that the inclusion $Core(v) \subseteq CC(v)$ holds in general because the utopia demand vector $\vec{M}^v$ and the minimum right vector $\vec{m}^v$ are well-known to be an upper and lower bound for the core, respectively. As a first main contribution, we provide an alternative proof of the following characterization of compromise stable TU games. For any non-empty coalition $T \subseteq N$ and any payoff vector $\vec{z} = (z_k)_{k\in N} \in \mathbb{R}^N$, write $\vec{z}(T) = \sum_{k\in T} z_k$, where $\vec{z}(\emptyset) = 0$.

Theorem 1.4. ([11], Theorem 3.1, page 494) A TU game $v : \mathcal{P}(N) \to \mathbb{R}$ is compromise stable if and only if

$$v(S) \leq \max\left[\sum_{k\in S} m_k^v, \quad v(N) - \sum_{k\in N\setminus S} M_k^v\right] \qquad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{1.5}$$

Alternative proof. (i) Suppose (1.5) holds. We prove the coincidence $CC(v) = Core(v)$. It suffices to prove the inclusion $CC(v) \subseteq Core(v)$. Suppose $\vec{y} = (y_k)_{k\in N} \in CC(v)$. Then $m_i^v \leq y_i \leq M_i^v$ for all $i \in N$. Let $S \subseteq N$, $S \neq \emptyset$. Clearly, $\vec{y}(S) \geq \vec{m}^v(S)$, whereas $\vec{y}(S) = v(N) - \vec{y}(N\setminus S) \geq v(N) - \vec{M}^v(N\setminus S)$. Hence, $\vec{y}(S) \geq \max\left[\vec{m}^v(S), \quad v(N) - \vec{M}^v(N\setminus S)\right]$. Due to (1.5), $\vec{y}(S) \geq v(S)$ for all $S \subseteq N$, $S \neq \emptyset$, and so, $\vec{y} \in Core(v)$, provided $\vec{y} \in CC(v)$.
(ii) In order to prove the converse statement, suppose the coincidence $CC(v) = Core(v)$. We aim to prove (1.5). Let $S \subseteq N$, $S \neq \emptyset$. We distinguish two cases.
Case 1. Assume $v(N) - \vec{M}^v(N\setminus S) < \vec{m}^v(S)$. We prove $v(S) \leq \vec{m}^v(S)$. For that purpose, construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ such that $y_i = m_i^v$ for all $i \in S$ and $y_i = m_i^v + \frac{v(N) - \vec{m}^v(N)}{(\vec{M}^v - \vec{m}^v)(N\setminus S)} \cdot (M_i^v - m_i^v)$ for all $i \in N\setminus S$. Then $m_i^v \leq y_i \leq M_i^v$ for all $i \in N\setminus S$ due to our assumption. So, $\vec{y} \in CC(v)$ and so, $\vec{y} \in Core(v)$, Thus, $\vec{y}(S) \geq v(S)$ or equivalently, $\vec{m}^v(S) \geq v(S)$.
Case 2. Assume $v(N) - \vec{M}^v(N\setminus S) \geq \vec{m}^v(S)$. We prove $v(S) \leq v(N) - \vec{M}^v(N\setminus S)$. We distinguish two subcases. Put $g^v(N) = \vec{M}^v(N) - v(N)$.
Subcase 2.1. Suppose there exists at least a player $i \in S$ with $M_i^v - m_i^v \geq g^v(N)$. Construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ such that $y_i = M_i^v - g^v(N)$ and $y_j = M_j^v$ for all $j \in N\setminus\{i\}$. Then $m_j^v \leq y_j \leq M_j^v$ for all $j \in N$. So, $\vec{y} \in CC(v)$ and so, $\vec{y} \in Core(v)$. Thus, $\vec{y}(S) \geq v(S)$ or equivalently, $v(S) \leq \vec{M}^v(S) - g^v(N) = v(N) - \vec{M}^v(N\setminus S)$.
Subcase 2.2. Suppose $M_i^v - m_i^v < g^v(N)$ for all $i \in S$. Without loss of generality, write $S = \{i_1, i_2, \ldots, i_s\}$ such that $M_{i_1}^v - m_{i_1}^v \leq M_{i_2}^v - m_{i_2}^v \leq \ldots\ldots \leq M_{i_s}^v - m_{i_s}^v$. Then there exists $2 \leq t \leq s$ such that

$$\sum_{k=1}^{t-1}\left[M_{i_k}^v - m_{i_k}^v\right] < g^v(N) \quad \text{and} \quad \sum_{k=1}^{t}\left[M_{i_k}^v - m_{i_k}^v\right] \geq g^v(N).$$

Construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ such that $y_i = M_i^v$ for all $i \in N\setminus S$, and $y_{i_k} = m_{i_k}^v$ for all $i_k \in S$, $k < t$ and $y_{i_k} = M_{i_k}^v$ for all $i_k \in S$, $k > t$, and $y_{i_t} = M_{i_t}^v + \sum_{k=1}^{t-1}\left[M_{i_k}^v - m_{i_k}^v\right] - g^v(N)$. Then $m_j^v \leq y_j \leq M_j^v$ for all $j \in N$. So, $\vec{y} \in CC(v)$ and so, $\vec{y} \in Core(v)$. Thus, $\vec{y}(S) \geq v(S)$ or equivalently, $v(S) \leq \vec{M}^v(S) - g^v(N) = v(N) - \vec{M}^v(N\setminus S)$. This completes the alternative proof. □

Remark 1.5. With every TU game $v : \mathcal{P}(N) \to \mathbb{R}$, there is associated its *gap function* $g^v : \mathcal{P}(N) \to \mathbb{R}$ defined by $g^v(S) = \vec{M}^v(S) - v(S)$ for all $S \subseteq N$, where $g^v(\emptyset) = 0$. An adapted version of (1.5) is well-known as the so-called 1-convexity constraint as follows:

$$v(S) \leq v(N) - \vec{M}^v(N \backslash S) \quad \text{or equivalently,} \quad g^v(N) \leq g^v(S) \quad \text{for all } S \subseteq N,\ S \neq \emptyset \tag{1.6}$$

In words, the TU game v is said to be 1-convex if its corresponding (non-negative) gap function g^v attains its minimum at the grand coalition. Clearly, the class of compromise stable TU games contains the subclass of 1-convex n-person games, as well as the 2-convex n-person games ([3]), and the big boss and clan games ([9], [10], [1]).

Further, from (1.3), we deduce that $M_i^v - m_i^v = \min\left[g^v(S)| \quad S \subseteq N, \quad i \in S\right]$ for all $i \in N$.

Thus, $m_i^v \leq M_i^v$ if and only if $g^v(S) \geq 0$ for all $S \subseteq N$ with $i \in S$. Particularly, $\vec{m}^v \leq \vec{M}^v$ if and only if $g^v(S) \geq 0$ for all $S \subseteq N$, $S \neq \emptyset$. Throughout the next section we tacitly assume a non-negative gap function.

2 The indirect function as a tool for the determination of the nucleolus of compromise stable TU games

Theorem 2.1. Let the TU game $v : \mathcal{P}(N) \to \mathbb{R}$ be compromise stable. Then its indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ satisfies the following properties:

(i) $\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k \in N} y_k\right]$ for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ with $\vec{m}^v \leq \vec{y} \leq \vec{M}^v$.

(ii) For all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ such that there exist unique $i, j \in N$ with $y_i < m_i^v$, $y_j > M_j^v$, and $m_k^v \leq y_k \leq M_k^v$ for all $k \in N \backslash \{i, j\}$,

$$\pi^v(\vec{y}) = \max\left[m_i^v - y_i, \quad v(N \backslash \{j\}) - \sum_{k \in N \backslash \{j\}} y_k\right] \tag{2.1}$$

$$= \max\left[m_i^v - y_i, \quad v(N) - \sum_{k \in N} y_k + y_j - M_j^v\right] \tag{2.2}$$

(iii) With any efficient payoff vector $\vec{x} = (x_k)_{k \in N} \in \mathbb{R}^N$ satisfying $\vec{m}^v \leq \vec{x} \leq \vec{M}^v$, and any pair $i, j \in N$ of players, and any transfer $\delta \geq 0$ from i to j, there is associated the adapted payoff vector $\vec{x}^{ij\delta} = (x_k^{ij\delta})_{k \in N} \in \mathbb{R}^N$ given by $x_i^{ij\delta} = x_i - \delta$, $x_j^{ij\delta} = x_j + \delta$, and $x_k^{ij\delta} = x_k$ for all $k \in N \backslash \{i, j\}$. Then, for $\delta \geq 0$ sufficiently large, it holds

$$\pi^v(\vec{x}^{ij\delta}) = \delta + \max\left[m_i^v - x_i, \quad x_j - M_j^v\right] \qquad \text{for all } i, j \in N,\ i \neq j. \tag{2.3}$$

(iv) For $\delta \geq 0$ sufficiently large, the pairwise equilibrium condition $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ is equivalent to

$$\min\left[x_i - m_i^v, \quad M_j^v - x_j\right] = \min\left[x_j - m_j^v, \quad M_i^v - x_i\right] \qquad \text{for all } i, j \in N,\ i \neq j. \tag{2.4}$$

The proof of Theorem 2.1 proceeds as follows. (i) From Theorem 1.4 we derive that for every vector $\vec{y} \in \mathbb{R}^N$ with $\vec{m}^v \leq \vec{y} \leq \vec{M}^v$ and every coalition $S \subseteq N$, $S \neq N$, $S \neq \emptyset$,

$$\begin{aligned} v(S) - \vec{y}(S) &\leq \max\Big[(\vec{m}^v - \vec{y})(S), \quad v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N \backslash S)\Big] \\ &\leq \max\Big[0, \quad v(N) - \vec{y}(N)\Big] \qquad \text{and so,} \\ \pi^v(\vec{y}) &= \max\Big[0, \quad v(N) - \vec{y}(N)\Big] \qquad \text{for all } \vec{m}^v \leq \vec{y} \leq \vec{M}^v. \end{aligned}$$

This completes the proof of part (i). In order to prove part (ii), let $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ be such that there exist $i, j \in N$ with $y_i < m_i^v$, $y_j > M_j^v$, and $m_k^v \leq y_k \leq M_k^v$ for all $k \in N \backslash \{i, j\}$. In order to study the excesses $e^v(S, \vec{y})$, $S \subseteq N$, $S \neq N$, $S \neq \emptyset$, we distinguish three cases.
Case 1. Assume $\vec{m}^v(S) \leq v(N) - \vec{M}^v(N \backslash S)$. Then it holds $v(S) \leq v(N) - \vec{M}^v(N \backslash S)$ and so,

$$\begin{aligned} v(S) - \vec{y}(S) &\leq v(N) - \vec{M}^v(N \backslash S) - \vec{y}(S) = v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N \backslash S) \\ &\leq v(N) - \vec{y}(N) + (y_j - M_j^v) = v(N \backslash \{j\}) - \vec{y}(N \backslash \{j\}) \end{aligned} \tag{2.5}$$

By (2.5), $e^v(S, \vec{y}) \leq e^v(N \backslash \{j\}, \vec{y})$ for all $S \subseteq N$ with $\vec{m}^v(S) \leq v(N) - \vec{M}^v(N \backslash S)$.
Case 2. Assume $\vec{m}^v(S) > v(N) - \vec{M}^v(N \backslash S)$. Then it holds $v(S) \leq \vec{m}^v(S)$. We distinguish two subcases.
Subcase 2.1. Assume $i \notin S$. Then we derive $v(S) - \vec{y}(S) \leq (\vec{m}^v - \vec{y})(S) \leq 0$.
Subcase 2.2. Assume $i \in S$. Then we derive $v(S) - \vec{y}(S) \leq (\vec{m}^v - \vec{y})(S) \leq m_i^v - y_i$.
In summary, $e^v(S, \vec{y}) \leq m_i^v - y_i$ for all $S \subseteq N$ with $\vec{m}^v(S) > v(N) - \vec{M}^v(N \backslash S)$. Particularly, $e^v(\{i\}, \vec{y}) = v(\{i\}) - y_i \leq m_i^v - y_i$. Notice that $m_i^v \geq v(N) - \vec{M}^v(N \backslash \{i\})$ because of $M_i^v - m_i^v \leq g^v(N)$. Without going into details, we claim that the upperbound $m_i^v - y_i$ coincides with the excess of an appropriately chosen coalition. Hence, (2.1) holds, or equivalently, (2.2). As a direct consequence, (2.3)–(2.4) hold. □

Without going into details ([8], [2]), we state that the pairwise equilibrium conditions $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for all pairs $i, j \in N$ of players and for $\delta \geq 0$ sufficiently large, fully determine the so-called *prekernel* of the TU game v ([6]). As a matter of fact, the set of efficient solutions of the non-linear system of equations (2.4) is unique and it is a so-called constrained equal award rule of the parametric form

$$x_i = m_i^v + \max\left[M_i^v - m_i^v - \lambda, \quad \frac{M_i^v - m_i^v}{2}\right] \qquad \text{for all } i \in N, \tag{2.6}$$

where the parameter $\lambda \in \mathbb{R}$ is determined by the efficiency constraint $\vec{x}(N) = v(N)$. This unique solution within the prekernel is well-known as the nucleolus of the TU game v. In [11], the approach to determine the nucleolus of compromise stable TU games is totally different and strongly based on the study of (convex) bankruptcy games ([11], Theorem 4.2, pages 497-498).

3 The indirect function and nucleolus of clan TU games

Definition 3.1. ([10], [9], [1], page 59)
An n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ is said to be a *clan game* if $M_i^v \geq v(\{i\})$ for all $i \in N$ and there exists a coalition $T \subseteq N$, called the *clan*, such that $v(S) = 0$ whenever $T \not\subseteq S$ and

$$v(S) \leq v(N) - \vec{M}^v(N \backslash S) \qquad \text{for all } S \subseteq N,\ S \neq \emptyset, \text{ with } T \subseteq S \tag{3.1}$$

A clan game v with an empty clan reduces to an 1-convex game, provided $g^v(N) \geq 0$. A clan game with the clan to be a singleton is known as a big boss game.

Throughout this section we suppose that the clan T consists of at least two players.

Theorem 3.2. Let the n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ be a clan game, Then its indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ satisfies the following properties:

(i) $\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k \in N} y_k\right]$ for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ with $y_i \geq 0$ for all $i \in N$ and $y_i \leq M_i^v$ for all $i \in N \backslash T$.

(ii) $\pi^v(\vec{y}) = \max\left[0, \quad v(N \backslash \{\ell\}) - \sum_{k \in N \backslash \{\ell\}} y_k\right] = \max\left[0, \quad v(N) - \sum_{k \in N} y_k + y_\ell - M_\ell^v\right]$ for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ such that there exists a unique $\ell \in N \backslash T$ with $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ for all $i \in N \backslash T$, $i \neq \ell$, and $y_i \geq 0$ for all $i \in N$.

(iii) $\pi^v(\vec{y}) = \max\left[-y_\ell, \quad v(N) - \sum_{k \in N} y_k\right]$ for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ such that there exists a unique $\ell \in N$ with $y_\ell < 0$, $y_i \geq 0$ for all $i \in N \backslash \{\ell\}$, and $y_i \leq M_i^v$ for all $i \in N \backslash T$.

(iv) $\pi^v(\vec{y}) = \max\left[-y_j, \quad v(N \backslash \{\ell\}) - \sum_{k \in N \backslash \{\ell\}} y_k\right] = \max\left[-y_j, \quad v(N) - \sum_{k \in N} y_k + y_\ell - M_\ell^v\right]$ for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ such that there exist unique $j \in N$, $\ell \in N \backslash T$ with $y_j < 0$, $y_i \geq 0$ for all $i \in N \backslash \{j\}$, and $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ for all $i \in N \backslash T$, $i \neq \ell$.

Proof. Let $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$.
(i) Suppose that $y_i \geq 0$ for all $i \in N$ and $y_i \leq M_i^v$ for all $i \in N \backslash T$. We distinguish two types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq 0$. In case $T \subseteq S$, then we derive from (3.1),

$$v(S) - \vec{y}(S) \leq v(N) - \vec{M}^v(N \backslash S) - \vec{y}(S) = v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N \backslash S) \leq v(N) - \vec{y}(N). \tag{3.2}$$

This proves part (i). In order to prove part (ii), suppose that there exists a unique $\ell \in N \backslash T$ with $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ for all $i \in N \backslash T$, $i \neq \ell$, and $y_i \geq 0$ for all $i \in N$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq 0$. In case $T \subseteq S$, together with $\ell \in S$, then $v(S) - \vec{y}(S) \leq v(N) - \vec{y}(N)$ as shown in (3.2). In case $T \subseteq S$, together with $\ell \notin S$, then we derive from (3.1)

$$\begin{aligned} v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + y_\ell + \vec{y}(N \backslash (S \cup \{\ell\})) \\ &\leq v(S) - \vec{y}(N) + y_\ell + \vec{M}^v(N \backslash (S \cup \{\ell\})) \\ &= v(S) - \vec{y}(N) + y_\ell - M_\ell^v + \vec{M}^v(N \backslash S) \\ &\leq v(N) - \vec{y}(N) + y_\ell - M_\ell^v = v(N \backslash \{\ell\}) - \vec{y}(N \backslash \{\ell\}). \end{aligned} \tag{3.3}$$

In this setting, the indirect function π^v attains its maximum either for $S = N$, $S = N\backslash\{\ell\}$ or $S = \emptyset$, but $S = N$ cancels. The similar proof of part (iii) is left for the reader.
(iv) Suppose that there exist unique $j \in N$, $\ell \in N\backslash T$ with $y_j < 0$, $y_i \geq 0$ for all $i \in N\backslash\{j\}$, and $y_\ell > M^v_\ell \geq 0$, $y_i \leq M^v_i$ for all $i \in N\backslash T$, $i \neq \ell$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq -y_j$. In case $T \subseteq S$, the proof proceeds similar to the proof of part (ii) and is left for the reader too. □

Corollary 3.3. For every n-person clan game $v : \mathcal{P}(N) \to \mathbb{R}$, with clan T, the following three statements concerning a payoff vector $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ are equivalent.

(i) $\vec{y} \in Core(v)$, i.e., $\vec{y}(N) = v(N)$ and $\vec{y}(S) \geq v(S)$ for all $S \subseteq N$, $S \neq \emptyset$

(ii) $\vec{y}(N) = v(N)$ and $\pi^v(\vec{y}) = 0$

(iii) $\vec{y}(N) = v(N)$ and $y_i \geq 0$ for all $i \in N$ and $y_i \leq M^v_i$ for all $i \in N\backslash T$

Theorem 3.4. Let the n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ be a clan game with clan T. From the explicit formula for the indirect function of clan games, as presented in Theorem 3.2 (ii)–(iv), we conclude that, for $\delta \geq 0$ sufficiently large, the pairwise equilibrium conditions $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for all pairs $i, j \in N$ of players reduce to the following system of equations:

Case	**Pairwise equilibrium equation** $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$
$i \in T,\ j \in T$	$\max\left[-(x_i - \delta),\quad 0\right] = \max\left[-(x_j - \delta),\quad 0\right]$
$i \notin T,\ j \in T$	$\max\left[-(x_i - \delta),\quad 0\right] = \max\left[-(x_j - \delta),\quad (x_i + \delta) - M^v_i\right]$
$i \notin T,\ j \notin T$	$\max\left[-(x_i - \delta),\quad (x_j + \delta) - M^v_j\right] = \max\left[-(x_j - \delta),\quad (x_i + \delta) - M^v_i\right]$

Case	**Resulting pairwise equation for** $\vec{x} = (x_k)_{k\in N} \in \mathbb{R}^N$
$i \in T,\ j \in T$	$x_i = x_j$
$i \notin T,\ j \in T$	$x_i = \min\left[x_j,\quad M^v_i - x_i\right]$
$i \notin T,\ j \notin T$	$\min\left[x_i,\quad M^v_j - x_j\right] = \min\left[x_j,\quad M^v_i - x_i\right]$

In summary, the unique solution is a so-called constrained equal reward rule of the form $x_i = \lambda$ for all $i \in T$ and $x_i = \min\left[\lambda,\quad \frac{b^v_i}{2}\right]$ for all $i \in N\backslash T$, where the parameter $\lambda \in \mathbb{R}$ is determined by the efficiency condition $\vec{x}(N) = v(N)$.

It is shown in [2] that the indirect function is a helpful tool for the determination of the nucleolus for the subclasses of big boss games as well as 1-convex and 2-convex n-person games.

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