Status-seeking in coalitional matching problems

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Abstract

We study group formation where agents' preferences over group memberships are defined on the *identity* of the other agents in the same group. We define identity to be two-dimensional: on the one-hand, it is determined by the agent's *type*, e.g., her race, nationality, ethnic background; on the other hand, it is determined by the agent's *quality*, e.g., her skill level or material endowment. Specifically, we consider two types of agents and view group formation as a coalitional two-sided matching problem where agents of each type may form coalitions which can be matched to each other. Agents' preferences are dictated by status-seeking. Status can be either local or global. In this setting, we consider four specific preference domains, show that on all of them core stable coalitional matchings always exist, and fully characterize the corresponding cores.

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1 Introduction

In many situations studied by social scientists and economists in particular, an individual's decision about taking a certain action hinges upon who the other individuals making the same choice are. Membership in social groups, fan clubs, or internet forums, choice of residence and fashion, political party affiliation, and even religion are all examples in which an individual's preference may be well-defined on the *identities* of the individuals with whom she will be associated after making a choice.

The notion of identity is arguably a complex one.[1] Traditionally, the individual in economic analysis is identified by her preferences. This modeling of an individual, however, is

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[1] Recently, Jeitschko, O'Connell, and Pecchenino (2007) provide a review of the treatment of identity in economics literature.

being criticized for missing on the broader social context within which an individual functions as an economic agent (see Sen, 1977 for an early critique). A remedy is offered by behavioral economics. Akerlof and Kranton (2000), for example, illustrate how including in economic analysis an individual's sense of belonging to a salient social category may rationalize observed facts and change the predictions of the classical theoretical model. A call for making a further distinction between the individual's identity as represented by her group membership, and as reflected by her position within the group is coming from the sociological literature (e.g., Baron and Hannan, 1994). Finally, Podolny (1993) argues that applying the distinction between role and position to producers of different quality helps explaining the generation of hierarchies in a market.

To address this critique, we distinguish between two components of one's identity: on the one hand, we use the *type* of an individual to define identity in terms of belonging to a pre-defined larger group (one can think of this as gender, nationality, ethnic background, language group, or skill type); and on the other hand, we take the *quality* of the individual to dictate one's self-perception relative to others in the same group (here quality may be thought of as skill level or material endowment).[2] With these two dimensions of identity in mind, we investigate what groups will form in a society of two distinct types of agents who differ in terms of quality.

For this purpose we define a new notion, *coalitional matching*, that allows us to study the formation of groups that may be homogeneous in type, i.e., contain agents of the same type, or heterogeneous, i.e., contain both types of agents; differ in the quality of their members; and may vary in size. What distinguishes coalitional matching from other settings studied in the literature on group formation is the possibility to have simultaneous group formation of agents of each type and matching of (groups of) agents of distinct types.

When discussing how individuals choose their affiliation with groups, socially-referenced preferences are a natural candidate. That people tend to compare themselves in terms of possessions, well-being, or achievement, to those in their immediate surrounding has been

[2]For the purpose of our work both type and quality are considered to be objective rather than subjective measures of identity. Furthermore, in line with the literature on group formation, identity is treated as an endowed characteristic rather than a choice. For an example of endogenous identity formation in the work place, see Akerlof and Kranton (2005).

acknowledged theoretically and substantiated empirically (cf. Frank, 1985; Clark and Oswald, 1998; Blanchflower and Oswald, 2004; and Bowles and Park, 2005). In our work preferences are founded on the *status* each agent attains by taking part in a coalitional matching. On the one hand, an agent strives to attain a high *local status* as defined by her relative position with respect to the other agents of the same type in the group. In this respect, we assume that an agent's local status is higher, the greater her quality is relative to the average quality of the agents of the same type in the group. On the other hand, our agents seek a high *global status* by being matched to a group in which the average quality of the agents of the other type is as high as possible.

As agents are unconstrained in the type and size of groups they may form, and as they are only motivated by status seeking, we employ the notion of the *core* to study the partitioning of agents into groups. We discuss four cases of agents' preferences over groups: when agents seek only local status; when they seek only global status; and when both local and global status affect preferences, we distinguish between the cases when they are substitutes and complements. In all these cases, we show that the core is non-empty. More specifically, we show that when preferences are defined on local-status, the core consists of partitions in which each group has at most a single agent of each type. The same result holds when local and global status are complements in agents' preferences. In the case when global status preferences dictate agents choice, stable partitions contain groups in which at least one type of agents has only one representative. Finally, when global and local status are substitutes in preferences, groups of more than one agent of each type may be stable. Moreover, we show that segregated society, i.e., one in which high-quality individuals and low-quality individuals form separate groups, is core stable under very special conditions. Instead, fully integrated partitions, where all agents are part of a coalitional matching which has the same average quality for agents of each type, when individually rational, are stable. This result is surprising as seeking either local or global status would intuitively lead to the formation of elitist groups. In this case, however, the trade-off between local and global status seen in the fact that high local status for an agent of a certain type implies a lower global status for the agents of the other type with whom the former is matched, sustains stable integration.

This work has its place within the vast literature on group formation when agents' preferences over groups of which they can be members depend on the identity of the other members of the group. Group formation by heterogeneous types of agents has been analyzed in two-sided matching problems (here, Shapley and Shubik, 1972, is a seminal reference). The hedonic coalition formation literature (cf. Drèze and Greenberg, 1980) studies group formation when agents are homogeneous and their preferences depend on group membership only. Our work may be viewed as marrying these two strands of the literature.[3] Another strand of the literature that combines matching and coalition formation is the literature on effective coalitions (cf. Kaneko and Wooders, 1982). Like that literature, we use the notion of core to study stability, however, we do not impose any restrictions on the type of coalitions that may form.

Within the matching literature, our work is closely related to the class of papers on many-to-one matchings with peer effects (see Dutta, and Massó, 1997; and more recently Echenique and Yenmez, 2007; Pycia, 2007; and Revilla, 2007). The difference between our work and theirs is that in our framework group formation occurs on both sides of the market while in theirs it happens on one side of the market only. Our paper is also related to the work of Kaneko and Kimura (1992) who study group formation by heterogeneous types agents, black and white, whose preferences over groups depend on the size of the group. Similarly, Karni and Schmeidler (1990) study the splitting of the population which contains two types of agents into three groups when preferences depend on the relative size of each group. In contrast, in our work peer effects are not size-based.

We use the notion of the core to study stability where identity is conceptualized as a hedonic trait, thus our work is related to the literature on hedonic coalition formation. Banerjee et al. (2001), Bogomolnaia and Jackson (2002), and Ihlé (2007), among others, introduce various notions of stability and provide sufficient conditions for the existence of stable partitions in hedonic games. In this literature, however, identity is summarized in the index of each agent and authors do not discuss heterogeneous types of agents. Moreover, the preference profiles studied here differ from those usually analyzed in the literature such as separable, size-based,

[3] In a different paper, Dimitrov and Lazarova (2008), we study the necessary and sufficient conditions that guarantee non-emptiness of the core when the preference profiles are lexicographic.

and symmetric preferences.

As we employ socially referenced preferences, our work is also related to the literature inspired by Schelling (1978) that studies group formation where identity is summarized by the spacial location of an agent. More closely, our work is related to Milchtiach and Winter (2002) who discuss the issues of segregation and integration in group formation when preferences of individuals depend on the distance of each individual from the group average. In their framework, an agent prefers to be in a group in which his quality is closer to the group average quality. While there are many situations where these preferences are a good proxy for reality, e.g., voting on the level of public good, there are other situations in which being higher than the average is desirable, e.g., when reward is based on relative performance. More importantly, our work differs from that of Milchtiach and Winter (2002) in that we study agents who have heterogeneous types. Furthermore, unlike these authors, we do not restrict the number of partition elements that agents may form in their group formation exercise.

Our notions of local and global status are adopted from the work of Watts (2007). As in Watts (2007), agents in our setting prefer to have a higher local status measured by their relative position in the group; while we measured relative position as the distance to the average, she captures that by the rank of the individual in the group. While global status in her work is measure by the average quality of agents in the group, here, global status is given by the average quality of agents of the other type members of the same group. Therefore, an agent's quality affects the group global status directly in Watts's sense, but it affects it only in strategic terms here. Like Milchtiach and Winter (2002), Watts (2007) studies group formation with a restriction on the number of groups that may be formed when agents are of a homogeneous type. Finally, as both Milchtiach and Winter (2002) and Watts (2007) study group formation where the number of groups that may be formed is restricted, the notion of the core is not applicable for their works. Both sets of authors also find that segregating outcomes are stable, while in our framework integrated outcomes are more easily stable.

Finally, this paper is related to to the literature on local public goods (cf. Tiebout, 1956; and, more recently, Conley and Wooders, 2001) as we, too, study group membership by heterogeneous types of agents. We, however, do not discuss public group production and the size of the partition in our model is not restricted as in the case of jurisdictions.

The rest of the paper is organized as follows. The next section introduces the basic concepts used in our analysis. In Section 3 we characterize the set of core stable coalitional matchings when agents look only at the groups of their own type by taking the distance between their own quality and the average quality of the group or only at groups of the opposite type seeking a match to a group with a higher average quality. We consider in Sections 4 and 5 the situations in which, given a coalitional matching, each agent perceives the two types of groups – the one he is member of and the one he is matched to – as being either perfect substitutes or perfect complements. We show that a core stable coalitional matching in both cases always exists, and fully characterize the corresponding sets of core stable coalitional matchings. We conclude in Section 6 with some final remarks.

2 Notation and definitions

Let $N^\alpha = \{1^\alpha, 2^\alpha, \ldots, m^\alpha\}$ and $N^\beta = \left\{1^\beta, 2^\beta, \ldots, n^\beta\right\}$ with $m \leq n$ be two disjoint and finite sets of agents of type α and type β, respectively. For $\gamma \in \{\alpha, \beta\}$, each agent $i^\gamma \in N^\alpha \cup N^\beta$ is endowed with quality level q_i^γ. Without loss of generality, we assume and index the agents in such a way that $q_1^\alpha > q_2^\alpha > \ldots > q_m^\alpha > 0$ and $q_1^\beta > q_2^\beta > \ldots > q_n^\beta > 0$; thus, 1^γ is the member of N^γ with the highest quality, 2^γ is the member of N^γ with the second highest quality, and so on.

An α-group A is a non-empty subset of N^α and a β-group B is a non-empty subset of N^β. We denote by 2^{N^α} the set of all α-groups, and by N_i^α the set of all α-groups containing agent $i^\alpha \in N^\alpha$. Correspondingly, 2^{N^β} stands for the set of all β-groups, while N_i^β is the set of all β-groups containing agent $i^\beta \in N^\beta$.

Each agent seeks an α-group and a β-group. Thus, each agent $i^\alpha \in N^\alpha$ has a complete and transitive preference $\succeq_{i^\alpha}$ defined over $N_i^\alpha \times 2^{N^\beta}$, and each agent $i^\beta \in N^\beta$ has a complete and transitive preference $\succeq_{i^\beta}$ defined over $2^{N^\alpha} \times N_i^\beta$. The corresponding strict preference and indifference relations are denoted, for $\gamma \in \{\alpha, \beta\}$, by $\succ_{i^\gamma}$ and $\sim_{i^\gamma}$, respectively.

A *coalitional matching* is a function μ from $N^\alpha \cup N^\beta$ into subsets of $N^\alpha \cup N^\beta$, such that for $\gamma \in \{\alpha, \beta\}$ and for all $i^\alpha \in N^\alpha$ and $i^\beta \in N^\beta$:

(1) $\mu(i^\alpha) \in N_i^\alpha \times 2^{N^\beta}$;

(2) $\mu\left(i^\beta\right) \in 2^{N^\alpha} \times N_i^\beta$;

(3) If $\mu(i^\gamma) = (A, B)$ for some $i^\gamma \in N^\alpha \cup N^\beta$, then $\mu(j^\gamma) = (A, B)$ for all $j^\gamma \in A \cup B$.

In what follows, we write $\mu(i^\gamma) = \left(\mu(i^\gamma)_\alpha, \mu(i^\gamma)_\beta\right)$ to denote the match of agent $i^\gamma \in N^\alpha \cup N^\beta$ under μ.

We say that a pair (C, μ'), where $C \subseteq N^\alpha \cup N^\beta$ and μ' is a coalitional matching, is *blocking* μ if for $\gamma \in \{\alpha, \beta\}$ and all $i^\gamma \in C$,

(1) $\mu'(i^\gamma)_\alpha \cup \mu'(i^\gamma)_\beta \subseteq C$;

(2) $\mu'(i^\gamma) \succ_{i^\gamma} \mu(i^\gamma)$.

Thus, the pair (C, μ') is blocking μ if each agent in C strictly prefers his corresponding match under μ' (which contains only agents belonging to C) over his match under μ. A coalitional matching is *core stable* if it cannot be blocked.

3 Local status vs global status

We start our analysis by considering two extreme cases of how the quality levels of the agents guide their preferences over compositions of α- and β-groups. In the first extreme case we assume that agents are interested in local status only, i.e., when judging two such compositions, each agent looks only at the distance between his own quality and the average quality of the group of agents of his own type in such a way that he prefers to be the member with the highest quality in a group in which the average quality is as low as possible. In the second case we suppose that agents are interested in global status only, i.e., they look at the groups of the opposite type and prefer to be matched to a group with a highest average quality.

3.1 Local status

If agents look only at the groups of their own type and are guided by the distance between their own quality and the average quality of the group, their preferences over compositions of α- and β-groups are formally specified as follows:

- for all $i^\alpha \in N^\alpha$ and all $(A,B),(A',B') \in N_i^\alpha \times 2^{N^\beta}$, $(A,B) \succeq_{i^\alpha} (A',B')$ iff $q_i^\alpha - \frac{\sum_{\ell \in A} q_\ell^\alpha}{|A|} \geq q_i^\alpha - \frac{\sum_{\ell \in A'} q_\ell^\alpha}{|A'|}$.
- for all $i^\beta \in N^\beta$ and all $(A,B),(A',B') \in 2^{N^\alpha} \times N_i^\beta$, $(A,B) \succeq_{i^\beta} (A',B')$ iff $q_i^\beta - \frac{\sum_{\ell \in B} q_\ell^\beta}{|B|} \geq q_i^\beta - \frac{\sum_{\ell \in B'} q_\ell^\beta}{|B'|}$.

In this case is easily seen that the core consists only of matchings μ for which the following holds:

- for all $i^\alpha \in N^\alpha$, $\mu(i^\alpha)_\alpha = \{i^\alpha\}$ and $\mu(i^\alpha)_\beta \in \left\{\left\{i^\beta\right\}, \emptyset\right\}$ for some $i^\beta \in N^\beta$;
- for all $i^\beta \in N^\beta$, $\mu\left(i^\beta\right)_\alpha \in \{\{i^\alpha\}, \emptyset\}$ for some $i^\alpha \in N^\alpha$ and $\mu\left(i^\beta\right)_\beta = \left\{i^\beta\right\}$.

In other words, in a core stable coalitional matching there are no two members (of the same type) who are grouped together - if this were the case, then the agent with the lowest quality in the group would prefer to stay single and hence, to block the corresponding matching.[4]

3.2 Global status

Consider next the other extreme case in which there are no peer effects and each agent looks only at groups of the opposite type seeking a match to a group with a highest average quality.[5] Agents' preferences are thus specified as follows:

- for all $i^\alpha \in N^\alpha$ and all $(A,B),(A',B') \in N_i^\alpha \times 2^{N^\beta}$, $(A,B) \succeq_{i^\alpha} (A',B')$ iff $\frac{\sum_{\ell \in B} q_\ell^\beta}{|B|} \geq \frac{\sum_{\ell \in B'} q_\ell^\beta}{|B'|}$.

[4]It is straightforward to see that the core of an analogous hedonic game which consists of either α- or β-type agents contains only the partition of singletons.

[5]Note that this type of problem has not been previously studied in the matching literature because here coalition formation happens on *both sides* of the market.

- for all $i^\beta \in N^\beta$ and all $(A,B), (A',B') \in 2^{N^\alpha} \times N_i^\beta$, $(A,B) \succeq_{i^\beta} (A',B')$ iff $\dfrac{\sum_{\ell \in A} q_\ell^\alpha}{|A|} \geq \dfrac{\sum_{\ell \in A'} q_\ell^\alpha}{|A'|}$.

The core in this case is again non-empty as for instance the following three coalitional matchings are core stable.

$$\mu' : \{(\{1^\alpha\}, N^\beta), (N^\alpha \setminus \{1^\alpha\}, \emptyset)\},$$
$$\mu'' : \{(N^\alpha, \{1^\beta\}), (\emptyset, N^\beta \setminus \{1^\beta\})\},$$
$$\mu''' : \{(\{1^\alpha\}, \{1^\beta\}), (\{2^\alpha\}, \{2^\beta\}), \ldots, (\{m^\alpha\}, \{m^\beta\}), (\emptyset, N^\beta \setminus \{1^\beta, \ldots, m^\beta\})\}.$$

Clearly, the coalitional matching μ' is the most preferred one by the β-agents as they are all matched to the α-group with the highest average quality. Similarly, μ'' is the most preferred core stable coalitional matching by the α-agents. One can think of the third element μ''' as a "fair" coalitional matching as the best set of α-agents are matched to the best set of β-agents.

Keeping these three examples in mind, let us now fully describe the set of core stable coalitional matching for this extreme case. We precede by first providing an algorithm which delivers a partition π of the set of agents $N^\alpha \cup N^\beta$ into compositions of α- and β-groups.

Algorithm 1

- Set $N^a := N^\alpha$, $N^b := N^\beta$, and $\pi := \emptyset$.

- Repeat the following until $N^a \cup N^b = \emptyset$:

  - Find a group $A \cup B$ with $A \subseteq N^a$ and $B \subseteq N^b$ s.t.

  either

  $A = \left\{ i^a \in N^a : q_i^\alpha \geq q_j^\alpha \text{ for all } j^\alpha \in N^a \right\}$ and

  $B \in \left\{ B' \subseteq N^b : \dfrac{\sum_{i \in B'} q_i^\beta}{|B'|} \geq \max\left\{ q_i^\beta : i^\beta \in N^b \setminus B' \right\} \right\}$,

  or

  $A \in \left\{ A' \subseteq N^a : \dfrac{\sum_{i \in A'} q_i^\alpha}{|A'|} \geq \max\left\{ q_i^\alpha : i^\alpha \in N^a \setminus A' \right\} \right\}$ and

  $B = \left\{ i^\beta \in N^b : q_i^\beta \geq q_j^\beta \text{ for all } j^\beta \in N^b \right\}$.

  - Set $N^a := N^a \setminus A$, $N^b := N^b \setminus B$ and $\pi := \pi \cup \{A \cup B\}$.

- Return π.

We denote by Π the set of all partitions delivered by the algorithm and, for any partition π of $N^\alpha \cup N^\beta$, we define the coalitional matching μ^π by $\mu^\pi(i^\gamma) = \left(\pi(i^\gamma) \cap N^\alpha, \pi(i^\gamma) \cap N^\beta\right)$ for all $i^\gamma \in N^\alpha \cup N^\beta$.

Proposition 1 *A coalitional matching μ is core stable for the above matching problem if and only if $\mu \in \{\mu^\pi : \pi \in \Pi\}$.*

Proof. Let $\pi = \{A_1 \cup B_1, A_2 \cup B_2, \ldots, A_P \cup B_P\} \in \Pi$ and μ^π be as defined above. We show that μ^π is core stable.

Notice first that by construction the average quality of the groups A_p and B_p, $p = 1, \ldots, P$, is non-negative. Suppose now that (C, μ') is blocking μ^π. Then it has to be the case that $C \cap N^\alpha \neq \emptyset$, $C \cap N^\beta \neq \emptyset$ with $|\mu'(i^\gamma)_\alpha| \geq 1$ and $\left|\mu'(i^\gamma)_\beta\right| \geq 1$ for all $i^\gamma \in C$. Let $\overline{p} = \min\{p : (A_p \cup B_p) \cap C \neq \emptyset\}$.

Case 1 ($A_{\overline{p}} \cap C \neq \emptyset$ and $B_{\overline{p}} \cap C = \emptyset$): Take $i^\alpha \in A_{\overline{p}} \cap C$ and let $i_*^\beta \in \mu'(i^\alpha)_\beta$ be the agent with the highest quality level in $\mu'(i^\alpha)_\beta$. Since μ' is blocking μ^π we have

$$q_{i^*}^\beta > \frac{\sum_{i^\beta \in \mu'(i^\alpha)_\beta} q_i^\beta}{\left|\mu'(i^\alpha)_\beta\right|} > \frac{\sum_{i^\beta \in B_{\overline{p}}} q_i^\beta}{|B_{\overline{p}}|}. \tag{1}$$

Note in addition that $\mu'(i^\alpha)_\beta \subseteq N^\beta \setminus \left(\cup_{p=1}^{\overline{p}} B_p\right)$ and that, by construction, we have either

$$\frac{\sum_{i^\beta \in B_{\overline{p}}} q_i^\beta}{|B_{\overline{p}}|} = q_{i^{**}}^\beta \tag{2}$$

with $q_{i^{**}}^\beta$ being the β-agent with the highest quality level in $N^\beta \setminus \left(\cup_{p=1}^{\overline{p}-1} B_p\right)$, or

$$\frac{\sum_{i^\beta \in B_{\overline{p}}} q_i^\beta}{|B_{\overline{p}}|} \geq \max\left\{q_i^\beta : i^\beta \in N^\beta \setminus \left(\cup_{p=1}^{\overline{p}} B_p\right)\right\}. \tag{3}$$

By $q_{i^*}^\beta \in \mu'(i^\alpha)_\beta \subseteq N^\beta \setminus \left(\cup_{p=1}^{\overline{p}} B_p\right) \subseteq N^\beta \setminus \left(\cup_{p=1}^{\overline{p}-1} B_p\right)$ and combining (1) with either (2) or (3), we have a contradiction to the construction of the partition π.

Case 2 ($A^{\overline{p}} \cap C = \emptyset$ and $B_{\overline{p}} \cap C \neq \emptyset$)): The proof is analogous to the one in Case 1.

Case 3 ($A^{\overline{p}} \cap C \neq \emptyset$ and $B_{\overline{p}} \cap C \neq \emptyset$)): The proof is again analogous to the one in Case 1 with the additional remark that $\mu'(i^\alpha)_\beta \subseteq N^\beta \setminus \left(\cup_{p=1}^{\overline{p}-1} B_p\right)$.

We conclude that μ^π is a core stable coalitional matching.

Suppose now that μ is core stable but $\mu \notin \{\mu^\pi : \pi \in \Pi\}$. That is, there is a partition $\overline{\pi} \notin \Pi$ with $\mu = \mu^{\overline{\pi}}$. Notice first that if there is $C \in \overline{\pi}$ containing at least two α-agents and at least two β-agents, then $\mu^{\overline{\pi}}$ will be not core stable as the higher quality α- and β-agents in C would form a blocking pair. Thus, $A \cup B = C \in \overline{\pi}$ implies either $|A| \in \{0,1\}$ and $|B| \geq 1$, or $|A| \geq 1$ and $|B| \in \{0,1\}$. Order the elements of $\overline{\pi}$ in such a way that $\overline{\pi} = \{A_1 \cup B_1, A_2 \cup B_2, \ldots, A_R \cup B_R\}$ with

$$\frac{\sum_{i^\beta \in B_r} q_i^\beta}{|B_r|} \geq \frac{\sum_{i^\beta \in B_{r+1}} q_i^\beta}{|B_{r+1}|}$$

for $r = 1, \ldots, R-1$ with the average quality of the empty set being equal to zero.

Take $A_1 \cup B_1$ and consider the following possible cases.

Case 1 ($A_1 = \emptyset$): The pair $\left(\left\{1^\alpha, 1^\beta\right\}, \mu'\right)$ where μ' is defined by $\mu'(i^\alpha) = \mu'\left(i^\beta\right) = \left(\{1^\alpha\}, \{1^\beta\}\right)$ is blocking μ since $q_1^\beta > \frac{\sum_{i^\beta \in B_1} q_i^\beta}{|B_1|} \geq \frac{\sum_{i^\beta \in B_r} q_i^\beta}{|B_r|}$ and $q_1^\alpha > \frac{\sum_{i^\alpha \in A_r} q_i^\alpha}{|A_r|}$ hold for all $r = 1, \ldots, R$. Thus, we have a contradiction to the core stability of μ.

Case 2 ($|A_1| = 1$): If $A_1 \neq \left\{i^a \in N^\alpha : q_i^\alpha \geq q_j^\alpha \text{ for all } j^\alpha \in N^\alpha\right\} = \{1^\alpha\}$ and $A_1 \notin \left\{A' \subseteq N^\alpha : \frac{\sum_{i \in A'} q_i^\alpha}{|A'|} \geq \max\{q_i^\alpha : i^\alpha \in N^\alpha \setminus A'\}\right\} \ni \{1^\alpha\}$, then, by the same reasoning as in Case 1, the pair $\left(\left\{1^\alpha, 1^\beta\right\}, \mu'\right)$ is blocking μ, a contradiction. Hence, we conclude that A_1 has to have the structure as indicated in the above algorithm.

Furthermore, if $A_1 = \left\{i^a \in N^\alpha : q_i^\alpha \geq q_j^\alpha \text{ for all } j^\alpha \in N^\alpha\right\} = \{1^\alpha\}$ and $B_1 \notin \left\{B' \subseteq N^\beta : \frac{\sum_{i \in B'} q_i^\beta}{|B'|} \geq \max\left\{q_i^\beta : i^\beta \in N^\beta \setminus B'\right\}\right\} \ni \left\{1^\beta\right\}$, then the pair $\left(\left\{1^\alpha, 1^\beta\right\}, \mu'\right)$ with μ' as in Case 1 is blocking μ since $q_1^\beta > \frac{\sum_{i^\beta \in B_1} q_i^\beta}{|B_1|}$ and $q_1^\alpha > \frac{\sum_{i^\alpha \in A_r} q_i^\alpha}{|A_r|}$ hold for all $r = 2, \ldots, R$ (note that $\mu\left(1^\beta\right)_\alpha = A_r$ for some $r \in \{2, \ldots, R\}$). Thus, we have again a contradiction to the core stability of μ.

The case in which $|B_1| = 1$ can be treated similarly. In an analogous way one can show that all elements of $\overline{\pi}$ have the structure provided by the above algorithm. We conclude that the core stability of μ implies $\mu \in \{\mu^{\pi} : \pi \in \Pi\}$. ■

4 Local and global status as substitutes

In this section we study the structure of core stable coalitional matching when agents look both at the groups of their own type and at the groups of the opposite type. More precisely, we assume that, given any coalitional matching μ, each agent $i^{\gamma} \in N^{\alpha} \cup N^{\beta}$ perceives the two types of groups – the group $\mu(i^{\gamma})_{\alpha}$ of α-agents and the group $\mu(i^{\gamma})_{\beta}$ of β-agents as being substitutable. Agents' preferences are thus specified as follows:

- for all $i^{\alpha} \in N^{\alpha}$ and all $(A,B),(A',B') \in N_i^{\alpha} \times 2^{N^{\beta}}$, $(A,B) \succeq_{i^{\alpha}} (A',B')$ iff $q_i^{\alpha} - \frac{\sum_{i^{\alpha}\in A} q_i^{\alpha}}{|A|} + \frac{\sum_{\ell\in B} q_{\ell}^{\beta}}{|B|} \geq q_i^{\alpha} - \frac{\sum_{i^{\alpha}\in A'} q_i^{\alpha}}{|A'|} + \frac{\sum_{\ell\in B'} q_{\ell}^{\beta}}{|B'|}$.
- for all $i^{\beta} \in N^{\beta}$ and all $(A,B),(A',B') \in 2^{N^{\alpha}} \times N_i^{\beta}$, $(A,B) \succeq_{i^{\beta}} (A',B')$ iff $q_i^{\beta} - \frac{\sum_{\ell\in B} q_{\ell}^{\beta}}{|B|} + \frac{\sum_{\ell\in A} q_{\ell}^{\alpha}}{|A|} \geq q_i^{\beta} - \frac{\sum_{\ell\in B'} q_{\ell}^{\beta}}{|B'|} + \frac{\sum_{\ell\in A'} q_{\ell}^{\alpha}}{|A'|}$.

In order to state our next result we will need the following additional notation. For any $A \subseteq N^{\alpha}$ and $B \subseteq N^{\beta}$ let $\lambda_{AB} := \frac{\sum_{i^{\alpha}\in A} q_i^{\alpha}}{|A|} - \frac{\sum_{i^{\beta}\in B} q_i^{\beta}}{|B|}$ (if either A or B is empty, we set the corresponding average quality level to be equal to zero). Given a coalitional matching μ, we write λ_{AB}^{μ} for the difference in the average qualities of the groups A and B with $\mu(i^{\gamma}) = (A,B)$ for all $i^{\gamma} \in A \cup B$. Moreover, for any coalitional matching μ, we let $I_0^{\mu} := \left\{ i^{\gamma} \in N^{\alpha} \cup N^{\beta} : \mu(i^{\gamma})_{\alpha} = \emptyset \text{ or } \mu(i^{\gamma})_{\beta} = \emptyset \right\}$ be the set of agents that are matched under μ to the empty set.

Theorem 1 *An individually rational coalitional matching μ is core stable for the above matching problem if and only if the following two conditions are satisfied:*

(1) $I_0^{\mu} \cap N^{\alpha} = \emptyset$ or $I_0^{\mu} \cap N^{\beta} = \emptyset$.

(2) For any two non-empty α- and β-groups A' and B' with $\mu(i^{\alpha})_{\beta} \nsubseteq B'$ for all $i^{\alpha} \in A'$ the following two implications hold:

(2.1) $\lambda_{A'B'} > \max_{B'\cap B\neq\emptyset} \lambda_{AB}^{\mu} \Rightarrow \lambda_{A'B'} \geq \min_{A'\cap A\neq\emptyset} \lambda_{AB}^{\mu}$.

(2.2) $\lambda_{A'B'} < \min_{A'\cap A\neq\emptyset} \lambda_{AB}^{\mu} \Rightarrow \lambda_{A'B'} \leq \max_{B'\cap B\neq\emptyset} \lambda_{AB}^{\mu}$.

Proof. Let μ be a coalitional matching satisfying (1) and (2). We show that it is core stable.

Suppose not, i.e., there is a pair (C, μ') with $C = A \cup B$ that blocks μ. That is, we have

$$q_i^\alpha - \lambda_{\mu'(i^\alpha)_\alpha \mu'(i^\alpha)_\beta} > q_i^\alpha - \lambda_{\mu(i^\alpha)_\alpha \mu(i^\alpha)_\beta}$$

for all $i^\alpha \in A$, and

$$q_i^\beta + \lambda_{\mu'(i^\beta)_\alpha \mu'(i^\beta)_\beta} > q_i^\beta + \lambda_{\mu(i^\beta)_\alpha \mu(i^\beta)_\beta}$$

for all $i^\beta \in B$.

Suppose first that $A = \emptyset$. Notice then that the lowest quality agent in B can attain at most zero utility in the blocking coalition. As μ is individually rational, a coalition consisting of β-type agents only cannot be blocking μ. For a similar reason, a coalition which consist of only α-type agents cannot be blocking μ either.

Next, suppose that the blocking coalition consists of both α- and β-type agents, and that there are $i^\alpha \in C \cap N^\alpha$ and $i^\beta \in C \cap N^\beta$ such that $i^\beta \in \mu(i^\alpha)_\beta$. Simple algebra shows that the above two inequalities cannot hold simultaneously for these two agents.

Last, suppose that the blocking coalition consists of both α- and β-type agents such that there are no two agents of two distinct types who are matched to each other under μ. Such blocking possibilities are ruled out by item (2) in the statement of the theorem. To see this notice that agent i^α gets under μ exactly $q_i^\alpha - \lambda_{\mu(i^\alpha)_\alpha \mu(i^\alpha)_\beta}$. Similarly, any agent i^β gets $q_i^\beta + \lambda_{\mu(i^\beta)_\alpha \mu(i^\beta)_\beta}$ under μ. Hence, for the incentives of agents i^α and i^β to be part of a blocking coalition C, it must be that $\lambda_{\mu(i^\beta)_\alpha \mu(i^\beta)_\beta} < \lambda_{\mu'(i^\alpha)_\alpha \mu'(i^\alpha)_\beta} < \lambda_{\mu(i^\alpha)_\alpha \mu(i^\alpha)_\beta}$. Therefore, item (2) guarantees that there is an α-agent (condition (2.1)) or a β-agent (condition (2.2)) for which such $\lambda_{\mu'(i^\alpha)_\alpha \mu'(i^\alpha)_\beta}$ cannot be found.

As to show that items (1) and (2) are also necessary for a coalitional matching to be core stable, let μ be core stable and do not satisfy (1). This implies the existence of $i^\alpha \in N^\alpha$ and $i^\beta \in N^\beta$ with $\mu(i^\alpha) = (\{i^\alpha\}, \emptyset)$ and $\mu(i^\beta) = (\emptyset, \{i^\beta\})$. Notice however that the pair $(\{i^\alpha, i^\beta\}, \mu')$, where μ' is defined by $\mu'(i^\alpha) = \mu'(i^\beta) = (\{i^\alpha\}, \{i^\beta\})$ is blocking μ in contradiction to its core stability.

Suppose finally that μ is core stable and does not satisfy (2). Consider first the case in which there are α- and β-groups A' and B' with $\mu(i^\alpha)_\beta \nsubseteq B'$ for all $i^\alpha \in A'$ such that $\lambda_{A'B'} >$

$\max_{B' \cap B \neq \emptyset} \lambda^{\mu}_{AB}$ and $\lambda_{A'B'} < \min_{A' \cap A \neq \emptyset} \lambda^{\mu}_{AB}$ hold (i.e., (2.1) is violated). Consider then the pair $(A' \cup B', \mu')$, where μ' is defined by $\mu'(i^{\gamma}) = (A', B')$ for all $i^{\gamma} \in A' \cup B'$. To see that this pair is blocking μ, notice that all $i^{\beta} \in B'$ get under μ' exactly $q^{\beta}_i + \lambda_{A'B'} > q^{\beta}_i + \lambda_{\mu(i^{\beta})_{\alpha} \mu(i^{\beta})_{\beta}}$ (as $\lambda_{A'B'} > \max_{B' \cap B \neq \emptyset} \lambda^{\mu}_{AB}$ holds). Furthermore, all $i^{\alpha} \in A'$ get $q^{\alpha}_i - \lambda_{A'B'} > q^{\alpha}_i - \lambda_{\mu(i^{\beta})_{\alpha} \mu(i^{\beta})_{\beta}}$ because of $\lambda_{A'B'} < \min_{A' \cap A \neq \emptyset} \lambda^{\mu}_{AB}$. Similarly, one can show how A' and B' can be used to form a blocking pair if condition (2.2) is violated. ■

The significance of Condition (2) in Theorem 1 is illustrated in the example below.

Example 1 *Let $N^{\alpha} = \{1^{\alpha}, 2^{\alpha}, 3^{\alpha}\}$ and $N^{\beta} = \{1^{\beta}, 2^{\beta}\}$ with $q^{\alpha}_1 = 4$, $q^{\beta}_1 = 3$, $q^{\alpha}_2 = q^{\beta}_2 = 2$, and $q^{\alpha}_3 = 1$. Consider the coalitional matching μ with $\mu(1^{\alpha}) = \mu(1^{\beta}) = (\{1^{\alpha}\}, \{1^{\beta}\})$, $\mu(2^{\alpha}) = \mu(2^{\beta}) = (\{2^{\alpha}\}, \{2^{\beta}\})$, and $\mu(3^{\alpha}) = (\{3^{\alpha}\}, \emptyset)$. This coalitional matching is not stable as it is blocked by the pair $(\{1^{a}, 3^{\alpha}, 2^{\beta}\}, \mu')$ where μ' is such that $\mu'(1^{\alpha}) = \mu'(3^{\alpha}) = \mu'(2^{\beta}) = (\{1^{\alpha}, 3^{\alpha}\}, \{2^{\beta}\})$. Clearly, $\mu' \succ_{1^{\alpha}} \mu$ as $q^{\alpha}_1 - q^{\alpha}_1 + q^{\beta}_1 = 3 < 3,5 = q^{\alpha}_1 - \dfrac{q^{\alpha}_1 + q^{\alpha}_3}{2} + q^{\beta}_2$. Similarly, one can show that both agents 3^{α} and 2^{β} strictly prefer matching μ' to matching μ.*

Special classes of core-stable coalitional matchings can be derived as corollaries to Theorem 1.

Corollary 1 *Let $\lambda \in \left[-q^{\beta}_n, q^{\alpha}_m\right]$ and π be a partition of $N^{\alpha} \cup N^{\beta}$ s.t. $\lambda_{AB} = \lambda$ for all $A \subseteq N^{\alpha}$ and $B \subseteq N^{\beta}$ with $A \cup B \in \pi$. Then the coalitional matching μ^{π} is core stable.*

The proof is easy to see. The condition $-q^{\beta}_n \leq \lambda \leq q^{\alpha}_m$ ensures that the coalitional matching is individually rational, while the fact that the corresponding α- and β-groups have equal average quality ($= \lambda$) guarantees that conditions (1) and (2) of Theorem 1 hold.

Furthermore, Corollary 1 describes conditions under which a segregating coalitional matching is in the core. Here segregation refers to a situation in which coalitions of the same type of agents may have different average quality. Formally[6], a coalitional matching μ is segregated if (1) given any three agents $i^{\gamma}, j^{\gamma}, k^{\gamma} \in N^{\gamma}$ with $\gamma = \{\alpha, \beta\}$ such that $j^{\gamma} \in \mu(i^{\gamma})_{\gamma}$ and $q_{k^{\gamma}} \in (q_{i^{\gamma}}, q_{j^{\gamma}})$, we have $k^{\gamma} \in \mu(i^{\gamma})_{\gamma}$; and (2) given any four agents $i^{\gamma}, j^{\gamma}, k^{\gamma}, \ell^{\gamma} \in N^{\gamma}$ with $\gamma = \{\alpha, \beta\}$ where $q_{i^{\gamma}}, q_{j^{\gamma}} \geq q'$ and $q_{k^{\gamma}}, q_{\ell^{\gamma}} \leq q''$ with $q'' < q'$, it cannot be that $k^{\gamma} \in \mu(i^{\gamma})_{\gamma}$, $\ell^{\gamma} \in \mu(q_{j^{\gamma}})_{\gamma}$ and $j^{\gamma} \notin \mu(q_{i^{\gamma}})_{\gamma}$. Corollary 1 then states that such segregated coalitional

[6]Here we adopt Definition 3 of segregated partitions in Watts (2007).

matching are in the core if the difference between the average quality of the α- and β-groups matched to one another is the same for all elements in the matching. This result implies that in a stable segregated matching an α-group with higher average quality than another α-group in the coalitional matching has to be matched to a β-group with higher average quality than the β-group with which the latter α-group is matched. It is not only that higher ranked agents on each side of the market are matched to each other under this condition, but also that a certain fairness requirement is satisfied: the average quality of each α-group in the coalitional matching exceeds/falls under the average quality of the β-group with which it is matched by the same amount.

To illustrate the significance of Corollary 1 for the stability of segregating outcomes, we refer again to Example 1 above. In this example, we study a segregated outcome in which the highest ranked individuals from each type are matched to each other, the second highest individuals of each type are also matched to each other, and the lowest ranked α-agent is matched to the empty set. As the analysis shows this segregated matching is not in the core, and indeed Corollary 1's condition, the differences between the average quality of α- and β-groups matched to each other must be equal, is not satisfied for this coalitional matching: $\lambda_{\{1^\alpha\},\{1^\beta\}} = 1$, and $\lambda_{\{2^\alpha\},\{2^\beta\}} = 0$. The following example shows a coalitional matching problem in which the core contains a segregated outcome.

Example 2 *Let $N^\alpha = \{1^\alpha, 2^\alpha, 3^\alpha\}$ and $N^\beta = \{1^\beta, 2^\beta\}$ with $q_1^\alpha = 4$, $q_1^\beta = 3$, $q_2^\alpha = 2$, and $q_2^\beta = q_3^\alpha = 1$. Consider the coalitional matching μ with $\mu(1^\alpha) = \mu(1^\beta) = (\{1^\alpha\}, \{1^\beta\})$, $\mu(2^\alpha) = \mu(2^\beta) = (\{2^\alpha\}, \{2^\beta\})$, and $\mu(3^\alpha) = (\{3^\alpha\}, \emptyset)$. It is easy to see that μ is in the core of this coalitional matching problem as there exists no blocking pair. Notice that $\lambda_{\{1^\alpha\},\{1^\beta\}} = \lambda_{\{2^\alpha\},\{2^\beta\}} = \lambda_{\{3^\alpha\},\emptyset} = 1$.*

The next corollary describes conditions under which a fully integrated coalitional matching is stable. A coalitional matching μ is fully integrated if for any two agents $i^\gamma, j^\gamma \in N^\gamma$ with $\gamma \in \{\alpha, \beta\}$, we have that $\dfrac{\sum_{k \in \mu(i^\gamma)_\alpha} q_k^\alpha}{|\mu(i^\gamma)_\alpha|} = \dfrac{\sum_{k \in \mu(j^\gamma)_\alpha} q_k^\alpha}{|\mu(j^\gamma)_\alpha|}$, and $\dfrac{\sum_{k \in \mu(i^\gamma)_\beta} q_k^\beta}{|\mu(i^\gamma)_\beta|} = \dfrac{\sum_{k \in \mu(j^\gamma)_\beta} q_k^\beta}{|\mu(j^\gamma)_\beta|}$.

Corollary 2 *Let $q_m^\alpha - \frac{\sum_{i^\alpha \in N^\alpha} q_i^\alpha}{|N^\alpha|} + \frac{\sum_{i^\beta \in N^\beta} q_i^\beta}{|N^\beta|} \geq 0$, $q_n^\beta - \frac{\sum_{i^\beta \in N^\beta} q_i^\beta}{|N^\beta|} + \frac{\sum_{i^\alpha \in N^\alpha} q_i^\alpha}{|N^\alpha|} \geq 0$, and $K \leq m$. Let $\pi = \{A_1 \cup B_1, \ldots, A_K \cup B_K\}$ be a partition of $N^\alpha \cup N^\beta$ s.t. $\frac{\sum_{i^\alpha \in A^k} q_i^\alpha}{|A^k|} = \frac{\sum_{i^\alpha \in A^{k+1}} q_i^\alpha}{|A^{k+1}|}$*

and $\frac{\sum_{i^\beta \in B^k} q_i^\beta}{|B^k|} = \frac{\sum_{i^\beta \in B^{k+1}} q_i^\beta}{|B^{k+1}|}$ *for all* $k = 1, \ldots, K-1$. *Then the coalitional matching* μ^π *is core stable.*

Notice here that the condition that all α- and β-groups in the partition π have the same average quality implies that this average quality equals the (positive) average quality of N^α and N^β, respectively. Therefore, the conditions $q_m^\alpha - \frac{\sum_{i^\alpha \in N^\alpha} q_i^\alpha}{|N^\alpha|} + \frac{\sum_{i^\beta \in N^\beta} q_i^\beta}{|N^\beta|} \geq 0$ and $q_n^\beta - \frac{\sum_{i^\beta \in N^\beta} q_i^\beta}{|N^\beta|} + \frac{\sum_{i^\alpha \in N^\alpha} q_i^\alpha}{|N^\alpha|} \geq 0$ imply that this type of coalitional matching is individually rational. Furthermore, $\frac{\sum_{i^\alpha \in A_k} q_i^\alpha}{|A_k|} = \frac{\sum_{i^\alpha \in A_{k+1}} q_i^\alpha}{|A_{k+1}|}$ and $\frac{\sum_{i^\beta \in B_k} q_i^\beta}{|B_k|} = \frac{\sum_{i^\beta \in B_{k+1}} q_i^\beta}{|B_{k+1}|}$ for all $k = 1, \ldots, K-1$ guarantees that condition (2) of Theorem 1 is satisfied as well. In other words, condition (2) of Theorem 1 is satisfied in all fully integrated coalitional matchings, and, therefore for such a matching to be in the core, only the individually rationality condition may be a constraining factor.

As an example of a coalitional matching problem in which a fully integrated outcome is in the core, consider again Example 1. The coalitional matching in which all α-agents are matched as a group to the group consisting of all β-agents is fully integrated and it is in the core.

Our next and last result finally shows that under perfect substitutability of the α- and β-groups, there always exists a core stable coalitional matching.

Theorem 2 *There always exists a core stable coalitional matching for the above matching problem.*

Proof. Consider the following algorithm for delivering a coalitional matching.

Algorithm 2

We initialize the algorithm by setting $A_0 = N^\alpha$, $B_0 = N^\beta$, $\overline{A}_0 = \emptyset$, and $\overline{B}_0 = \emptyset$. In the k^{th} step of the algorithm, we set $A_k = A_{k-1} \setminus \overline{A}_{k-1}$, $B_k = B_{k-1} \setminus \overline{B}_{k-1}$, $\overline{A}_k = \overline{A}_{k-1} \cup \{i^\alpha \in A_k : q_i^\alpha - \frac{\sum_{i^\alpha \in A_k} q_i^\alpha}{|A_k|} + \frac{\sum_{i^\beta \in B_k} q_i^\beta}{|B_k|} < 0\}$, and $\overline{B}_k = \overline{B}_{k-1} \cup \{i^\beta \in B_k : q_i^\beta - \frac{\sum_{i^\beta \in B_k} q_i^\beta}{|B_k|} + \frac{\sum_{i^\alpha \in A_k} q_i^\alpha}{|A_k|} < 0\}$. The algorithm stops when $\overline{A}_\ell = \overline{A}_{\ell-1}$ and $\overline{B}_\ell = \overline{B}_{\ell-1}$ and we set $K = \ell$. Define the coalitional matching μ by $\mu(i^\gamma) = (A_K, B_K)$ for all $i^\gamma \in A_K \cup B_K$, $\mu(i^\alpha) = (\{i^\alpha\}, \emptyset)$ for all $i^\alpha \in \overline{A}_K = N^\alpha \setminus A_K$, and $\mu(i^\beta) = (\emptyset, \{i^\beta\})$ for all $i^\beta \in \overline{B}_K = N^\beta \setminus B_K$.

We show that μ is core stable.

First, we will show that K is finite, and, in particular that it is an integer at most equal to $n+1$. Notice that either $\overline{A}_1 = \emptyset$ or $\overline{B}_1 = \emptyset$; otherwise there is an agent with negative quality, which is not possible. For ease of exposition, suppose that $\overline{A}_1 = \emptyset$. Since $q_i^\beta - \frac{\sum_{i^\beta \in B_1} q_i^\beta}{|B_1|} + \frac{\sum_{i^\alpha \in A_1} q_i^\alpha}{|A_1|} < 0$ for some $i^\beta \in N^\beta$, it is clear that $q_i^\beta < \frac{\sum_{i^\beta \in B_1} q_i^\beta}{|B_1|}$ and, therefore, $\frac{\sum_{i^\beta \in B_2} q_i^\beta}{|B_2|} \geq \frac{\sum_{i^\beta \in B_1} q_i^\beta}{|B_1|}$. This is why for all α-agents $q_i^\alpha - \frac{\sum_{i^\alpha \in A_2} q_i^\alpha}{|A_2|} + \frac{\sum_{i^\beta \in B_2} q_i^\beta}{|B_2|} \geq 0$. Similarly, one can show that $A_K = N^\alpha$ and $\overline{A}_K = \emptyset$. The above analysis and the fact that N^β is finite proves that K is finite. Moreover, as $\frac{\sum_{i^\alpha \in N^\alpha} q_i^\alpha}{|N^\alpha|} > 0$ and $\frac{\sum_{i^\beta \in N^\beta} q_i^\beta}{\left|N^\beta\right|} > 0$, implies that $A_K \neq \emptyset$ and $B_K \neq \emptyset$, and, therefore $K \leq n+1$.

Next, we will show that there is no pair (C, μ') that blocks the constructed matching μ. Suppose, on the contrary, that such a pair exists. First, suppose that C consists of homogeneous type agents, i.e., $C \subseteq N^\alpha$ or $C \subseteq N^\beta$. Notice that by construction all agents in A_K and B_K have at least zero utility under μ. Furthermore, all agents in $\overline{A}_K$ and $\overline{B}_K$ have also zero utility under μ. Since the agents with the lowest quality in C can obtain at most zero utility under μ', the pair (C, μ') cannot be blocking.

Suppose next that there are at least two agents $i^\alpha, i^\beta \in C$ who are matched to each other under both μ and μ'. For (C, μ') to be blocking μ it must be that

$$q_i^\alpha - \frac{\sum_{i'^\alpha \in \mu'(i^\alpha)_\alpha} q_{i'}^\alpha}{|\mu'(i^\alpha)_\alpha|} + \frac{\sum_{i^\beta \in \mu'(i^\alpha)_\beta} q_i^\beta}{\left|\mu'(i^\alpha)_\beta\right|} > q_i^\alpha - \frac{\sum_{i^\alpha \in A_K} q_i^\alpha}{|A_K|} + \frac{\sum_{i^\beta \in B_K} q_i^\beta}{|B_K|}$$

and

$$q_i^\beta - \frac{\sum_{i'^\beta \in \mu'(i^\beta)_\beta} q_{i'}^\beta}{\left|\mu'(i^\beta)_\beta\right|} + \frac{\sum_{i^\alpha \in \mu'(i^\beta)_\alpha} q_i^\alpha}{|\mu'(i^\beta)_\alpha|} > q_i^\beta - \frac{\sum_{i^\beta \in B_K} q_i^\beta}{|B_K|} + \frac{\sum_{i^\alpha \in A_K} q_i^\alpha}{|A_K|}.$$

Simple algebra shows that the above two inequalities cannot hold simultaneously.

Last suppose that there are at least two agents $i^\alpha, i^\beta \in C$ who are matched to each under μ' but not under μ. W.l.o.g., suppose that $i^\alpha \in \overline{A}_K$ and $i^\beta \in B_K$. It is easy to see that the agent with the highest quality level in $\overline{A}_K$, is one who is in $\overline{A}_K$ (and therefore in $\overline{A}_{K-1}$) but not in $\overline{A}_{K-2}$. Denote this agent by i_*^α. Then, by construction, we have

$$q_{i^*}^\alpha < \frac{\sum_{i^\alpha \in A_{K-2}} q_i^\alpha}{|A_{K-2}|} + \frac{\sum_{i^\beta \in B_{K-2}} q_i^\beta}{|B_{K-2}|} < \frac{\sum_{i^\alpha \in A_K} q_i^\alpha}{|A_K|} - \frac{\sum_{i^\beta \in B_K} q_i^\beta}{|B_K|}. \tag{4}$$

Furthermore, notice that by definition of i_*^α, $\frac{\sum_{i^\alpha \in \tilde{A}} q_i^\alpha}{|\tilde{A}|} \leq q_{i^*}^\alpha$ for all $\widetilde{A} \subseteq \overline{A}_K$. Therefore, for (C, μ') to be blocking μ it must be that for the β-agent in C with the lowest quality, denoted by i_*^β, it must hold that

$$q_{i^*}^\beta - \frac{\sum_{i^\beta \in B_K} q_i^\beta}{|B_K|} + \frac{\sum_{i^\alpha \in A_K} q_i^\alpha}{|A_K|} < q_{i^*}^\beta - \frac{\sum_{i'^\beta \in \mu'\left(i_*^\beta\right)_\beta} q_{i'}^\beta}{\left|\mu'\left(i_*^\beta\right)_\beta\right|} + \frac{\sum_{i^\alpha \in \mu'\left(i_*^\beta\right)_\alpha} q_i^\alpha}{\left|\mu'\left(i_*^\beta\right)_\alpha\right|} \leq q_{i^*}^\beta + q_{i^*}^\alpha, \tag{5}$$

where the last inequality follows from $\mu'\left(i_*^\beta\right)_\alpha \subseteq \overline{A}_K$ (note that $\mu'\left(i_*^\beta\right)_\alpha \cap A_K \neq \emptyset$ would mean that there are a β-agent (i_*^β) and an α-agent who are matched to each other under both μ and μ' implying, as shown above, that (C, μ') is not blocking μ). Clearly, expressions (4) and (5) lead to a contradiction. ■

5 Local and global status as complements

Let us now consider the situation in which agents perceive the two types of groups – the one they are members of and the one they are matched to – as being complements. Agents' preferences over compositions of α- and β-groups are then formally specified as follows:

- for all $i^\alpha \in N^\alpha$ and all $(A, B), (A', B') \in N_i^\alpha \times 2^{N^\beta}$, $(A, B) \succeq_{i^\alpha} (A', B')$ iff
$$\min\left\{q_i^\alpha - \frac{\sum_{\ell \in A} q_\ell^\alpha}{|A|}, \frac{\sum_{\ell \in B} q_\ell^\beta}{|B|}\right\} \geq \min\left\{q_i^\alpha - \frac{\sum_{\ell \in A'} q_\ell^\alpha}{|A'|}, \frac{\sum_{\ell \in B'} q_\ell^\beta}{|B'|}\right\}.$$
- for all $i^\beta \in N^\beta$ and all $(A, B), (A', B') \in 2^{N^\alpha} \times N_i^\beta$, $(A, B) \succeq_{i^\beta} (A', B')$ iff
$$\min\left\{\frac{\sum_{\ell \in A'} q_\ell^\alpha}{|A'|}, q_i^\beta - \frac{\sum_{\ell \in B} q_\ell^\beta}{|B|}\right\} \geq \min\left\{\frac{\sum_{\ell \in A'} q_\ell^\alpha}{|A'|}, q_i^\beta - \frac{\sum_{\ell \in B'} q_\ell^\beta}{|B'|}\right\}.$$

It is then easily seen that, as in the case when agents look only at the groups of their own type and are guided by the distance between their own quality and the average quality of the group, the core consists only of matchings μ for which the following holds:

- for all $i^\alpha \in N^\alpha$, $\mu(i^\alpha)_\alpha = \{i^\alpha\}$ and $\mu(i^\alpha)_\beta \in \{\{i^\beta\}, \emptyset\}$ for some $i^\beta \in N^\beta$;
- for all $i^\beta \in N^\beta$, $\mu(i^\beta)_\alpha \in \{\{i^\alpha\}, \emptyset\}$ for some $i^\alpha \in N^\alpha$ and $\mu(i^\beta)_\beta = \{i^\beta\}$.

Notice first that the utility obtained by each agent in a matching μ of the above type is zero. Thus, μ can be blocked by (C, μ') only if C contains at least two agents i^γ and i'^γ from the same type such that $\mu'(i^\gamma)_\gamma = \mu'(i'^\gamma)_\gamma$. As the agent with the lowest quality in $\mu'(i^\gamma)_\gamma$ will have a negative utility, (C, μ') cannot be blocking μ. By the reverse argument, no other matchings are core stable.

6 Conclusion

We study group formation when agents' preferences are dictated by the identity of the other agents in the group and in particular by the local and global status they may achieve in a group membership. Our theoretical results show that in all four cases: when agents only care about their local status; when the agents only care about their global status; when local and global status are treated as substitutes; and when the two types of status are treated as complements, a core stable outcome exists.

Furthermore, we can identify the types of outcomes which are stable in light of segregation and integration. As segregated we define those outcomes in which the higher quality agents of each type are matched to each other and there are at least two groups of agents containing each type. Corollary 1 shows that such segregated outcomes may be stable if and only if the difference in average quality between the groups of α- and β-agents matched with each other is the same for all elements in the partition. Whether or not this condition may be satisfied hinges crucially on the distribution of qualities of agents of each type. Corollary 2, instead, may be viewed as describing coalitional matchings characterized by full integration since all groups in the partition have the same average quality of their α-members and the same average quality of their β-members. This coalitional matching can also be interpreted in the light of 'social equality' between groups and one which is envy-free. Notice that the coalitional matching derived by the algorithm in the proof of Theorem 2 can be one of the type of matchings described in Corollary 2 in case the coalitional matching in which the grand coalitions from both market sides are matched to each other is individually rational for all agents. When this is not the case, this algorithm derives a stable coalitional matching of what we may call a 'hybrid' construct. In this matching all agents of one type are matched as a grand coalition to a strict subset of the agents of the other type, hence, these agents are in an integrated state. The other type of agents, instead, are in a segregated state because there is a quality threshold such that all agents of this type whose quality is higher than the threshold are matched as a group to the grand coalition of the former type and all those whose quality is lower stay single.

Furthermore, our results show that in all other cases but the one in which local and global status are substitutes stable outcomes involve a partition of the agent set in groups in which at least one of the types is represented by a single agent. This result does no longer hold when agents may substitute local for global status.

Finally, our results may be seen as providing an alternative mechanism to the one discussed by Frank (1985) for gluing individuals together in social groups when they care for local status. Frank argues that what keeps a low-ranked individual in a group with higher ranked individuals are transaction costs (see Frank, 1985, p. 10). These transaction costs outweigh the gains such an individual might reap from moving to another group where her local status will be higher. In our setting transaction costs are zero. What keeps low-ranked individuals in a group with higher ranked individuals is the access to a matching with another type of agents that this membership provides.

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