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Two-way Flow Networks with Small Decay*

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Abstract
The set of equilibrium networks in the two-way flow model of network formation (Bala and Goyal, 2000) is very sensitive to the introduction of decay. Even if decay is small enough so that equilibrium networks are minimal, the set of equilibrium architectures becomes much richer, especially when the benefit functions are non-linear. However, not much is known about these architectures. In this paper we remedy this gap in the literature. We characterize the equilibrium architectures. Moreover, we show results on the relative stability of different types of architectures. Three of the results are that (i) at most one players receives multiple links, (ii) the absolute diameter of equilibrium networks can be arbitrarily large, and (iii) large (small) diameter networks are relatively stable under concave (convex) benefit functions.

Keywords: Network formation, two-way flow model, decay, non-linear benefits.

JEL classification: C72, D85.

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Two-way flow networks with small decay

1 Introduction

Not much is known about the architectures of "strict Nash Networks" (SNNs) in the two way flow model (introduced in the seminal paper by Bala and Goyal, 2000a) when there is decay.[1] That is despite the fact that this seminal article demonstrates that the set of equilibrium architectures is quite sensitive to even very small levels of decay. The two main results which are known about this are the following. First, if there are constant marginal benefits of information (CMBI) all stars can be SNNs, as well as interlinked stars (Bala and Goyal, 2000a). Second, if there are decreasing marginal benefits of information (DMBI), then the only SNN is the "periphery sponsored star" (PSS) provided that two conditions are satisfied: (i) information cannot travel beyond a certain maximum distance, and (ii) the population size is large enough (Hojman and Szeidl, 2008). As interlinked stars have diameter 3 (meaning that no two players are more than three links apart), and stars have diameter two, a superficial reading of this literature might give the impression that equilibrium networks are necessarily small in diameter. As we will demonstrate, this impression would be false.

In this paper we study the two-way flow model with a small level of decay. We provide an extended characterization of SNNs for any population size and any increasing benefit function, not just CMBI or DMBI. Furthermore, we are able to state some results on the relative stability of different network architectures which are potential SNNs for three special cases: CMBI , DMBI and IMBI (increasing marginal benefits of information). In line with the above two papers, we find that the PSS is the most stable network architecture. However, the results also show a gap under small decay and DMBI: stable networks are either PSS (diameter two) or they have some minimal diameter d (where $d \geq 4$), where d may be arbitrarily large[2].

The two way flow model has been extensively studied and been modified in several directions such as player heterogeneity[3] and link reliability[4].Unlike those

[1] This paper is an extended version of our 2008 working paper Network Formation with Decreasing Marginal Benefits of Information (TKI discussion paper 08-16). It treats constant and increasing marginal benefits on top of the decreasing marginal benefits treated in our 2008 paper. Moreover, on top of the necessary conditions for stability of networks treated in our 2008 paper, the current paper treats sufficient conditions as well.

[2] Although for large diameter SNNs, the diameter is *relatively* small as compared to the population size. The reason is the minimum population needed to create a stable network of diameter d is increasing exponentially in d.

[3] For instance, Galeotti *et al.* (2006), and Kamphorst and Van der Laan (2007).

[4] See for instance Bala and Goyal (2000b) and Haller *et al.* (2005).

papers, this paper does not extend the two way flow model in any way. Instead it continues the analysis of decays where Bala and Goyal (2000a) stopped. We believe that this analysis provides worthwhile and unforeseen results.[5]

In Section 2 we will present the model, and some standard preliminary results. Section 3 provides the initial extended characterization of the SNNs. The relative stability of different candidate SNNs is then studied in Section 4 for three special types of benefit functions: CMBI, DMBI and IMBI. There we will also first demonstrate the gap in diameters of the set of SNNs for DMBI. We are able to extend the characterization further in Section 5 by looking at weakly smaller levels of decay. This results in a condition which we call the *balancing condition*. This balancing condition allows us to show that all networks which satisfy balancing condition as well as the properties derived in Section 3 are stable for some positive range of the parameters. Hence each such network is indeed relevant. These networks may have any diameter larger than two, provided that the population is large enough. Moreover, we show a stronger result on the diameter gap: only the maximal diameter candidate networks are stable next to the PSS. We end with a discussion of the results in Section 6.

2 The Model

Consider a population of n agents denoted by the set $\mathcal{N}$ with $n \geq 3$. Each player faces the choice to which of the other players he will sponsor a link. A link by player i (the sponsor) to player j (the recipient) is denoted by (i, j), or ij for short. The set of all links that a player i can possibly sponsor is given by

$$\mathcal{L}_i \equiv \{kj \in \mathcal{N} \times \mathcal{N} : k = i,\ j \neq i\}.$$

$\mathcal{L}$ is defined as the set of all possible links, meaning that

$$\mathcal{L} \equiv \bigcup_{i \in \mathcal{N}} \mathcal{L}_i = \{ij \in \mathcal{N} \times \mathcal{N} : i \neq j\}.$$

We typically denote the strategy of player i – the set of links that he sponsors – by g_i. His strategy space $\mathcal{G}_i$, where obviously $g_i \in G_i$, is therefore the collection of all subsets of $\mathcal{L}_i$, specifically:

$$\mathcal{G}_i \equiv \{g_i \subseteq \mathcal{N} \times \mathcal{N} : g_i \subseteq \mathcal{L}_i\}.$$

All links together form a network[6], typically denoted by g, so

$$g \equiv \bigcup_{i \in \mathcal{N}} g_i.$$

[5] Non-linear benefits have also been studied in other types of network formation models. See Vergara-Caffarelli (2004) for an example in the one-way flow model; Buechel (2007) for an example with two sided link formation; Goyal and Joshi (2006) for an example with two sided link formation and non-linearity of payoffs in the number of own links and the number of links by others; and Bloch and Dutta (2009) for an example with endogenous link strength (and non-linearity of benefits in link strengths).

[6] Observe that the strategy profile coincides with the network. In this paper we will refer to any strategy profile as a network. Similarly, we will refer to any (strict) Nash equilibrium as a (strict) Nash network.

The strategy space $\mathcal{G}$ is therefore the set of all possible networks, which is the collection of all subsets of the set of all possible links. Thus

$$\mathcal{G} \equiv \{g \subseteq \mathcal{N} \times \mathcal{N} : g \subseteq \mathcal{L}\},$$

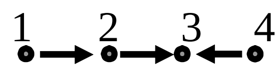

Figure 1: Example of a 4 player network g, where $g = \{12, 23, 43\}$.

We can depict such a network g in a graph, where the players are the nodes, and each link $ii' \in g$ is represented by an arrow (directed arc) from i to i'. For example, Figure 1 shows the network $\{12, 23, 43\}$.

Now we come to the (dis)incentives for players to sponsor links. The disincentives arise because sponsoring links is costly. The costs of a link ij are denoted by c_{ij}, and are incurred completely by the sponsor; the recipient incurs no costs. Because we wish to focus on the benefits of link formation we model the cost side as simple as possible: $c_{ij} = c$ for all $i, j \in N$. Let $N_i^S(g) \subset \mathcal{N}$ be the set of players to whom player i sponsors a link in g, so $N_i^S(g) \equiv \{j \in \mathcal{N} : ij \in g\}$. Hence the total costs for player i in network g are equal to$\left|N_i^S(g)\right| c$.

Players derive benefits from being connected to each other by a path of links. On this path, it does not matter who the sponsor of the links are. The benefits of a link 'flow in two directions'. To make this precise, we let $\overline{ij} \in g$ denote that $ij \in g$ or $ji \in g$ or both[7]. We say that in network g players i_0 and i_k are connected if there exists some subset of players $\mathcal{N}_{i_0 i_k} \subseteq \mathcal{N}$, $\mathcal{N}_{i_0 i_k} = \{i_0, ..., i_k\}$ such that for all $\ell \in \{1, ..., k\}$ we have that $\overline{i_{\ell-1} i_\ell} \in g$. When two players are connected, they exchange their private information. Let $N_i(g)$ denote the set of players to whom player i is connected in network g.

In this paper, we will assume that there is *decay*. In other words, as this information travels through the network it becomes less accurate or less complete. We assume that what is lost at each step is independent of the path the information travels. Hence only the shortest path between any two players is relevant. We say that the *distance* between players i and j in network g is the length (i.e. the number of links) of the shortest path between these two players. We denote this distance by $d_{ij}(g)$.[8] We follow the convention in assuming that every time the information is passed on a constant fraction $(1 - \delta)$ of the (remaining) information is lost, where $\delta \in (0, 1]$. Observe that decay gives players

[7] So $\overline{ij} \in g$ says that the intersection of $\{ij, ji\}$ and g is not empty.

[8] Note that by two-way flow we have that $d_{ij} = d_{ji}$ for all $i, j \in N$.

incentives to sponsor links to players to whom they are already connected for the purpose of reducing the distance between them. However, throughout this paper we assume that δ is large enough to ensure minimality of any Nash network[9], meaning that any two players are not connected by more than one path of links. So the amount of decay is limited.

We assume that each player has one unit of private information. The value of the information does not depend on the original owner of this information, so any two units of information are *a priori* equally valuable. Players derive benefits from the information which they gathered. To model the benefits we need a few more definitions.

Let $N_i^k(g) \subset \mathcal{N}$ be the set of players at distance k from player i in network g. So $N_i^k(g) \equiv \{j \in \mathcal{N} : d_{ij}(g) = k\}$. Due to decay, the total amount of information gathered by player i in network g is then

$$I_i(g) = \sum_{k=0}^{n-1} \left(\delta^k \left| N_i^k(g) \right| \right).$$

Note that by definition, $N_i^0(g) = 1$. We call $I_i(g)$ player i's ex-post information, as it is the information that i obtains after having decided to sponsor all links in g_i.

The benefits derived by player i from network g, $V_i(g)$, are an increasing function of $I_i(g)$, specifically

$$V_i(g) = f(I_i(g))$$

where $f' > 0$. In Section 4 we study three special types of benefit function, namely either $f''(I) = 0 \ \forall I \geq 0$ (CMBI), $f''(I) < 0 \ \forall I \geq 0$ (DMBI), or $f''(I) > 0 \ \forall I \geq 0$ (IMBI).

The utility which i obtains in g equals his benefits minus his costs. Formally,

$$U_i(g) = V_i(g) - \left| N_i^S(g) \right| c.$$

Define g_{-i} as all the links in g excluding the links sponsored by player i. A network g is a SNN if for each player $i \in \mathcal{N}$ and all $g_i' \in \mathcal{G}_i$, $g_i' \neq g_i$, we have

$$U_i(g) > U_i(g_{-i} \cup g_i').$$

Similarly, in a Nash network every player plays a best reply strategy. Denote by $BR_i^f(g)$ the set of best reply strategies of player i versus network g under function f. Formally, for any benefit function f

$$BR_i^f(g^*) = \left\{ g_i \in \mathcal{G}_i : U_i\left(g_{-i}^* \cup g_i\right) \geq U_i\left(g_{-i}^* \cup g_i'\right) \text{ for all } g_i' \in \mathcal{G}_i \right\}.$$

[9] For instance BG and Lemma 2 show that there exists some $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ this is indeed the case.

If in a Nash network the best reply set for each player is singleton then the network is a SNN.

Each network g partitions the population into *components* (of g), where two players belong to the same component if and only if they are connected. Component k is denoted as $C_k(g)$. Note that a player's information is also defined over a component, just as it is over a graph.

A network is *minimal* if the deletion of any link in that network will result in an increase of the number of components. A *cycle* is a set of links $\left\{\overline{j_0 j_1}, ..., \overline{j_{k-1} j_k}\right\}$ such that $j_0 = j_k$. This implies that a component (or network) is minimal if and only if it contains no cycles. We call a link which is not part of a cycle a *minimal link*, while any link which is part of a cycle is called a *non-minimal link*. A network may contain an *end sponsor* i, namely a player i who sponsors a link to a player j who does not have any links to players other than i; we call j an *end recipient*, and the link between i and j an *end link*.

We now define our concept of the best informed player. For a network g and for M, $M \subseteq N$, define network g_M as the set of links of network g of which both the sponsor and the recipient of the link belong to M. Formally:

$$g_M = \{ij \in g : i, j \in M\}.$$

Definition 1 *Let $M \subseteq \mathcal{N}$ be a connected subset of players in network g. Then player i, $i \in M$, is a best informed player of M if $I_i(g_M) \geq I_j(g_M)$ for all $j \in M$.*

Remark 1 *If in network g some player i is not part of component $C_k(g)$, then the additional information he receives from sponsoring a link to some player j, $j \in C_k(g)$, is $\delta I_j(g)$. Since all links cost the same, and because utility is strictly increasing in information the best link which player i can have into the component is to this component's best informed player*[10]*. Moreover, by the same arguments it follows that player i is indifferent between any two most informed players of a particular component.*

Let $ii' \in g$. Then we denote the set of players observed by player i exclusively via link ii' by $A_{ii'}(g)$. So

$$A_{ii'}(g) = \{j \in \mathcal{N} : j \in N_i(g) \text{ and } j \notin N_i(g \setminus \{ii'\})\}.$$

Example 2 *Consider the network in Figure 1. There $A_{12}(g) = \{2, 3, 4\}$, and $A_{23}(g) = \{3, 4\}$.*

[10] Note that without decay (so $\delta = 1$), every player in any connected set is a most valuable player in that set. This concept can also be useful when considering heterogeneous agents.

Denote by $|A_{ii'}(g)|$ the cardinality of this set, i.e. the number of players that i indirectly assesses by sponsoring a link to i'. Additionally, we define $A^I_{ii'}(g,\delta)$ as the set of best informed players in $A_{ii'}(g)$ given decay factor δ.[11] So

$$A^I_{ii'}(g,\delta) = \left\{ j \in A_{ii'}(g) : j \text{ is a best informed player in } g_{A_{ii'}(g)} \right\}.$$

Example 3 *Consider the network in Figure 1. There for any δ, $A^I_{12}(g,\delta) = \{3\}$. Player 3 gathers $1+2\delta$ in network $g\backslash\{12\}$, whereas players 2 and 4 only gather $1+\delta+\delta^2$ (which is less, since $\delta < 1$). Moreover $A^I_{23}(g,\delta) = \{3,4\}$, since in network $g\backslash\{23\}$ players 3 and 4 both collect $1+\delta$ information.*

Using the notation above, $g_{A_{ii'}(g)}$ is the component to which player i gains access by sponsoring a link to i'.

Moreover the set $A_{ii'}(g)$ is useful in defining the orientation of a link $ii' \in g$ with respect to some player $j \in N$.

Definition 2 *A link $ii' \in g$ is said to point to player j if $j \in A_{ii'}(g)$.*

A link $ii' \in g$ is said to point away from player j if j observes $A_{ii'}(g)$ via ii' and this set $A_{ii'}(g)$ is non-empty.

Two non-non-minimal links point towards each other if and only if the sponsors of these two links observe each other through these two links.[12]

Example 4 *In the network of Figure 1 link 23 points to players 3 and 4, and away from players 1 and 2. Moreover links 12 and 43 point towards each other.*

Before we start our actual characterization, we first provide two results assuring that networks connecting only part of the players, and networks containing redundant players, cannot be SNN. To derive the result that networks containing redundant players cannot be SNN, we first show that in the absence of decay, non-empty SNN are always minimal.

Lemma 1 *In the absence of decay, no player in any network g prefers to sponsor non-minimal links.*

Proof. Any non-minimal link yields a marginal benefit of zero. As linking is costly, a player sponsoring a non-minimal link is better off when deleting any one of his non-minimal links. ■

We now define two levels threshold levels of decay, and after that we show that both are smaller than one.

[11] Let $M \subseteq N$. The identity of the best informed player in g_M may depend on δ. This is one of the reasons why in Sections 5 and 6 we are able to get additional results by tightening the restrictions on δ.

[12] Formally: consider links $ii' \in g$ and $jj' \in g$, then these point towards each other if and only if $i \in A_{jj'}(g)$ and $j \in A_{ii'}(g)$.

Definition 3 *Let $\delta_M(c, n, f(I))$ be the lowest level of decay such that for all $\delta > \delta_M(c, n, f(I))$ no player in any network prefers to sponsor a non-minimal link.*

Definition 4 *Let $\delta_R(c, n, f(I))$ be the lowest level of decay such that for all $\delta > \delta_R(c, n, f(I))$ any SNN g is minimal.*

We now show that both thresholds are strictly below 1.

Lemma 2 *For any $c > 0$, $n \geq 3$, and $f(I) > 0$ we have $\delta_R(c, n, f(I)) \leq \delta_M(c, n, f(I)) < 1$.*

Proof. Note that the benefit function is continuous in δ. By this continuity it follows from Lemma 1 that $\delta_R(c, n, f(I)), \delta_M(c, n, f(I)) < 1$. Moreover the minimality of every SNN does not imply that there is no other network (for instance the line) where some player does prefer to sponsor a non-minimal link, hence $\delta_R(c, n, f(I)) \leq \delta_M(c, n, f(I))$. ■

Note that the minimality of every SNN does not imply that there is no other network (for instance the line) where some player does prefer to sponsor a non-minimal link. However, if $\delta > \delta_M(c, n, f(I))$, then it follows that no SNN is non-minimal.

Before starting our actual characterization, we have thus shown that for small decay all non-empty SNN must be minimal networks, i.e. trees or forests with directed links. Two highly stylized networks that play an important role in our analysis are the *center-sponsored star* (CSS) and the *periphery-sponsored star* (PSS). Stars are minimal connected networks where one player, the characterizing player, has a link (either as sponsor or as recipient) with every other player (see Networks A-C in Figure 2). The CSS is a special case of the star where the characterizing player sponsors each link (Network A), while the PSS s a special case of the star where the characterizing player sponsors each link (Network C). Stars have a small *diameter*, where the diameter of a minimal connected network is defined as the maximal distance that exists between two players in the network.

The role of decay is important, not only because the concept makes sense, but also because it greatly affects the set of SNNs. For large levels of decay this is obvious. Players have incentives to sponsore non-minimal links in order to reduce their distance to other players. Bala and Goyal (2000) showed moreover, that even at levels of decay where this effect plays no role, the influence of decay is profound. It makes homogeneous players differ in how attractive they are as recipients, by their position in a component. In fact, they showed that for no decay and any increasing benefit function only the CSS was a SNN, while with small amounts of decay[13] and a CMBI benefit function all star networks, but also networks of diameter 3 can be SNNs. However, they restricted themselves

[13] Decay small enough so that all SNNs are minimal.

to some examples and a CMBI benefit function. In the following section we will identify all networks which can be stable with such small decay, for any increasing benefit function. This will include networks of any diameter, provided the population size is large enough. We will also show that each such candidate network is indeed a SNN for some range of parameters and class of benefit functions.

3 Characterization

In this section we will present the characterization of SNN networks in the two way flow model with decay. We will start with a preliminary result on the recipients of links in minimal Nash networks. Second we will introduce two types of players given the network, which we will use to characterize the network. Then we will present our main proposition, Proposition 1, after which the seperate parts of the proposition will be proved in seperate subsections. We finish this section by showing that the characterizing players are either central or close to central in the network.In Section 5 we will further extend this characterization by looking at weakly smaller levels of decay.

We start by a straightforward preliminary result on the recipients of links in Nash equilibria. It is based on the observation that if in network g players i and j are not connected, then if i sponsors a link to j he receives additional information equal to $\delta I_j(g)$. Hence if a player wants to connect himself to some other component, his best option is a link to a best informed player of that component.

Lemma 3 *Let $\delta > \delta_R(c, n, f(I))$. If g is a Nash network, then $j \in A^I_{ij}(g)$ for all $ij \in g$. Furthermore, if g is a strict Nash network, then $\{j\} = A^I_{ij}(g)$ for all links $ij \in g$.*

Proof. By $\delta > \delta_R(c, n, f(I))$ we have that g is minimal. Hence every link, say ij, in g connects the sponsor to some component of $g \backslash \{ij\}$ which he is not part of. This component is $C_l(g \backslash \{ij\}) = A_{ij}(g)$. Hence we can use our arguments from Remark 1, which implies that it is optimal for for player i to sponsor the link to j only if j is a most informed player of that component, *i.e.* if $j \in A^I_{ij}(g)$. Since indifference is not allowed in a strict Nash network, we have the stronger condition that $\{j\} = A^I_{ij}(g)$ if g is a strict Nash network. ■

For the characterization it is necessary to identify to types of players, given the network.

Definition 5 *Define as a* ***multi-recipient player*** *a player who receives at least two links.*

Definition 6 *Define as a* ***non-recipient player*** *a player who does not receive any links from any player.*

We will prove that each SNN will have either a unique multi-recipient player or no multi-recipient player. If he exists, this unique multi-recipient player will be the *'characterizing player'* of the network. If he does not exist, it will follow that there is a unique non-recipient player. We will then define this player as the *'characterizing player'* of the network.

This enables us to characterize the SNN as follows.

Proposition 1 *Let g be a non-empty strict Nash network and $V_i(g) = f(I_i(g))$ for all $i \in \mathcal{N}$. Moreover let $\delta > \delta_R(c, n, f(I))$. Then g has the following properties:*

1. *Network g is minimal connected.*
2. *There is a unique characterizing player in g. If a multi-recipient player exists in g, then he is the unique multi-recipient player in g, and he will be the characterizing player of g. If no multi-recipient player exists in g, then there exists a unique non-recipient player in g and he will be the characterizing player of g.*
3. *Network links tend to be outward-oriented in the following way:*
   - *(a) if the characterizing player is a multi-recipient player, all links point away from him, except for those links of which the characterizing player is the recipient.*
   - *(b) if the characterizing player is a non-recipient player, then all links point away from him.*
4. *Every recipient of a link in a SNN either has no other links, or at least two other links.*

To simplify the formulations in this paper, we define 'candidate network' as follows.

Definition 7 *For a given $\delta > \delta_R(c, n, f(I))$, we say that any network satisfying both Lemma 3 and Properties in Proposition 1 is called a* candidate network*.*

Moreover, we say that a link is *outward-oriented* if it points away from the characterizing player.

Examples of candidate networks are shown in Figure 2, where multi-recipient characterizing players are indicated by triangles, while non-recipient characterizing players are indicated by a square.

These results will be proven in a number of subsections. Readers familiar with the literature will find Property 1 of Proposition 1 quite standard. These readers might want to focus on Subsections 3.2 to 3.4. Subsection 3.2 proves the most important parts of this characterization, which are Properties 2 and

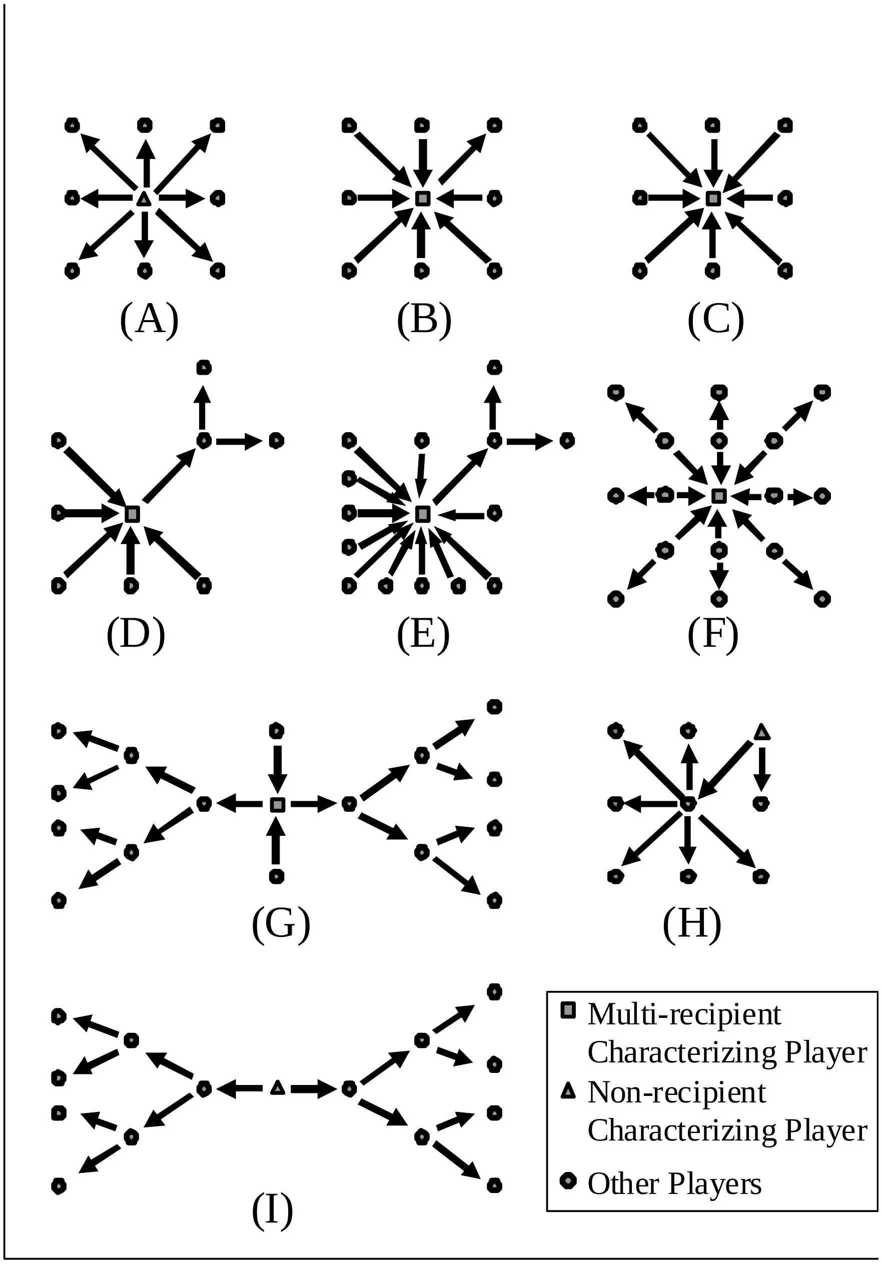

Figure 2: Example networks fitting Proposition 1.

3. These Properties, as well as Property 4 are quite selective among the set of minimal connected networks by imposing a lot of structure on the orientation of the links. For instance, none of the links in Networks (F) or (I) of Figure 2 could be reversed[14]. Also network architectures with larger diameters need enough 'branches', as shown by Property 4. This excludes for instance network architectures such as the line, for $n \geq 4$. Lastly note that the proposition above does not exclude the existence of large diameter SNNs as Proposition 9 will show in combination with Proposition 8 in Section 5.

Subsections 3.4 and 5.2 are relevant for the interpretation of the provided characterization, as discussed in the introduction of this Section.

3.1 Characterization of Part 1

The minimality of g follows immediately from Lemma 2 and $\delta > \delta_R(c, n, f(I))$. Connectedness will be proven by the following two Lemmas[15].

Lemma 4 *Let network g be a non-empty SNN and $\delta > \delta_R(c, n, f(I))$. Then g has no singleton component.*

Proof. Suppose not. Then there exists a player, say j, who is isolated. Moreover, by minimality and non-emptiness of g, there exists some player i who receives no links, but does sponsor a set of links himself, namely to all players in $\mathcal{N}_i^S(g)$. Note that $I_i(g_{-i}) = I_j(g_{-i}) = I_j(g)$. Now let player j consider strategy g'_j in which he sponsors a link to every recipient of links from player i. So $g'_j = \left(ji' \in \mathcal{L} : i' \in N_i^S(g)\right)$. This costs him the same as g_i costs player i. But the benefits to player j are strictly larger because he accesses the same players at the same distance as player i does and in addition he will be connected to player i. Hence if g_i is a best reply for player i to network g, then $g_j = \{\emptyset\}$ cannot be a best reply to player j. Since g is a SNN by assumption, this forms a contradiction. ■

Lemma 5 *Let network g be a non-empty SNN and $\delta > \delta_R(c, n, f(I))$. Then g is connected.*

Proof. Suppose not. Then by Lemma 4 there is a strict Nash network g which contains multiple non-singleton components. Without loss of generality we have $ii', jj' \in g$ such that i and i' belong to one component, say $C_1(g)$, and j and j' to another, say $C_2(g)$. Because g is a strict Nash network, player i prefers to sponsor a link to i' and not to any player in $C_2(g)$. So player i' has access to more information in $g\backslash\{ii'\}$ than j' in g. Hence we obtain that $I_{i'}(g) > I_{i'}(g\backslash\{ii'\}) > I_{j'}(g) > I_{j'}(g\backslash\{jj'\})$. Because g is Nash, we also have that $I_{i'}(g) < I_{j'}(g\backslash\{jj'\})$, which gives us a contradiction. Hence any Nash network has only one component and is therefore connected. ■

[14] By reversal it is meant that the recipient becomes the sponsor, and the sponsor the recipient.

[15] This result is similar to Bala and Goyal (2000a) Proposition 4.1.

So *all non-empty* SNNs are connected. Naturally, if costs are low enough the empty network is not a SNN, implying that *all* SNNs are connected. Since the minimal benefit an isolated player would derive from sponsoring a link is $f(1+\delta) - f(1)$, Lemma 5 implies the following corollary.

Corollary 1 *Let $c < f(1+\delta) - f(1)$, then any network g which is a SNN is connected.*

Bala and Goyal (2000) pointed out that there always exists a strict Nash network. Under these more general benefit functions this remains true. Note that for $c \geq f(1+(n-1)\delta) - f(1)$ the empty network is a SNN, while for $c < f(1+(n-1)\delta) - f(1)$ the PSS is a SNN.

3.2 Characterization of Parts 2 and 3

In this subsection we will first introduce a powerful lemma from which much of the characterization is derived. This lemma says that if in a SNN network both i and j observe each other via a link that they sponsor themselves (so their links point towards each other), the recipient of the link sponsored by i is the same player as the recipient of the link by j. From this result Parts 2 and 3 of Proposition 1 will follow.

Lemma 6 *Let g be a strict Nash network and $\delta > \delta_R(c, n, f(I))$. If $j \in A_{ii'}(g)$ and $i \in A_{jj'}(g)$ then $i' = j'$.*

Proof. We prove this by contradiction. Suppose not, so $i' \neq j'$. Note that by minimality there is one path connecting i and j, and this path goes via players i' and j'. Now observe that, with decay, the players who lose most if a link is deleted are the ones closest by. So

$$I_{i'}(g) - I_{i'}(g \backslash \{ii'\}) > I_{j'}(g) - I_{j'}(g \backslash \{ii'\}), \text{ and} \quad (1)$$
$$I_{i'}(g) - I_{i'}(g \backslash \{jj'\}) < I_{j'}(g) - I_{j'}(g \backslash \{jj'\}) \quad (2)$$

By Lemma 3, i' is more informed than j' in $g_{A_{ii'}(g)}$, implying that

$$I_{i'}(g \backslash \{ii'\}) > I_{j'}(g \backslash \{ii'\})$$

Applying Eq. 1 gives:

$$I_{i'}(g) > I_{j'}(g),$$

after which Eq. 2 tells us that

$$I_{i'}(g \backslash \{jj'\}) > I_{j'}(g \backslash \{jj'\}) \quad (3)$$

However Eq. 3 implies that $j' \notin A^I_{jj'}(g, \delta)$, which contradicts Lemma 3. Hence a contradiction arises. ■

Verbally the proof of this lemma is that player i' receives more information via the link ii' than player j' does, as i' is at least one link closer to i and the players behind i than j' is. Similarly, player j' receives more information than i' via the link jj'. Now, if g is a SNN, then i' is more informed than j' in $g_{A_{ii'}(g)}$, and therefore i' is also more informed than j' in $g_{A_{jj'}(g)}$. However that contradicts that g is Nash, since jj' is sponsored in g, while link ji' would have given him more information at the same cost.

Using this lemma, we can now show the properties of the characterizing players in the network (Parts 2 and 3 of Proposition 1).

Lemma 7 *Let g be a strict Nash network and $\delta > \delta_R(c, n, f(I))$. Then there is at most one multi-recipient player.*

Proof. Suppose that players i and j are both multi-recipient players. Because g is minimally connected, players i and j are connected by at most one path. This means that both players receive at least one link which is not part of the path connecting them. These links point toward each other which implies by Lemma 6 that $i = j$. Since i and j were arbitrarily chosen multi-recipient players, it follows that there cannot be more than one multi-recipient player. ■

Lemma 8 *Let g be a strict Nash network and $\delta > \delta_R(c, n, f(I))$. Then if a player is a multi-recipient player, all links which he does not receive point away from him.*

Proof. Let player i' be a multi-recipient player and let there be any link pointing towards i'. So there exists some $jj' \in g$ such that $i' \in A_{jj'}(g)$. Since player i' receives at least two links, there exists some player $i \in A_{jj'}(g)$ such that $ii' \in g$. It follows that $j \in A_{ii'}(g)$. By Lemma 6 it must be that $j' = i'$. ■

Lemmas 7 and 8 prove that at most one multi-recipient player exists in a strict Nash network, and that all links either point away from him, or are received by him.

We will now prove that for any non-empty SNN there is exactly one non-recipient player if it contains no multi-recipient player. Moreover, all links will point away from that non-recipient player.

Lemma 9 *Let g be a strict Nash network and $\delta > \delta_R(c, n, f(I))$. If there is no multi-recipient player in network g, then there is exactly one non-recipient player.*

Proof. Recall that g is minimal. Hence $n - 1$ links exists. If no player receives more than 1 link, then $n - 1$ players receive one link. Therefore there is a unique player who receives no links. ■

Lemma 10 *Let g be a strict Nash network and $\delta > \delta_R(c, n, f(I))$. If there is no multi-recipient player then all links point away from the non-recipient player.*

Proof. Suppose not, so there is no multi-recipient player. Moreover let i be the non-recipient player. Suppose that there exists a link $jj' \in g$ such that $i \in A_{jj'}(g)$. Because g is minimally connected and because i receives no links, there exists a link $ii' \in g$ such that $j' \in A_{ii'}(g)$. Lemma 6 then implies that $i' = j'$, which implies that player i' receives at least two links. Namely one from i and one from j. Thus i' is a multi-recipient player which contradicts our assumption that g has no multi-recipient players. ■

If g is a SNN, then we will call the unique multi-recipient player the characterizing player or, if no multi-recipient player exists, we will call the unique non-recipient player the characterizing player. Together with these labels, Lemmas 7 and 9 imply Part 2 of Proposition 1, while Lemmas 8 and 10 imply Part 3.

Now consider any candidate network which is not a PSS. Then by Part 3 of Proposition 1 there exists some link, say ij, which points away from the characterizing player. Now look at the player which is the furthest away from the characterizing player along a path containing ij. Then by Part 3 the final link of this path is outward-oriented, and its sponsor is an end sponsor. Hence Part 3 of Proposition 1 implies the following corollary.

Corollary 2 *Let network g be a SNN and $\delta > \delta_R(c, n, f(I))$. If g is not a PSS then g contains at least one end sponsor.*

This Corollary will prove useful in Section 4.

3.3 Characterization of Part 4

Finally we know that in equilibrium, every recipient of a link, say player j receiving link ij, has either at least two other links[16] or no other links. The reason is the following. If j has one other link, then player i observes at least two players through ij. If there are two players then he will be indifferent between them, hence g is not be a SNN. If there are more than two players, then j is certainly not the best informed player of the group, because he is at the periphery of that group. His neighbor in that group will be better informed. In this case i would strictly prefer to replace ij by some other link. This gives us, without further proof, the following Lemma.

Lemma 11 *Let g be a strict Nash network and $\delta > \delta_R(c, n, f(I))$. Then every recipient of a link in a SNN either has no other links, or at least two other links.*

[16] A player j has two links in network g if there exist two players $k, k' \in N$, where $j \neq k \neq k' \neq j$, such that both $\overline{jk} \in g$ and $\overline{jk'} \in g$.

Example 5 *Consider the network of Figure 1. Considering link* 23, *Example 3 shows that it does not matter to player 2 whether he sponsors a link to player 3 or to player 4. Second consider the link* 12. *Example 3 shows that in the set* $A_{12}(g)$ *the most informed player is player 3. This is because players 2 and 4 are peripheral in* $g_{A_{12}(g)}$. *So this network cannot be a SNN.*

This concludes the proof of Proposition 1.

3.4 The centrality of characterizing players

In Proposition 1 a characterizing player is defined. In this section we show that the characterizing player is quite central to the network, when centrality is measured in terms of information gathered (more central players acquire more information). It appears that characterizing players who are multi-recipient players are actually the central player of the network too. A non-recipient characterizing player either is himself a central player, or he sponsors a link to each central player in the network. In that sense the characterizing player is always central or almost central in the network.

Proposition 2 *Let* g *be a strict Nash network and* $\delta > \delta_R(c, n, f(I))$. *Moreover let* $I^{\max}(g)$ *be the amount of information of the most informed player in* g. *If*

1. *the characterizing player is a multi-recipient player, then he is the unique most informed player in* g.

2. *the characterizing player is a non-recipient player, then the most informed players are all either the characterizing player or one of the recipients of links sponsored by the characterizing player. As such, the characterizing player,* i_{ch}, *either has the most information in* g, *or his information is close to being most (in the sense that* $I_{i_{ch}}(g) > \delta I^{\max}(g)$)

Proof. Let player i_V be the most informed player and i_{ch} the characterizing player in g. Consider Part 1 first, which we will prove by contradiction. If i_{ch} is not the most informed player in g, so $i_{ch} \neq i_V$, then because i_{ch} receives multiple links, there exists at least one player i sponsoring a link to i_C such that $i_V \in A_{ii_{ch}}(g)$. This implies that $i_c \in A^I_{ii_{ch}}(g, \delta)$. Since player i_{ch} benefits more from the link ii_{ch} than player i_v does (by decay), this would imply that $I_{i_{ch}}(g) > I_{i_v}(g)$, which is a contradiction.

Now we consider Part 2, which we prove by construction. Note that there are two possibilities. Either i_{ch} is the most informed player, or i_{ch} is not the most informed player. The first case satisfies our claim. If it is the second case, then by connectedness (Lemma 5) i_{ch} sponsors a link, say $i_{ch}j \in g$, through which he observes i_V. Lemma 3 then implies that $\{j\} = A^I_{i_{ch}j}(g, \delta)$. Since player j benefits more from the link $i_{ch}j$ then any other player in $A_{i_{ch}j}(g)$, it follows that $I_j(g) > I_{j'}(g)$ for any $j' \in A_{i_{ch}j}(g)$. Hence, because $i_V \in A_{i_{ch}j}(g)$, we have $i_V = j$. This concludes the proof of Part 2. ■

We conclude this subsection with a remark on the identity of link recipients. Let us consider the link ij, where $ij \in g$ and g is a candidate network . If we accept the notion that information is a good measure for centrality, it follows from Lemma 3 of Proposition 1 that j is the central player in $A_{ij}(g)$. Hence links in a candidate network are sponsored to the central player in the recipient's component.

4 Marginal benefits of information and relative stability

In this section we will investigate the relative stability of the candidate networks for three types of benefit functions: CMBI, DMBI and IMBI. We say that a candidate network is more stable than another candidate network if, given δ, stability of the latter implies stability of the former but not the other way around.

We investigate these three special cases of the benefit functions, because they are especially salient to us. The CMBI benefit function is often solely considered in the network formation literature. For instance, Bala and Goyal (2000a) consider only CMBI when they discuss decay. DMBI is a natural result if all players get a private signal from a common distribution. The first signals are more informative than later signals. IMBI is a natural assumption (at least until some information level), when some pieces of information are hard to interpret without other pieces of information[17].

The results will show that for any increasing benefit function the PSS is the most stable network architecture. Moreover, for CMBI all other networks are equally stable (but less stable than the PSS). For DMBI we will show that each star which is not a PSS is the least stable candidate network. Moreover, relatively short diameter networks tend to be less stable. For instance, we will show that if the population is large enough such that there exist candidate networks with diameter 4, it can be shown that the most stable diameter 4 network is more stable than the most stable diameter 3 network[18]. In contrast, under IMBI benefit functions, shorter diameter networks tend to be most stable. For instance, all stars are more stable than non-star candidate networks.

We will first present the results for CMBI, DMBI and IMBI. Those results will be discussed at the end of the section.

Proposition 3 *Consider the set of candidate networks as identified by Proposition 1 and Lemma 3, let $\delta > \delta_M(c, n, f(I))$, and let there be CMBI. Then*

1. *the PSS is the unique most stable candidate network architecture. It is stable if and only if $c < f\left(1 + \delta + (n-2)\delta^2\right) - f(1)$.*

[17] See for instance Cohen and Levinthal (1989). There knowledge of old research results increases how much one can learn from new research results.

[18] In Section 5.4 we will strengthen this result.

2. *any other candidate networks are equally stable. These are stable if and only if* $c < f(1+\delta) - f(1)$.

3. *The empty network is a SNN if and only if* $c > f(1+\delta) - f(1)$.

Proof. First note that by Lemma 3 no player wishes to replace a link. By connectedness and $\delta > \delta_M(c, n, f(I))$, no player in a candidate network wants to add a link. Hence we are left with checking whether a player wants to delete links (in case of a candidate network).

ad 1. Observe that for any peripheral player i in a PSS $f\left(1+\delta+(n-2)\delta^2\right) - f(1)$ is the added benefit of a link. This proves that the PSS is stable if $c < f\left(1+\delta+(n-2)\delta^2\right) - f(1)$.

ad 2. In any other candidate network, we know that there is at least one end sponsor (Corollary 2). Among all links in a candidate network, the end links bring the lowest benefit to their sponsor, namely (by CMBI) $f(1+\delta) - f(1)$. The sponsors will therefore strictly prefer to keep the links if $c < f(1+\delta) - f(1)$. Because $n \geq 3$, we know that $f(1+\delta) - f(1) < f\left(1+\delta+(n-2)\delta^2\right) - f(1)$. Hence all non-empty candidate networks are equally stable, except for the PSS networks which are more (and most) stable.

ad 3. Finally note that the empty network is stable for $c > f(1+\delta) - f(1)$, because in the empty network any number of added links with the same sponsor have strictly larger costs than the benefits they generate. ■

We continue with some results on DMBI benefit functions. Note that under DMBI the added benefits of a link depends on two factors. First, the amount of information the sponsor would have without the link: the less information he already has, the higher the benefits of the link will be. And second, the amount of additional information received through the link: the more, the better.

Proposition 4 *Consider the set of candidate networks as identified by Proposition 1 and Lemma 3, let* $\delta > \delta_M(c, n, f(I))$*, and let there be DMBI. Then*

1. *the PSS is the most stable candidate network architecture. It is stable if and only if* $c < f\left(1+\delta+(n-2)\delta^2\right) - f(1)$.

2. *all other stars are the only least stable candidate network architectures. They are stable if and only if* $c < f(1+(n-1)\delta) - f(1+(n-2)\delta)$.

3. *any other candidate networks are strictly in between with regards to the stability.*

4. *The empty network is a SNN if and only if* $c > f(1+\delta) - f(1)$.

Proof. First note that by Lemma 3 no player wishes to replace a link, and by $\delta > \delta_M(c, n, f(I))$, no player in a candidate network wants to add a link.

ad 1. Observe that for any peripheral player i in a PSS $f\left(1+\delta+(n-2)\delta^2\right) - f(1)$ is the added benefit of a link. This proves that the PSS is stable if $c < f\left(1+\delta+(n-2)\delta^2\right) - f(1)$. Moreover, if in the PSS network one of the sponsors prefers not to sponsor a link, then no non-empty network can be stable. This is because a sponsor in the PSS obtains the maximal possible added benefit that can be obtained from sponsoring a link. Without the link the sponsor would have the minimal possible amount of information (namely 1) while the link itself gives the maximal amount of additional information which a link can give: access to one player at distance one and all other players at distance two, so $\delta + (n-2)\,\delta^2$. If this added benefit is too small, then so will be the marginal benefits of any link in any other candidate SNN. Hence if there is a non-empty g which is a SNN then any PSS is a SNN too.

ad 2. Observe that the characterizing player sponsors at least one link in such stars. The extra information through this link is the minimal possible. However, without the link the sponsor would have access to all but one of the other players at distance one. This is the maximal amount of information that any player can have in a non-connected network. Hence, if in some minimal network there exists a player who wants to delete his link, then so does every sponsor who is the characterizing player in a star. This makes all stars, other than the PSS, the least stable candidate networks. Moreover the arguments above show that any star other than the PSS is stable if and only if $c < f\left(1 + (n-1)\,\delta\right) - f\left(1 + (n-2)\,\delta\right)$.

ad 3. We show here that all other candidate networks are strictly less stable than the PSS and strictly more stable than the other stars. First, note that by Corollary 2 in any non-PSS candidate network there is at least one end sponsor. This player receives only δ additional information through his link. By connectedness of the candidate network implies that without the link he has more information than 1. Hence he has strictly less incentives to keep his link than any sponsor in the PSS. Moreover, if the network is not a star, then no end sponsor will have all other players at distance one from him. This means that although non-star candidate networks have end sponsors too. These end sponsors receive as little information through the end link as the characterizing player does in a star, but they have less information without it. Consequently each such sponsor has more reason to sponsor their link. Hence all other candidate networks are strictly more stable then any star which is not fully periphery sponsored.

ad 4. The proof is the same as in Proposition 3. ■

As shown in Proposition 4 the most compact networks (ignoring PSS), with diameter 2, are less stable than more dispersed, larger diameter networks. We now show that, if the parameters allow for candidate networks of diameter 4,[19] then the most stable diameter 3 network is less stable than some candidate networks of diameter 4.

Proposition 5 *Let there be candidate networks of diameter* 4 *and let* $\delta > \delta_M\left(c, n, f\left(I\right)\right)$. *Then some candidate networks of diameter 4 are more stable than the most stable candidate network of diameter* 3.

[19] Lemma 9 together with Proposition 8 will show that this is feasible if the population is sufficiently large.

Proof. Note first that by Lemma 3 no player has incentives to replace a link. Moreover, by $\delta > \delta_M(c, n, f(I))$ we have that all SNNs are minimal, and that in no candidate network a player wants to add a link. Thus we can focus on whether each link is strictly worthwhile to its sponsor. We will proceed by identifying the most stable candidate network of diameter 3, and calculating the added benefit of its end link(s). Then we will construct a candidate network of diameter 4, and show that the benefits of its end links are higher. Finally we show that the end links in these two networks are indeed the links with the lowest added benefit.

The most stable diameter 3 network is a PSS with one alteration, namely that one link by a peripheral player, say j, is deleted and replaced by a link from one non-isolated peripheral player i, $i \neq j$, to player j. This is true because the network must have an end sponsor (Corollary 2), and this puts the maximal number of players at distance 2 of the end sponsor. Hence his added benefit of the end link equals $f\left(1 + 2\delta + (n-3)\delta^2\right) - f\left(1 + \delta + (n-3)\delta^2\right)$. Note that any sponsor of a link which is not an end link, has less information than $f\left(1 + \delta + (n-3)\delta^2\right)$ without that link, and will gain more than δ information.

Now we will construct a diameter 4 network where the end link gives a greater added benefit. Take again the PSS and delete the links of two distinct peripheral players, j and j'. Consider two distinct other peripheral players, i and i' and let them form the links ij and $i'j'$. This network has diameter 4. The added benefit of any of the two end links is then $f\left(1 + 2\delta + (n-4)\delta^2 + \delta^3\right) - f\left(1 + \delta + (n-4)\delta^2 + \delta^3\right)$. Since $f\left(1 + \delta + (n-4)\delta^2 + \delta^3\right) < f\left(1 + \delta + (n-3)\delta^2\right)$, we obtain that the added benefit of the end link in the diameter 4 network is larger than in the most stable candidate network of diameter 3.

To conclude the proof note that any sponsor of a link which is not an end link, has less information than $f\left(1 + \delta + (n-4)\delta^2 + \delta^3\right)$ without that link, and will gain more than δ information. Hence the constructed diameter 4 candidate network is more stable than any candidate network of diameter 3. ■

Without further proof, this gives us the following result.

Corollary 3 *Under DMBI a gap in network diameters can appear. In this case all SNNs are either PSS (which have diameter 2), or they have a network diameter of at least* d^* ($d^* \geq 4$).

Finally we present our results for the case of IMBI. Afterwards we will discuss the intuition behind the results for DMBI and IMBI.

Proposition 6 *Consider the set of candidate networks as identified by Proposition 1 and Lemma 3, let* $\delta > \delta_M(c, n, f(I))$, *and let there be DMBI for all* $I \geq 0$. *Then*

1. *A PSS is stable if and only if* $c < f\left(1 + \delta + (n-2)\delta^2\right) - f(1)$.
2. *A CSS is stable if and only if* $(n-1)c < f(1 + (n-1)\delta) - f(1)$.

3. *Stars with* k, $k \in \{1, ..., n-2\}$ *, center sponsored links are stable if and only if* $kc < f(1 + (n-1)\delta) - f(1 + (n-1-k)\delta)$ *and* $c < f\left(1 + \delta + (n-2)\delta^2\right) - f(1)$ *.*

4. *The empty network is stable if and only if* $nc > f(1 + (n-1)\delta) - f(1)$ *.*

5. *If a star has less center-sponsored links, it is weakly more stable.*

6. *Any other candidate network is less stable than star networks.*

Proof. By Lemma 3 no player has incentives to replace a link. Moreover, by $\delta > \delta_M(c, n, f(I))$ we have that in no candidate network a player wants to add a link. Thus we can focus on whether each link is strictly worthwhile to its sponsor.

ad 1. For Part 1 the threshold values have already been proven in Proposition 4.

ad 2 and 3. By IMBI, if the center would prefer to delete any number of links, he would delete all his links. The reason is that all his links are end links – giving him δ additional information – and that the added benefit of these links is thus increasing. Hence the only possible optimal strategies for the characterizing player are to delete all these k links or none of them. For stars with periphery-sponsored links, there is also the condition that those links are stable. The relevant condition for that is the same as for a PSS. The stability conditions for Parts 2 and 3 follow.

ad 4. This follows the same reasoning as above: the best response to an empty network is either to sponsor links to all other players, or to sponsor no links at all. The condition follows.

ad 5. To prove Part 5 we note that the condition given in Parts 2 and 3, namely $kc < f(1 + (n-1)\delta) - f(1 + (n-1-k)\delta)$, is stricter for $k+1$ than for k, provided that $k \geq 1$, while the condition for any sponsor of a periphery-sponsored link remains $c < f\left(1 + \delta + (n-2)\delta^2\right) - f(1)$. Now we prove that a PSS cannot be less stable than any other star. If PSS is not stable, it follows that $c \geq f\left(1 + \delta + (n-2)\delta^2\right) - f(1)$, so any star with any periphery-sponsored links is unstable.

What about the CSS, which has only center-sponsored links? Following the reasoning above, if the PSS is unstable any of the sponsors have two potential best reply strategies. Either sponsor no links, or to sponsor links to all other players. He prefers the latter if $(n-1)c < f(1 + (n-1)\delta) - f(1)$, which is the stability condition for the CSS. However the latter is ruled out by our assumption $\delta > \delta_M(c, n, f(I))$, which ensures that no player prefers to sponsor non-minimal links.

ad 6. Any non-star candidate network, as well as the CSS, has end links. The added benefit of such an end link increases in the amount of information the sponsor would have without that link. In the non-star candidate network, end sponsors have less information than the unique end sponsor in the CSS. Hence, any non-star candidate network (note that such networks have diameters of at least 3) is less stable than the (weakly) least stable star network. This ends the proof. ■

Propositions 4, 5 and 6 suggest the following intuition. There are two forces that can drive a network to be stable, in giving the players incentives to sponsor costly links. *First*, players may sponsor links because ex ante they have little connections, but can access a lot of information by sponsoring a link to a well-informed player. This effect is equally at work for CMBI, DMBI and IMBI, and is strongest in the PSS, explaining why overall this is the most stable network. *Second*, players may sponsor a link even though they are already well-connected, if their marginal benefit of doing so is large. In the case of DMBI, an already well-connected player still has a large marginal benefit of sponsoring an extra link if the players to whom he was already connected are at a relatively large distance from him. Because of decay, he then does not have much information, and still has a high marginal benefit of sponsoring a link. For this reason, large-diameter networks with outward point links may be stable on top of small-diameter networks with inward pointing links. There is thus a gap in the diameters covered by the stable networks.

In the case of IMBI, a well-connected player still has a large marginal benefit of sponsoring a link if the players to whom he is connected are at a relatively small distance from him. Because decay has little effect then, he has a lot of information, which under IMBI gives him incentives to acquire even more information by sponsoring a link. Thus, in the case of IMBI, the two different forces both lead to stable networks with a small diameter.

The reader may not yet be convinced by the argument, because the characterization limits itself to showing cost levels exist such that the smallest-diameter networks on top of the PSS which are stable are diameter-4 networks, and cost levels exist such that under IMBI all stars are stable, but diameter-3 networks are. The reasons we are not able to derive further results at this point are the following. *First*, we did not derive a general rule assuring that no sponsor wants to change networks satisfying Proposition 1. Specifically, we have not derived a rule identifying the most-informed player in larger-diameter components, to which a sponsor should then connect. The main difficulty is that the identity of the most informed player in components with larger diameters can depend on delta. Thus, the reason why in Proposition 5 we have limited our attention to networks of diameter 4 and smaller is that (*i*) there is only a small number of architectures of candidate networks, and (*ii*) it is clear who the best-informed player in any component is. However, in Section 5 we will derive a simple rule identifying the most-informed player in any component for a range of δ close to 1.

Second, we did not derive a general rule assuring that no player wants to delete a link. In order to assure this, we need to identify the crucial sponsor(s) in any candidate SNN. In Section 5.4, we show that for a range of δ close to 1, any crucial sponsor (the sponsor least willing to maintain his links) is always an end sponsor. Once we know this, we are able to identify which of the candidate SNN remain stable. Our results will confirm the intuition of a gap in the diameters covered by the set of SNN under DMBI, and of no such gap in the case of IMBI.

In each of these two sections, the further assumptions made to make the characterization more detailed are to again impose small decay. While this approach has its limits (see Section 6), it identifies the impact which decay has even when it would not lead to non-minimality of the network. Recall the well-known result that in the absence of decay only the CSS is a SNN. In this sense, our results as well as Proposition 5.3 and 5.4 by Bala and Goyal (2000) clearly show that the result under the absence of decay that only small diameter networks are stable is not robust to the introduction of even a small level of decay. However from Bala and Goyal (2000) and Hojman and Szeidl (2008) a reader might get the impression that with small decay the diameters of SNNs remain small. This impression would be false. In fact, we will show in Section 5.4 that under DMBI it can be the case that, next to the small diameter PSS, only the candidate networks[20] with the maximal diameter are stable. Under IMBI, however, it does continue to be the case under decay that small-diameter networks are more stable.

Inasfar as the Bala and Goyal two-way flow model is an accurate description of some network, we can note on the basis of this section that the small world hypothesis is most consistent with IMBI which tends to favor low diameter networks, and less with DMBI which is in comparison unlikely to give rise to small diameter networks.

5 Small decay and no change in links: balancing condition

We now show that for small levels of decay, we can derive a simple rule identifying in any component the most informed player, and at the same time assuring that any component to which a player sponsors a link has a unique most-informed player. In this way, we can characterize the larger diameter minimal connected networks with a multi-recipient characterizing player or with a non-recipient characterizing player meeting the condition that no player wants to change a link as those that meet the derived rule. Intuitively, for small levels of decay, any player sponsors a link "in the middle" of any component that he faces, leaving about the same players on one side and on the other side of the node he sponsors a link to, where because of small decay, the distance of these players on one and on the other side does not matter much. Because of this tendency of any sponsor of leaving about the same players on one side and on the other side, we refer to this as the *balancing condition*.

Moreover we show that all candidate networks satisfying the balancing condition are SNN for some range of parameters. Hence every such network is indeed relevant: the characterization contains no networks which cannot be a SNN after all.

[20] These candidate networks satisfy Lemma 3, Proposition 1, and a 'balancing condition' as will be defined in Section 5.

Then, in Section 5.3, we calculate the maximum diameter of candidate networks satisfying the balancing condition as a function of n. An implication of the result there is that there is no maximum diameter if the population size can be chosen freely to accommodate that diameter. From the last two results it follows that that there are SNN with quite large diameters, although the diameter remains relatively small with respect to the population size[21].

In Section 5.4 we show that for small decay the player most likely to delete his links will be the end sponsor. Since by small decay and the balancing condition no player wants to add or replace links, the player most willing to delete his link(s) is the 'crucial player'. This allows us, combined with the results in Section 5.3, to derive a last result, which says that the diameter gap can be very large: namely that all SNNs either have diameter two (then they are a PSS) or the maximal diameter of all candidate networks satisfying the balancing condition.

5.1 The balancing condition

In this section we introduce the balancing condition. It basically says that if δ is close enough to 1 and the network is a connected Nash network, then the recipient of each link is in the middle of the group that the sponsor connects to through the link. More specifically, if he would sponsor the link to some neighbor of the recipient instead, there are less players that he would get closer to than players that he would get further away from. To formalize this result, we will first define the balancing condition and then a new threshold decay level.

Definition 8 *Network g satisfies the balancing condition if for any $ij \in g$ we have that $|A_{ij}(g)| > 2\left|A_{\overline{jk}}(g)\right|$ for all $\overline{jk} \in g$.*

Definition 9 *Let $\delta_B(c, n, f(I))$ be the lowest level of decay such that for all $\delta > \delta_B(c, n, f(I))$ all candidate networks which are SNN satisfy the balancing condition.*

We now show $\delta_B(c, n, f(I)) < 1$.

Proposition 7 *For any $c > 0$, $n \geq 3$, and $f(I) > 0$ we have $\delta_B(c, n, f(I)) < 1$.*

Proof. Denote by i_x^d a player at distance d from player i in component $g_{A_{ii'}(g)}$. Denote by $d_{\max}$ the maximal distance between i and any other player in $g_{A_{ii'}(g)}$. Finally, denote by $N_i^s\left(g_{A_{ii'}(g)}, i_x^d\right)$ the set of players at distance d from player i in network $g_{A_{ii'}(g)}$ to which player i_x^d gives access.

We now derive the condition under which a player i sponsoring a link to i' in g does not instead want to sponsor a link to any player i_x^2 at distance 2

[21] The minimum population size needed for a diameter d candidate network which satisfies the balancing condition is exponentially increasing in the diameter.

from player i in network $g_{A_{ii'}(g)}$. The marginal information gain to player i of sponsoring a link to player $i' = i_x^1$ in $g_{A_{ii'}(g)}$ equals

$$\sum_{s=1}^{d_{\max}} \delta^s \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^1 \right) \right| .$$

Consider next the marginal information gain to player i of sponsoring a link to a player i_x^2 at distance 2 from player i in network $g_{A_{ii'}(g)}$. Note that $N_i^r \left(g_{A_{ii'}(g)}, i_x^r \right) = 1$, and $N_i^s \left(g_{A_{ii'}(g)}, i_x^r \right) = 0$ if $s < r$. The marginal information gain equals

$$\sum_{s=1}^{d_{\max}} \left\{ \delta^{s+1} \left[\left| N_i^s \left(g_{A_{ii'}(g)}, i_x^1 \right) \right| - \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right) \right| \right] + \delta^{s-1} \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right) \right| \right\} .$$

This formula follows from observing that all players in set $N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right)$ are one step closer than before by switching the link from ii_x^1 to ii_x^2, while the remainder of the players in $A_{ii_x^1}(g)$, namely $\left[\left| N_i^s \left(g_{A_{ii'}(g)}, i_x^1 \right) \right| - \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right) \right| \right]$, are now one step further away. So i strictly prefers to keep his link, rather than replacing it with a link to i_x^2 if the cost of getting one step further away from some agents ($\left[\left| N_i^s \left(g_{A_{ii'}(g)}, i_x^1 \right) \right| - \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right) \right| \right]$), outweighs the benefits of getting one step closer to some other agents ($N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right)$). This is the case if and only if

$$\sum_{s=1}^{d_{\max}} \left\{ \left[\delta^s - \delta^{s+1} \right] \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^1 \right) \right| + \left[\delta^{s+1} - \delta^{s-1} \right] \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right) \right| \right\} > 0 \tag{B}$$

Define as $\delta_B < 1$ the largest root of the polynomial on the left-hand side of (B) that is smaller than 1. Another root of (B) is $\delta = 1$, reflecting the fact that in the absence of decay, it does not matter where the player connects. If the derivative with respect to δ of the left-hand side of (B) is negative at $\delta = 1$, then a range of large δs exists such that the player prefers to connect to i_x^1 rather than to any i_x^2. This is the case if

$$- \sum_{s=1}^{d_{\max}} \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^1 \right) \right| + 2 \sum_{s=1}^{d_{\max}} \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right) \right| < 0$$

or

$$\sum_{s=1}^{d_{\max}} \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^1 \right) \right| > 2 \sum_{s=1}^{d_{\max}} \left| N_i^s \left(g_{A_{ii'}(g)}, i_x^2 \right) \right|$$

However, this only shows that the condition stated in the proposition is a necessary condition, as we have only derived a condition assuring that no player wants to reconnect to a player at distance 2 in the accessed component. To see that this condition is also a sufficient condition, consider a path $\left\{\overline{i_x^1 i_x^2}, \overline{i_x^2 i_x^3}, ..., \overline{i_x^p i_x^{p+1}}, \overline{i_x^{p+1} i_x^{p+2}}, ..., \overline{i_x^{z-1} i_x^z}\right\}$ in $g_{A_{ii'}(g)}$, where node i_x^z has no other links then $\overline{i_x^{z-1} i_x^z}$. Then by the given condition, player i prefers connecting to i_x^p rather than to i_x^{p+1} if $\left|A_{ii_x^p}(g)\right| > 2\left|A_{\overline{i_x^p i_x^{p+1}}}(g)\right|$. Note now that $\left|A_{ii_x^p}(g)\right| = |A_{ii'}(g)|$, whereas $\left|A_{\overline{i_x^p i_x^{p+1}}}(g)\right| < \left|A_{\overline{i'j}}(g)\right|$. Thus, it follows from the condition that player prefers linking to any player p rather than to player $(p+1)$ along the path. The condition is thus sufficient. ■

The balancing condition translates Lemma 3 into a structural property of the network, provided that there is little enough decay. Let $\delta > \delta_B(c, n, f(I))$), and take any minimal network g which satisfies Lemma 3. Then for any link $ij \in g$ we find that $j \in A_{ij}^I(g, \delta)$ if and only if the balancing condition holds for link ij. We say that a network g satisfies the balancing condition if for any $ij \in g$ in a minimal connected SNN g, $|A_{ij}(g)| > 2\left|A_{\overline{jk}}(g)\right|$ for any $jk \in g$.

This gives us the following corollary.

Corollary 4 *Let $\delta > \max\{\delta_R(c, n, f(I)), \delta_B(c, n, f(I))\}$. A given network satisfies Lemma 3 if and only if it meets the balancing condition.*

We say that a network g which satisfies the balancing condition is a *balanced network*. A candidate network which satisfies the balancing condition is thus a *balanced candidate network*.

We continue in the following section by proving that any balance candidate network is indeed a SNN for some positive range of parameters.

5.2 Are balanced candidate networks indeed SNN?

The main problem with having only necessary conditions for SNNs in a characterization is that the characterization may include networks which are not SNN. In this subsection we show that this problem does not apply to balanced candidate networks. In other words: for each balanced candidate network there exists some feasible positive range of parameters and some class of benefit functions such that this balanced candidate network is in fact a SNN.

Proposition 8 *Consider a network g which satisfies the Properties of Proposition 1 and the Balancing Condition. Then there exists some range of parameters δ and c and a value function $f(I)$, $f' > 0$, such that g is a SNN and $\delta > \delta_R(c, n, f(I))$.*

Proof. We prove this by construction. For simplicity, consider a benefit function of the following form: $f(I) = I^k$, where $k > 0$. If $k > 1$, we have IMBI, for $k = 1$ we have CMBI and for $k < 1$ we have DMBI. We will first look at the incentives to delete a link. After that we will consider the incentives to add a link, and to the incentives to replace a link.

Note that the addition of a minimal link gives not less than δ additional information. This implies that the added benefit of such a link is at least $f(1+\delta) - f(1)$ if $k \geq 1$, while it is at least $f(1+(n-1)\delta) - f(1+(n-2)\delta)$ for $k < 1$. A player has a strict preference not to delete a link, if its costs are strictly less than its benefits. The minimum benefits are, as note above,

$$\begin{aligned} b^{\min}(\delta, f(I)) &= \min\left\{f(1+\delta) - f(1),\ f(1+(n-1)\delta) - f(1+(n-2)\delta)\right\} \\ &= \min\left\{(1+\delta)^k - 1,\ (1+(n-1)\delta)^k - (1+(n-2)\delta)^k\right\}. \end{aligned}$$

Then, picking some arbitrary $\delta^* \in (0,1)$, and choosing $c < b^{\min}(\delta^*, f(I))$, we obtain that no player prefers to delete a link in network g.

Second, we will look at the incentives to add links. We start by noting that, regardless of k, $b^{\min}(\delta)$ is non-decreasing in δ.[22] So if $\delta > \delta^*$, and $c < b^{\min}(\delta^*)$ then $c < b^{\min}(\delta)$. Hence we can pick a $\delta \geq \delta^*$ close enough to 1, specifically $\delta > \max\{\delta^*, \delta_M(c, n, f(I))\}$, such that no player wishes to sponsor any non-minimal link. Because $b^{\min}(\delta, f(I))$ is non-decreasing in δ, it remains to be true that no player wishes to delete a link.

Thirdly, we consider the incentives to replace a link. Network g satisfies the balancing condition, hence by Proposition 7 there exists some $\delta_B(c, n, f(I))$ such that for all $\delta > \delta_B(c, n, f(I))$ we have that $j \in A^I_{ij}(g, \delta)$ for any $ij \in g$. This implies that no player is willing to replace that link because doing so would reduce the information gathered without saving any costs.

Summarizing:

- given an arbitrary $\delta^* \in (0,1)$ and
- a value function $f(I)$ such that $b^{\min}(\delta)$ increases in δ (e.g. $f(I) = I^k$, $k > 0$),
- there exists a range of c (namely $c < b^{\min}(\delta^*)$)
- and δ (specifically $\delta > \max\{\delta^*, \delta_M(c, n, f(I)), \delta_B(c, n, f(I))\}$)
- for which no player wants delete a link, add or replace a link.

This proves the Proposition. ■

Having shown that each balanced candidate networks is indeed a SNN for the right parameters, we are interested in the relationship between the maximal di-

[22] It is obvious that $(1+\delta)^k - 1$ is increasing in δ. The other requirement is that $\frac{\partial}{\partial \delta}\left((1+(n-1)\delta)^k - (1+(n-2)\delta)^k\right) \geq 0$, which simplifies to $(n-1)^{\frac{1}{1-k}} - (n-2)^{\frac{1}{1-k}} \geq (n-1)(n-2)\delta\left((n-2)^{\frac{k}{1-k}} - (n-1)^{\frac{k}{1-k}}\right)$. Since the LHS is positive and the RHS is negative, this is satisfied, and $b^{\min}(\delta, f(I))$ is non-decreasing in δ.

ameter of balanced candidate networks and the population size. We investigate this in the following section.

5.3 Maximal diameter of balanced candidate networks

The balancing condition implies that large-diameter networks may be SNN. The PSS is then only one out of many possible networks. To illustrate the possibility of large diameter networks, we now derive the maximal-diameter SNN for all $n \geq 5$ (for smaller n, only stars are SNN). We start by deriving a Lemma that we need to find the maximal-diameter SNN.

Lemma 12 *Consider a minimally connected balanced network g. Let $ij \in g$ be such that it gives access to an end recipient at distance s from player i. Then*

1. *If i gets access to the end recipient at distance s through a link sponsored by player j, then in the component to which player j gives access, player i has at least $\sum_{l=1}^{x+1} 2^l$ players at distance $(s-x)$ or larger from her, where $0 \leq x \leq (s-2)$.*

2. *For $s \geq 3$, if i gets access to the end recipient at distance s through a link received by j other than ij, then in the component to which player j gives access (a) if additionally $s \geq 4$, player i has at least $\sum_{l=1}^{x+1} 2^l$ players at distance $(s-x)$ or larger from her, where $0 \leq x \leq (s-4)$; (b) Player i has at least $1 + \sum_{l=1}^{s-3} 2^l$ players at distance 3 or larger in component $g_{A_{ij}(g)}$; and (c) at least $4 + 2\sum_{l=1}^{s-3} 2^l$ players at distance 2 or larger in component $g_{A_{ij}(g)}$.*

Proof. We first prove 1. and 2(a). If an end sponsor k receives a link from a player i, then by Property 4, k should sponsor at least two links.

Let an end sponsor k receiving a link from a player l sponsor exactly two end links. Then, if player l is himself a sponsor, then by the balancing condition (see Proposition 7), player l's links should point to at least 6 players at distance 1 or 2 from her, and at least 2 players at distance 2 from her.

Let player l's links point to exactly 6 players, at distance 1 or 2 from her, and let player l herself receive a link from player h. Then, if player h is himself a sponsor, his links should point to at least 14 players at distance 1, 2 or 3 from her, at least 6 players at distance 2 or 3 from her, and at least 2 players at distance 3 from her. And so forth.

We next prove 2(b). This follows simply from the fact that there must be at least one player at distance 3, and as already shown, at least $\sum_{l=1}^{s-3} 2^l$ players at distance 4 or larger.

In order to prove 2(c), note first that given the above, there is at least one player k connected to player j who gives access to at least $1+\sum_{l=1}^{s-3}2^{l}$ players. This means that $\left|A_{\overline{jk}}(g)\right| = 2+\sum_{l=1}^{s-3}2^{l}$, meaning that by the balancing condition, $|A_{ij}(g)| > 4+2\sum_{l=1}^{s-3}2^{l}$. ■

We now derive the maximal diameter networks that can be achieved for any n. In order to do this, we derive the minimal number of nodes n that are needed to achieve a network of given diameter d. For this n, and for a range of populations just above it, this is then also the maximal diameter network that can be achieved.

Proposition 9 *Consider any odd x, with $x \geq 3$. For n such that $1+2^{(x-1)/2}+2\sum_{l=1}^{(x-1)/2}2^{l-1} \leq n < 4+3\sum_{l=1}^{(x-1)/2}2^{l-1}$ (where $(4+3\sum_{l=1}^{(x-1)/2}2^{l-1})-(1+2^{(x-1)/2}+2\sum_{l=1}^{(x-1)/2}2^{l-1}) = 2$), the maximal-diameter balanced candidate networks have diameter $d = x$. For n such that $4+3\sum_{l=1}^{(x-1)/2}2^{l-1} \leq n < 1+2^{(x+1)/2}+2\sum_{l=1}^{(x+1)/2}2^{l-1}$ (where $(1+2^{(x+1)/2}+2\sum_{l=1}^{(x+1)/2}2^{l-1})-(4+3\sum_{l=1}^{(x-1)/2}2^{l-1}) >> 2$), the maximal-diameter balanced candidate networks have diameter $d = (x+1)$.*

Proof. It suffices to derive the minimal number of nodes $n_1(d_1)$ needed to achieve a network with any even diameter d_1, and the minimal number of nodes $n_2(d_2)$ needed to achieve a network with any odd diameter d_2. Once we have these results, it immediately follows that for any n with $n_1(d_1) < n < n_2(d_2)$, the maximal achievable diameter is d_1, and for any n with $n_2(d_2) < n < n_1(d_1)$, the maximal achievable diameter is d_2.

Even diameter and minimal number of nodes. We here show that in order to construct a candidate network that meets the balancing condition and has even diameter d_1 with $d_1 \geq 4$, one needs at least $n = 4+3\sum_{l=1}^{d_1/2-1}2^{l-1}$ nodes.

We *first* show that the minimal number of nodes needed to achieve a network with a *multi*-recipient characterizing player of even diameter $d_1 \geq 4$ is the given n. By the definition of a multi-recipient player, at least two of the nodes at distance 1 from such a characterizing player must sponsor links towards him

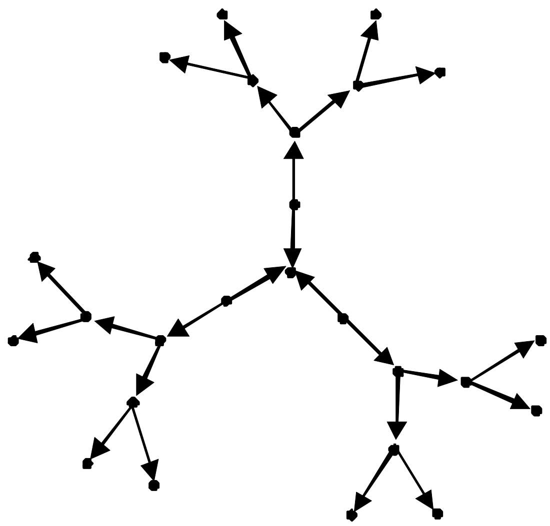

Figure 3: Maximal-diameter network for n = 25.

(though the characterizing player may sponsor links himself as well). Furthermore, it follows from Lemma 12 and from the fact that $4+2\sum_{l=1}^{s-3} 2^l < \sum_{l=1}^{s-1} 2^l$ that in order to put an end recipient at any distance s from a sponsor i of the multi-recipient characterizing player j using a minimal number of nodes, it should be that i accesses this end recipient through a sponsor k of the multi-recipient characterizing player, and not through a link sponsored by the characterizing player. Thus, in order to construct with a minimal number of nodes a network with a multi-recipient characterizing player where two end recipients l and m are at distance d_1 from each other, we can limit ourselves to networks where sponsor i (respectively sponsor k) of the characterizing player j accesses end recipient l (m) through sponsor k (i) of the characterizing player.

Consider in particular such a diameter-d_1 with two end recipients each at distance $d_1/2$ of the multi-recipient characterizing player, and let us derive the minimal number of nodes necessary to construct such a network. By Lemma 12, sponsor i (respectively k) of the characterizing player must give sponsor k (i) access to at least $1 + \sum_{l=1}^{d_1/2-1} 2^{l-1}$ nodes (where i (k) is included himself). This means that sponsor k (respectively i) has this same number of nodes at distance 2 or more in the component that includes i (k) and the links sponsored by i (k). Applying the balancing condition now, it follows that k (i) should have at least this same number of links at distance 2 or more in components to which characterizing player j gives access, but to which i (k) does not give access. It follows that $n = 4 + 3 \sum_{l=1}^{d_1/2-1} 2^{l-1}$ is the minimal number of nodes necessary to construct a diameter-d_1 with two end recipients at opposite sides each at distance $d_1/2$ from the characterizing player.

To show that a balanced candidate network with such a number of nodes indeed exists, consider a two-sponsor network where exactly three players sponsor a link to the multi-recipient characterizing player. There are exactly $3 * 2^{l-1}$ players at distance l from the characterizing player, with $2 \leq l \leq d_1/2-1$, whose links point away from the characterizing player. Such a network indeed meets Property 4 of Proposition 1, and meets the balancing condition. An example is the network in Figure 3 for $n = 25$ and $d_1 = 8$.

Next, we show that alternative diameter-d_1 networks with a multi-recipient characterizing player, where at one end of the network an end recipient is at distance smaller than $d_1/2$ of the characterizing player, and at the other end of the network an end recipient is at distance larger than $d_1/2$, use a larger number of nodes. By Lemma 12, due to the exponentially increasing number of nodes needed to increase the maximal distance between the characterizing player and an end recipient, decreasing the distance between the characterizing player and an end recipient on one side of the network and increasing this distance at the other side of the network while maintaining diameter d_1 is only possible when more nodes are used.

Second, we show that with the given $n = 4 + 3 \sum_{l=1}^{d_1/2-1} 2^{l-1}$, one cannot construct a network with a *non*-recipient characterizing player that achieves a higher diameter than d_1. In order to show this, we show that, for even d_1, the minimal number of nodes needed to achieve a diameter-d_1 network with a *non*-recipient characterizing player is always at least as high as the minimal number of nodes needed to achieve a diameter-d_1 network with a multi-recipient characterizing player. Consider a diameter-d_1 network with a non-recipient characterizing player with two end recipients at distance d_1 from one another, and each at distance $d_1/2$ from the characterizing player. In such a network, the characterizing player sponsors at least two links. By Lemma 12, the characterizing player has at least $2 \sum_{l=1}^{d_1/2-1} 2^{l}$ nodes at distance 2 or more from him. Together with the minimum of two players at distance 1, this means a minimum of $2 + 2 \sum_{l=1}^{d_1/2-1} 2^{l} = 2 \sum_{l=1}^{d_1/2} 2^{l-1}$ nodes at distance 1 or more. Thus, to construct such a network, we need a minimum of $n = 1 + 2 \sum_{l=1}^{d_1/2} 2^{l-1}$ nodes.

To show that a network using this number of nodes indeed exists, consider a symmetric network where there are exactly 2^{l-1} nodes at distance l from the characterizing player, with $1 \leq l \leq d_1/2$, and note that this network meets Property 4 of Proposition 1, and meets the balancing condition. By Lemma 12, decreasing the distance between the characterizing player and an end recipient at one side of characterizing player in order to increase the distance between the characterizing player and an end recipient on the other side of the characterizing player is only possible with the use of more nodes. It follows that $n = 1+2\sum_{l=1}^{d_1/2} 2^{l-1}$ is the minimal number of nodes with which we can construct a diameter-d_1 network with a non-recipient characterizing player.

Finally, note that for $d_1 \geq 4$, $1+2\sum_{l=1}^{d_1/2} 2^{l-1} \geq 4+3\sum_{l=1}^{d_1/2-1} 2^{l-1}$ (where equality is obtained only for $d_1 = 4$). It follows that to construct a network with even diameter $d_1 \geq 4$, one needs at least $4 + 3\sum_{l=1}^{d_1/2-1} 2^{l-1}$ nodes.

Odd diameter and minimal number of nodes. Next, we show that in order to construct a candidate network that meets the balancing condition and has odd diameter d_2 with $d_2 \geq 3$, one needs at least $n = 1 + 2^{(d_2-1)/2} + 2\sum_{l=1}^{(d_2-1)/2} 2^{l-1}$ nodes. Consider a diameter-d_2 network with two end recipients at distance d_2 from each other, where one end recipient is at distance $(d_2-1)/2$ of the non-recipient characterizing player, and the other end recipient at distance $(d_2+1)/2$. In such a network, the characterizing player has at least two players at distance 1. By Lemma 12, the characterizing player has at least $\sum_{l=1}^{(d_2-1)/2-1} 2^l + \sum_{l=1}^{(d_2+1)/2-1} 2^l$ at distance 2 or more from him. Together with the minimum of two nodes at distance 1, this means a minimum of $\sum_{l=1}^{(d_2-1)/2} 2^{l-1} + \sum_{l=1}^{(d_2+1)/2} 2^{l-1}$ nodes at distance 1 or more. Thus, in order to construct a diameter-d_2 network of the given type, we need at least $n = 1 + 2^{(d_2-1)/2} + 2\sum_{l=1}^{(d_2-1)/2} 2^{l-1}$ nodes. To show that a network using this number of nodes indeed exists, consider a network where the non-recipient accesses two components. In each of these components, there are exaclty 2^{l-1} nodes at distance l from the characterizing player, with $1 \leq l \leq (d_2-1)/2$. In one of the two components, there are additionally exactly $2^{(d_2-1)/2}$ players at distance $(d_2-1)/2$ from the characterizing player. An example is the network in Figure 4 for $n = 23$ and $d_2 = 7$. Note that such a network meets Property 4 of Proposition 1, and meets the balancing condition.

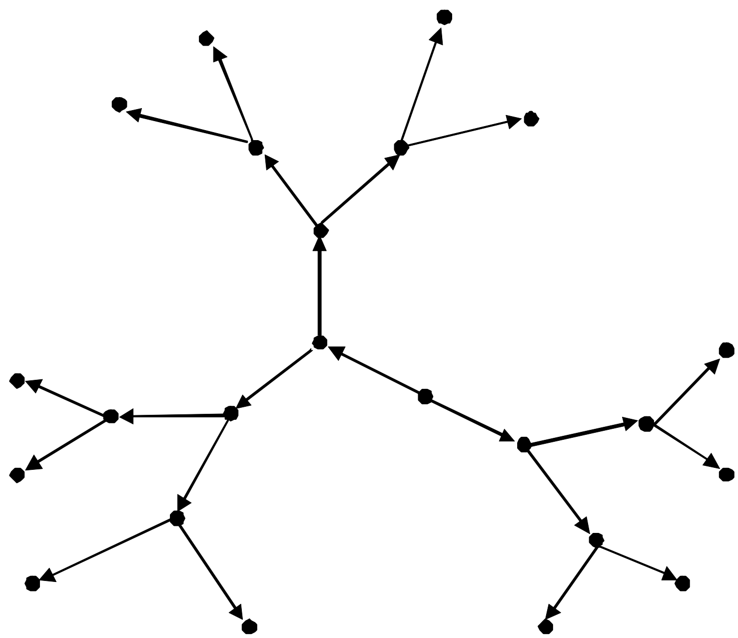

Figure 4: Maximal diameter network for $n = 23$.

Next, note that by Lemma 12, due to the exponentially increasing number of nodes needed to increase the maximal distance between the characterizing player and an end recipient, decreasing the distance between the characterizing player and an end recipient on one side of the network and increasing this distance at the other side of the network while maintaining diameter d_2 is only possible when additional nodes are used. It follows that the given n is the minimal number of nodes need to construct a diameter-d_2 network with a non-recipient characterizing player.

Second, we show that in order to achieve a network with odd diameter d_2 in a network with a *multi*-recipient characterizing player, one needs more nodes than in the specified network with a non-recipient characterizing player. By the analysis above, in order to achieve a network with even diameter $(d_2 - 1)$, the minimal number of nodes is achieved in e.g. the two-sponsor network with a multi-recipient characterizing an example of which is given in Figure 3. Note now that applying the balancing condition to the sponsors of the characterizing player, in order to increase the diameter of such a network to d_2, one needs to add at least the same number of nodes as are needed to increase the diameter to (d_2+1). As $4+3\sum_{l=1}^{(d_2+1)/2-1} 2^{l-1} - 1 - 2^{(d_2-1)/2} - 2\sum_{l=1}^{(d_2-1)/2} 2^{l-1} = 2$, this means that more nodes are needed than to achieve diameter d_2 for the network with a non-recipient characterizing player.

Other numbers of nodes. Note finally that for n such that $1+2^{(x-1)/2}+2\sum_{l=1}^{(x-1)/2} 2^{l-1} \leq n < 4+3\sum_{l=1}^{(x-1)/2} 2^{l-1}$, one can achieve diameter $d=x$ e.g. by letting the non-recipient characterizing player sponsor additional links. For n such that $4+3\sum_{l=1}^{(x-1)/2} 2^{l-1} \leq n < 1+2^{(x+1)/2}+2\sum_{l=1}^{(x+1)/2} 2^{l-1}$, one can achieve diameter $d=(x+1)$ e.g. by giving the multi-recipient characterizing player extra sponsors. ■

We conclude from this section that networks of any diameter can be balanced candidate networks, provided that n is large enough. Proposition 8 implies that any such network can indeed be a SNN. Therefore SNNs may have any finitely large diameter, if the population size can be freely chosen. This result differs significantly from the small diameter networks which were reported by earlier studies on this subject, namely the stars and interlinked stars (diameter 3) in Bala and Goyal (2000), and the PSS in Hojman and Szeidl (2008).

5.4 Diameter gap revisited

In this section, we show that the results of Proposition 5 and Corollary 3 extend to larger diameter networks. Specifically, we show that if there is little enough decay then cost levels exist such that under DMBI only the PSS and the maximal-diameter network are stable. We show that for small enough decay, the sponsor who is most likely to delete a link is an end sponsor. As such we will call him the *crucial sponsor*. Note that we established before that for small enough decay (i) no player wants to add a non-minimal link, and (ii) no player wants to change his link if the network is balanced. Consequently, in order to check that no player wants to delete a link in any candidate SNN that is not a PSS, it suffices to focus on the end sponsors in this network. With this in mind, we then compare the marginal benefits of the crucial end sponsors across all non-PSS candidate SNN. For DMBI, the networks with the least-informed crucial end sponsor are the most stable non-PSS networks, in that we can then find a cost level c such that the only stable networks are the PSS and these networks.

We start by showing that, for small decay, any crucial sponsor is an end sponsor. In order to show this, we first show this to be the case in the absence of decay.

Lemma 13 *Let $\delta=1$. Then in any minimal connected network g, any end sponsor i has a smaller marginal benefit of sponsoring an end link ii', $ii' \in g$, than any sponsor j has to sponsor non-end link $jj' \in g$.*

Proof. In the absence of decay, the marginal benefit of sponsoring a link of any sponsor i sponsoring a link to j equals $f(N) - f(N-x)$, where $x \geq 1$ is the number of players that player i accesses by sponsoring a link to j. Given that $f' > 0$, the sponsor has a larger marginal benefit the more players he accesses, and any end sponsor thus has the smallest marginal benefit. ∎

By continuity, it follows that for sufficiently small decay, any non end sponsor has a larger marginal benefit than any end sponsor.

Lemma 14 *A level of decay $\delta_E(c, n, f(I))$ exists such that for all $\delta > \delta_E(c, n, f(I))$ and for every minimally connected network g, any end sponsor i has a smaller marginal benefit of sponsoring a single link than any non end sponsor j.*

Proof. Given Lemma 13, in the absence of decay, the marginal benefit of sponsoring an end link in *any* network with end links equals $f(N) - f(N-1)$, and is strictly smaller than the marginal benefit of sponsoring any non end link. It follows by continuity that for levels of δ close enough to 1 this continues to be true. ∎

For DMBI, it follows that in order to check whether no player wants to delete any links, it suffices to check that a single end sponsor, the crucial end sponsor, does not want to delete a single link. By DMBI, the crucial end sponsor of a network is the end sponsor who has the most information. Each network has at least one crucial end sponsor. After identifying a crucial end sponsor for each candicate balanced network, the benefits of these crucial end sponsors of sponsoring a single end link can then be compared across these networks. The size of this benefit depends on the crucial end sponsor's ex ante information. We now look for the network whose crucial end sponsor has the smallest possible ex ante information, as this is the network where the end sponsor is most motivated to keep his link. Intuitively, the crucial end sponsor in a network tends to have less ex ante information the larger the diameter of the network. We show that this intuition is valid for a particular n, where the SNN with maximal diameter has the crucial end sponsor with the least ex ante information across all non-PSS SNN.

Lemma 15 *Consider the balanced candidate networks. Then for $n = 4 + 3\sum_{l=1}^{d_1/2-1} 2^{l-1}$ (with d_1 even, and $d_1 \geq 4$), a level of decay $\delta_D(c, n, f(I))$ exists such that for every $\delta > \delta_D(c, n, f(I))$, the networks with the crucial end sponsors with the least ex-ante information have the maximal diameter that can be achieved for this n.*

Proof. Consider for the given n the two-sponsor maximal-diameter network architecture described in the proof of Proposition 9 (see Figure 3 for an example). Below we will show that the crucial end sponsors of lower diameter networks have more information than the crucial end sponsors in this two-sponsor network. Because this two-sponsor network has the maximal diameter it follows that the network with the crucial end sponsors with the least ex-ante information have the maximal diameter that can be achieved for this n.

In order to compare the information of crucial end sponsors across networks, compare in general the ex ante information of a crucial end sponsor i in one candidate SNN, and the ex ante information of a crucial end sponsor k in another candidate SNN. Crucial end sponsor i's ex ante information takes the form $N_i^1\delta + N_i^2\delta^2 + N_i^3\delta^3 + ...$, and crucial end sponsor k's ex ante information takes the form $N_k^1\delta + N_k^2\delta^2 + N_k^3\delta^3 + ...$ Crucial end sponsor i has less ex ante information than crucial end sponsor k if

$$(N_i^1 - N_k^1)\delta + (N_i^2 - N_k^2)\delta^2 + (N_i^3 - N_k^3)\delta^3 + ... < 0.$$

For $\delta = 1$, the two crucial end sponsors have exactly the same information, and the expression equals zero. The slope of the polynomial $(N_i^1 - N_k^1)\delta + (N_i^2 - N_k^2)\delta^2 + (N_i^3 - N_k^3)\delta^3 + ...$as a function of δ, valued at 1, is larger than zero if

$$(N_i^1 - N_k^1) + 2(N_i^2 - N_k^2) + 3(N_i^3 - N_k^3) + ... > 0. \quad (4)$$

If inequality (4) is valid, then for a range of δ close to 1 exists such that crucial end sponsor i has less ex ante information than crucial end sponsor k. Given the coefficients in this inequality, this means that for small decay, the number of nodes far away from a player matter more in determining a player's comparative level of ex ante information than the number of nodes close by.

We now apply this result to compare the information of the crucial end sponsor in the two-sponsor network to the information of an end sponsor in a balanced candidate network with diameter smaller than d_1. We can consider any smaller diameter network as a network in which one or more nodes have come closer to crucial end sponsor i in the two-sponsor network. We focus on the comparison with balanced candidate networks with diameter $(d_1 - 1)$. In any network of the latter type, end sponsor i will now have $2 * 2^{d_1/2-2}$ nodes (namely the end nodes at distance $(d_1 - 1)$ from him in the two-sponsor network) closer by. Let us next look at the number of nodes that could be further from him. The $2 * 2^{d_1/2-3}$ nodes at distance $(d_1 - 2)$ from end sponsor i in the two-sponsor network cannot be put further away, since otherwise we would again obtain a network with diameter d_1. Finally, in order to maintain diameter $(d_1 - 1)$, of the nodes at distance smaller than $(d_1 - 1)$ to the end sponsor, along a line, at least $(d_1 - 1)$ nodes should remain at the same distance of the end sponsor (including the end sponsor himself, who is at a distance of zero from himself). It follows that the end sponsor i can have at most $(n - 2 * 2^{d_1/2-3} - 2 * 2^{d_1/2-2} - (d_1 - 1))$ nodes further away from him than in the original network.

As $(n - 2 * 2^{d_1/2-3} - 2 * 2^{d_1/2-2} - (d_1 - 1)) < 2 * 2^{d_1/2-2}$, by the decrease in diameter, the end sponsor gets more nodes closer by than further off. Moreover, the nodes that come closer are the ones that used to be furthest away. Given that in the inequality $(N_i^1 - N_k^1) + 2(N_i^2 - N_k^2) + 3(N_i^3 - N_k^3) + ... > 0$, nodes further away count more, it follows that for small decay the end sponsor has more information in the modified network. It is easy to see now that this effect is even more pronounced for larger decreases in diameter away from d_1: even more nodes will then come close to an end sponsor, and even less nodes can be put further away. ■

An example of a maximal-diameter network with crucial end sponsors with the smallest possible ex-ante information is the network in Figure 3 for the case $n = 25$. We can now use Lemma 15 to show that for the specified n, levels of c exist such that the only stable networks are the PSS and networks with maximal diameter. Thus, e.g. for $n = 25$, relatively large levels of linking costs exist such that the only two stable network architectures are the PSS (diameter 2), and the maximal-diameter network, such as Figure 3 (diameter 8) for $n = 25$.

Proposition 10 *Consider the set of balanced candidate networks. Moreover let the benefit function be a DMBI function and let* $\delta > \max\{\delta_R(c,n,f(I)), \delta_B(c,n,f(I)), \delta_E(c,n,f(I)), \delta_D(c,n,f(I))\}$. *Consider the case* $n = 4+3\sum_{l=1}^{d_1/2-1} 2^{l-1}$ *(with* d_1 *even). Consider now the crucial end sponsor of the network with the crucial end sponsors with the least ex-ante information. Let* MB^* *denote the marginal benefit to this crucial end sponsor of one of his end links. Then*

1. *For* $MB^* \leq c < f(1+\delta+(N-2)\delta^2) - f(1)$*, the only non-empty SNN are the PSS.*

2. *A cost level* c^* *strictly smaller than* MB^* *exists such that for* $c^* \leq c < MB_{\max}$*, the only SNN are the PSS and at least one of the maximal-diameter networks.*

Proof. By Lemma 15, for the specified n, the network with the least ex-ante informed crucial end sponsor has the maximal diameter among the set of balanced candidate networks. Under DMBI, this means that this network has the crucial end sponsor with the largest marginal benefit of sponsoring an end link, MB^*. By Proposition 4, the marginal benefit of a link in the PSS is strictly larger than that of any other link in any other network. It follows that both cost levels must exist where the PSS are the only SNN, and where the specified maximal-diameter networks and the PSS are the only SNN. ■

Propositions 4 and 10 together show the following for small enough decay and DMBI. For small linking costs, a wide range of networks is stable, including large diameter networks. As the cost of linking is gradually increased, the first non-empty networks to become unstable are the non-PSS stars. For specific n, as the cost of linking is further increased, the one-but-last type of networks to leave the set of non-empty SNN are the maximal-diameter networks where the distance between any two end sponsors is either 2 or is equal to the maximal diameter. Eventually, the last type of non-empty networks to leave the set of SNN are the PSS, after which the set of non-empty SNN is empty.

Yet, one should not conclude from this that there is a monotone relationship between the diameter of non-PSS balanced candidate networks and stability.

For example, take the network in Figure 3, and in order to construct a stable network for the case $n = 26$, modify the network by adding one player j who sponsors a link to the characterizing player in the network, i. Then the end sponsors continue to all be crucial end sponsors, and this large-diameter network continues to be relatively stable under DMBI. Yet, if we modify this network now only by letting i sponsor the link to j instead of the other way around, player i becomes the crucial end sponsor, and has more information than the other end sponsors. This small change, which leaves the diameter of the network unchanged, significantly reduces the range of c for which this modified network is stable under DMBI.

6 Conclusion

The purpose of this paper is to investigate what happens when slightly deviating from a standard network formation model without decay by introducing small levels decay. We wish to highlight three of the contributions of this paper. A *first* contribution has been to characterize such networks. This characterization shows for instance that there is at most one player receiving multiple links, that links tend to be outward oriented and that large diameter networks can be stable, even though these networks are inefficient.

A *second* contribution of this paper is to show a sufficiency result, namely that every network satisfying a certain set of properties is a strict Nash network for some range of parameters.

A *third* contribution of this paper is to analyze the relative stability of different network architectures. A general result is that if any non-empty network is stable, then all periphery-sponsored stars are stable. This is because the characterizing player in this case has the highest possible quality of information, and connecting to him is an all-or-nothing decision. Under constant marginal benefits of information, if linking costs are decreased sufficiently to make the periphery-sponsored star stable, then all possible networks including large-diameter networks simultaneously become stable. Under decreasing marginal benefits of information, the second-most stable networks are large-diameter networks. Intuitively, it is the fact that players have relatively little information in a large-diameter network that gives them strong incentives to sponsor their links. Under increasing marginal benefits of information, the second-most stable networks are other star networks. Intuitively, it is the fact that the characterizing player already has a lot of information that under increasing marginal benefits of information still gives him an incentive to sponsor another link.

While these results are intuitive, we should still formulate some caveats . *First*, our analysis shows that in order to construct a larger-diameter network, one needs an exponentially increasing number of players. This is because each recipient should sponsor at least two links. Thus, within the parameters of our analysis, this puts a bound on the diameters that can be reached. For instance, the maximal diameter of any stable network for 10 players is 4, but the maximal diameter for 100 players is only 12. *Second*, for larger levels of decay than in our analysis, when any players are at a large distance from a given player in a network, this player may find it worth to form a non-minimal link, thus bringing this player closer. For larger levels of decay, this puts a further bound on the maximal diameter of networks. *Third*, the larger the number of players, the larger the diameter of the maximal-diameter the network, and the more incentives the player would seem to have to sponsor a non-minimal link. In this sense, for a large number of player, the stability of the maximal-diameter networks would not seem to be plausible. Yet, as we have shown, while the maximal diameter increases with the number of nodes, it increases at a sharply decreasing rate. Also, under decreasing marginal benefits, for which the stability of maximal-diameter networks was argued, increasing the number of players also means giving a player less incentives to sponsor a non-minimal link. Indeed, adding nodes to a network without increasing its diameter, under DMBI players have less incentives to sponsor links. *Fourth*, part of our analysis is based on the result that for small levels of decay, players always sponsor "in the middle" of any component, leaving the same number of players on one side and on the other side of the sponsored node. The degree of individual nodes in this case plays no role. However, for larger levels of decay, what matters most to a sponsor is how many players he has nearby. This again limits the maximal diameter that a network may achieve, as it creates a tendency for sponsoring to a characterizing player. *Fifth*, part of our analysis is based on the fact that for small levels of decay, the crucial sponsor in a network is always an end sponsor, as this sponsor accesses the information of only a single player. Yet, for larger levels of decay and under DMBI, a sponsor may be crucial because he is already very well-informed. Thus, under DMBI, the player with the most direct links may become crucial, and a network may then become more stable by letting such a player additionally access a lot of information through his link. This again reduces the relevance of large-diameter networks. Still, even in an analysis that takes into account such countervailing effects, the effect that under DMBI (IMBI), increasing (decreasing) the diameter of the network gives players more incentives to sponsor a link will continue to apply.

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