Non-renewable Resources and Economic Growth: Comparing the classics to new models of endogenous technology and growth

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Abstract
The aim of the lecture is to revisit the classic question about the relationship between resource scarcity and economic growth. The new element is that we explicitly study the role of endogenous technological change. In the models we study, growth is driven by R&D-efforts as in endogenous growth theory. We examine under what conditions depletion of non-renewable resources limits economic growth, how substitution between man-made inputs and resources can overcome scarcity limits, and how substitution possibilities and resource depletion affect the incentives to innovate.
We first review models of growth driven by non-renewable resource use and exogenous technological change. These models directly build on the classic contributions by Dasgupta/Heal, Solow, and Stiglitz published in 1974. We discuss the role of the elasticity of substitution and that of the rate of technological change. We then turn to new models of endogenous technological change.
We examine whether physical capital alone is sufficient to guarantee positive growth, or whether technological change is needed. We also study whether investments in physical capital and in new technology are sustained, such that no exogenous technological change is needed to sustain growth.

1. Introduction

There has been a remarkable shift in attention in thinking about growth and natural resources. In the 1970s, we were afraid of running out of all vital resources. Especially oil reserves seemed to be much too small to keep the economy running for more than a few decades. Nowadays, it seems as if there is so much oil and other fossil fuels at low prices, that it has become a serious threat to the world climate system. At the same time, oil reserves are limited, and predictions of the peaking of oil production from mining engineers still attract attention, oil price have recently risen, and dependence on oil imports is regularly debated.

One interpretation of this shift in attention is the – rather optimistic – claim that scarcity of resources triggers counter reactions. Energy conservation policies and market responses to high energy-prices and forecasts of shortages made firms change their production process. Substitution towards less fossil fuel intensive production processes and investments in new energy efficient technologies might have changed the role of energy resources in the economy. Others interpretations give rise to less optimistic stories: prices might be distorted so that they do not reflect true scarcity, the fall in prices might be temporary because of improvements in extraction technology, substitution and technological change may become insufficient in the long-run to offset scarcity.

This paper gives an overview of economic growth models in which non-renewable resources are an essential input in production. Our main questions are:

- How does depletion of essential non-renewable resources impose a *drag on growth*?
- How can investment in *physical capital* offset this drag on growth?
- How can investment in *new technologies* offset the resource drag?
- How are incentives to invest in capital of new technologies affected by resource depletion?

Our purpose is to summarize the most recent literature on (endogenous) growth and non-renewable resources. We try to point out the underlying common elements, mechanisms and principles in the main contributions to the literature. We show that it is useful to go back to the classics in this literature: the Review of Economic Studies Symposium issue with contributions by Dasgupta/Heal, Solow and Stiglitz. We point out the links between these papers and developments in recent growth theory. Then we investigate how the classics, which all rely on exogenous technological change, relate to new models of resource scarcity and growth that include endogenous technological change. We pay attention in particular to the recent contributions by Schou (1999), Scholz/Ziemes (1999), Aghion/Howitt (1998), Grimaud/Rougé (2003), Groth/Schou (2002a, 2002b), Bretschger/Smulders (2004).

The first part of this chapter studies the driving forces behind growth in the case that good substitutes are available for the non-renewable resource, without making the resource unnecessary. We start with the simplest possible model and then gradually add more complexity. However, the central message is that, if we stick to constant elasticity specifications of production and utility functions, the most complex models can be reduced to the (almost) simplest model, which therefore captures most essential elements. Moreover, the basic model is shown to be isomorphic to the simplest endogenous growth model in the literature, the "AK-model" (Rebelo, 1991).

The second part studies the role of substitution possibilities. It turns to production functions with non-constant production elasticities.

2. The framework

2.1. A sketch of the problem

The problem we will study concerns the depletion of a non-renewable resource and its effect on growth of production. The resource provides inputs that are necessary to produce consumption goods. Our question is under what conditions we can sustain production (maintain a non-vanishing flow of production) over an infinite horizon. We go beyond this question by looking for conditions under which growth is unbounded, that is, production can grow at a strictly positive rate forever, even though the non-renewable resource is gradually depleted.

We make the distinction between feasibility of unbounded growth and optimality of unbounded growth. Although it may be technically feasible to let output grow indefinitely, it may be better for intertemporal welfare to abstain from unbounded growth. Of course, feasibility of unbounded growth is a prerequisite of unbounded optimal growth. Most of our discussion will concentrate on balanced growth path. By definition, on a balanced growth path, all variables grow at constant growth rates, which may be zero (in which case the variable in question is constant) or negative.

Before starting the formal modelling, let us give a first impression of some of the essential details of the problem at hand. The limited available stock of a necessary resource input implies that cumulative resource input over the infinite horizon is bounded. Cumulative production need not be bounded, however. The resource input is just one input, other inputs may be growing and substitute for the resource. We must distinguish between three types of inputs:

(i) resource inputs, which must (ultimately) decline,

(ii) exogenous inputs or production factors, possibly growing (we consider labour inputs and exogenous technological change),

(iii) endogenous inputs "man-made capital" (we study "physical capital" and "knowledge capital"), also possibly growing.

To model the problem of feasible production and optimal growth, we need the following elements:

- How consumption goods can be produced using resource inputs, exogenous inputs and man-made inputs (*production technology*).
- How availability of resource inputs evolves over time (resource stock dynamics)
- How availability of exogenous inputs evolves over time (labour, technology).
- How availability of man-made inputs evolves over time (*investment technology*)
- How consumption goods contribute to welfare, and in particular, how much current consumption contributes to welfare relative to future consumption (*welfare criterion*).

We focus on the technology side of the model: we compare different structures for production, but we stick to one and the same welfare criterion and specification of the resource stock dynamics. The literature on sustainability literature provides a discussion of the role of the welfare criterion (see Pezzey and Toman 2002). The resource stock dynamics are kept extremely simple by leaving out extraction costs, uncertainty and exploration.

To study feasibility of production and growth, we consider different specifications of the production and investment technology and check whether it is feasible to choose decision variables (such as depletion rates and investment rates) for which the growth rate of output is a positive constant. It should be noted that this exercise is independent of the welfare criterion. The latter is needed only when we turn to

optimal growth. We then look for a (the) specific feasible growth path from the total set of feasible growth paths, such that welfare is maximized according to the criterion. Everything depends on production function and accumulation technologies.
We also pay attention to the feasibility of one special growth path, viz the path on which consumption per capita is constant over time and at the maximum sustainable level. Some writers have such a "constant consumption path" as desirable for reasons of fairness towards future generations (Rawls, Solow, Hartwick), and it is often refered to in discussions about "sustainability".
Finally we need to contrast the optimal growth path to the growth path that results in the market equilibrium. This requires definition of property rights and markets, assumptions about behaviour (profit maximization, utility maximization) and expectations (rational expectations) and solving for equilibrium prices.
We now turn to the formal modelling. We first present the model in general terms and specify all elements except the production technology and research technology. Alternative specification of these will be presented in the rest of the paper.

2.2. The model

Non-renewable resource extraction
Let $S(t)$ be the stock of non-renewable resources available at time t, and $R(t)$ the rate of extraction of this resource at time t. It implies that the stock at time t equals the stock at time zero, minus what has been extracted cumulatively between time zero and t. In mathematical terms[1]:

$$(2.1) \qquad S(t) = S(0) - \int_0^t R(\tau)d\tau$$

Extraction can at most run down the stock completely, i.e. $S(t) \geq 0$ for all t. This implies that the total amount of resources that can be extracted over time is bounded by the initial resource stock (evaluate the formula over for $t \to \infty$ and impose $S(\infty) \geq 0$):

$$(2.2) \qquad \int_0^\infty R(\tau)d\tau \leq S(0)$$

In all models considered here, $R(t)$ is a continuous function of time, so that we find the following expression after differentiating (2.1) with respect to time:

$$(2.3) \qquad \dot{S}(t) = -R(t) \leq 0$$

where a dot over a symbol denotes the time derivative ($\dot{S} \equiv dS/dt$). The expression simply says that the stock decreases over time with the rate of extraction. It follows immediately that the growth rate of the stock equals:

$$(2.4) \qquad \hat{S}(t) \equiv \dot{S}(t)/S(t) = -R(t)/S(t) \equiv -u(t) \leq 0$$

[1] Dasgupta and Heal (1979), p. 154.

where hats are used to denote growth rates and u denotes the fraction of the stock being extracted. In a balanced growth path, the growth rate of S and R are constant by definition. This requires $0 = \hat{R} - \hat{S} = \hat{R} + R/S = \hat{R} + u$ and $\hat{R}$ is constant. So we have:

(2.5) Along a BGP, $\hat{S} = \hat{R} = -R/S = -u$ and $\dot{u} = 0$

So, along a BGP, a constant fraction u of the resource stock is extracted (i.e. $R = uS$), or, in other words, the resource stock decreases at a constant rate u (i.e. $\hat{S} = -u$). In order to clearly distinguish between R and u, we will try to consistently call R the resource flow and u the rate of depletion.

Population

Population and labour supply, L^S, grows at an exogenous rate n, which may be zero:

(2.6) $\hat{L}^S = n$

Utility and welfare

Throughout the chapter, the benchmark welfare criterion will be:

(2.7) $$W(0) = \int_0^\infty \left(\frac{1}{1-1/\sigma} C(t)^{1-1/\sigma} \right) e^{-\rho t} dt$$

where C is consumption, ρ is the utility discount rate and σ is the intertemporal elasticity of substitution. The term in parentheses is the intratemporal (or instantaneous) utility function of an individual agent. Note that the utility function is more curved if σ is small. In that case, marginal utility falls quickly with an increase in consumption ($1/\sigma$ is the elasticity of marginal utility). In other words, there is satiation in consumption (and lower σ means more stronger satiation): when the consumption has grown large, an additional unit of consumption is worth only a little. Ceteris paribus, this will imply a low incentive to shift consumption from the presence to the future if consumption goods are relatively more abundant in future than at present. Hence, we say that the smaller σ is, the lower the elasticity of intertemporal substitution. (Below, we will give an alternative interpretation in terms of "flexibility").

In the formulation (2.7), C is aggregate consumption. Usually, we think welfare depends on per capita consumption, rather than aggregate consumption, so in case of representative consumer, not only aggregate consumption, but also population size and population growth matter. However, it can be shown that as long as population growth is exogenous the above specification is sufficient. Consider the following specification of the intertemporal welfare function:

$$W(0) = \int_0^\infty \left(\frac{c(t)^{1-1/\sigma}}{1-1/\sigma} \right) L^S(t)^{\upsilon_L} e^{-\rho_L t} dt$$

where c is per capita consumption of the representative agent, L^S is population size, ρ_L is the utility discount rate and σ is the intertemporal elasticity of substitution.
Population size enters the welfare criterion if $\upsilon_L > 0$ to reflect the notion that welfare aggregated over population (or households) may matter, rather than individual welfare. (If $\upsilon_L = 1$, we have a Benthamite intertemporal utility function; if $\upsilon_L = 0$, we have a Millian utility function; see Canton and Meijdam (1997)).
Denoting aggregate consumption by $C = cL^S$, normalizing population size at time zero to one ($L(0) = 1$), and assuming that population grows at a constant exogenous rate n, $L^S(t) = e^{nt}$, we may rewrite the welfare function as in (W), if we define $\rho \equiv \rho_L - (\upsilon_L - 1 + 1/\sigma)n$ as the population-growth-adjusted utility discount rate.

Production and investment technology
The economy produces output, Y, by combining resource inputs R, labour inputs L, capital inputs K, and "technology inputs" A, according to a neoclassical production function:

(2.8) $Y = F(K, L, R, A_T, A_L, A_R)$

Below we will consider several specifications of the production function: (i) one without labour and capital inputs, the "cake production function", (ii) one with constant production elasticities, the "CD RES production function", and (iii) one with a constant elasticity of substitution and factor augmentation, the "CES RES production function". Details will be discussed below.
Capital K is accumulated by foregoing consumption:

(2.9) $\dot{K} = Y - C$

To model technological change, we allow technology variables A_T, A_R, A_L (or, for short A_i, with $i = T,R,L$) to change over time, either exogenously or as the result of costly investment in new technologies. In the former case of exogenous technological change we assume that $\hat{A}_i$ ($i = T, R, L$) is a constant. In the latter case of endogenous technological case, we assume the following investment function:

(2.10) $\dot{A}_i = G_i(A_i, L_{Ai})$

We will restrict ourselves to a constant-elasticity specification:

(2.11) $\dot{A}_i = \xi_i A^{\varphi_i} L_{Ai}{}^{\lambda_i}$

In this formulation, which we will discuss in detail below, technological change requires labour input L_{Ai} (think of researchers working in the R&D lab).
Total labour inputs in production and research activities cannot exceed labour supply as expressed in the following labour market constraint:

(2.12) $L^S = L + \Sigma L_{Ai}$

2.3. Optimal growth conditions

Optimal growth is the path of consumption that maximizes the welfare criterion subject to the resource and technology constraints. Below, we will derive the long-run optimum growth path in closed form for various cases. However, to point out the common elements it is useful to derive the optimality conditions in their most general form from the complete model. The maximization problem reads:

$$\text{Max} \quad W(0) = \int_0^\infty U(C(t))e^{-\rho t}\,dt$$

$$\text{subject to} \qquad \dot{K} = F(K, L, R, A_T, A_L, A_R) - C$$

$$\dot{S} = -R, \quad S \geq 0$$

$$\dot{A}_i = G(A_i, L_{Ai})$$

$$L^S = L + \Sigma L_{Ai}$$

The following optimality conditions with respect to resource extraction, capital accumulation, and research, respectively, can be derived:

(2.13) $\rho - \hat{U}_C = \hat{F}_R$

(2.14) $\rho - \hat{U}_C = F_K$

(2.15) $\rho - \hat{U}_C = \dfrac{F_{Ai}}{F_L / G_L} + G_{Ai} + F_L / G_{L_{Ai}}$

where $U_C \equiv \partial U / \partial C$, $F_X \equiv \partial F / \partial X$, and $G_{Xi} \equiv \partial G_i / \partial X_i$ denote partial derivatives. We now simplify these expressions in two steps. First we exploit the fact that the elasticities of the utility function U and the knowledge production function G are constant. For the production function F we denote the production elasticity of factor X by θ_X so that $F_X = \theta_X Y / X$, where θ_X may change over time. Now we can rewrite the optimality conditions as :

$$\rho + \frac{1}{\sigma}\hat{C} = \hat{Y} - \hat{R} + \hat{\theta}_R$$

(2.16) $$= \theta_K \frac{Y}{K}$$

$$= \frac{\theta_{Ai}}{\theta_L}\frac{\lambda_i \hat{A}_i L}{L_{Ai}} + \hat{Y} - \hat{L} + \hat{\theta}_L + (1 - \lambda_i)\hat{L}_{Ai}$$

3. Good substitution (Cobb-Douglas)

3.1. The production technology with "good substitution"

We start with a Cobb-Douglas specification of (2.8), which is characterized by constant production elasticities:

$$Y = a_0 K^{\beta} L^{\alpha} R^{\nu} A_L^{\eta_L} A_R^{\eta_R} A_T^{\eta_T}$$

As is well known, in the Cobb-Douglas production function there is no reason to distinguish between input-augmenting technological change and total factor productivity growth. We therefore define $A_{TFP} = A_L^{\eta_L} A_R^{\eta_R} A_T^{\eta_T}$ and write the production function as:

(3.1) $$Y = a_0 K^{\beta} L^{\alpha} R^{\nu} A_{TFP}$$

The Cobb Douglas specification is the most often studied specification in the literature, but depending on whether α, β, or ν is set to zero and depending on whether these parameters add up to one or not, we can distinguish different cases.

3.2. Cake eating, constant returns to scale

The simplest model of growth and non-renewable resources assumes that output is derived only from non-renewable resource inputs, R, of which the productivity, A_{TFP}, grows at a constant and exogenous rate, denoted by a:

(3.2) $$C(t) = Y(t) = e^{at} R(t)$$

It is often called the "cake-eating model" or "hard-tack model". Suppose $a = 0$. Then at time t there is a cake of given size S(t) which can be eaten or stored for consumption later. If $a < 0$ there is decay of the cake. If $a > 0$, the cake grows. In the latter cake the analogy breaks down a bit. Then more appropriate is the interpretation that S is the stock of ingredients to process a cake, and the cake-processing technology becomes more efficient at rate a.

Feasibility.
Let us first examine whether it is possible to keep output at least constant. Differentiating (3.2) with respect to time, we find:

(3.3) $$\hat{Y} = a + \hat{R}$$

Since all output is consumed, this also gives the growth rate of consumption. On a balanced growth path, $\hat{R} = -u < 0$. Since the resource stock gets depleted over time, resource inputs ultimately have to fall over time. This imposes a drag on growth, which is counterbalanced by the boost on output from productivity growth ($a > 0$).

Hence, on a balanced growth consumption grows at a positive rate if $\hat{C} = a - u > 0$. It immediately follows that positive growth of consumption is feasible by choosing a constant rate of depletion below the rate of technology growth: $u = R/S < a$, which is feasible for any positive rate of technological progress. Note that per capita consumption growth requires that the rate of technological progress exceeds the rate of population growth: $C/L = \hat{C} - n = a - n - u > 0$ and $u > 0$ requires $a > n$.

Rawls

Suppose, we choose per capita consumption to be constant over time. How large is this level of consumption? Constant per capita consumption requires $R/S = u = a - n$. The level of consumption is then determined from the production function (CE.1): $C(t)/L(t) = C(0)/L(0) = R(0)/L(0) = uS(0)/L(0)$. So we have:

$$c_{Rawls} = \max\left\{0, (a-n)\frac{S(0)}{L(0)}\right\}$$

So the maximum sustainable level of per capita consumption increases with technological progress and with the per capita resource endowment, but it decreases with population growth. Without technological progress (a = 0), per capita income can only be kept constant or growing if population shrinks (if a = 0, n < 0 we need $0 < u < -n$). Intuitively, by letting the resource decline (deplete) at a slower rate than the rate at which population declines, the resource endowment per capita increases.

Optimality.

Maximize (W) subject to (CE.1) and (S3).

The Hamiltonian and associated first-order conditions read:

$$\begin{aligned}
&H = U(C) + \lambda_Y[e^{at}R - C] - \lambda_S R \\
&H_C = U_C - \lambda_Y = 0 \\
&H_R = \lambda_Y e^{at} - \lambda_S = 0 \\
&H_S = 0 = \rho\lambda_S - \dot{\lambda}_S \\
&\lim_{t\to\infty} e^{-\rho t}\lambda_S S = 0
\end{aligned}$$

Substituting the utility function and eliminating the Lagrange multipliers and co-state variables, we find: $\hat{U}_C = -(1/\sigma)\hat{C} = \hat{\lambda}_Y = \hat{\lambda}_S - a = \rho - (\hat{Y} - \hat{R})$, or:

$$\rho + \frac{1}{\sigma}\hat{C} = \hat{Y} - \hat{R} \tag{3.4}$$

This equation is a fundamental optimality condition. We will show below that it holds for all optimal growth problems with non-renewable resources if utility and production functions are characterized by constant elasticities.

The condition characterizes the optimal timing of resource extraction over time. It equates the marginal cost of postponing consumption (at the left-hand side) to the marginal benefits of conserving a unit of resources (at the right-hand side). The equality states that in the optimum, no net welfare gains can be reaped through

shifting extraction from a certain date to a future date (or the other way around). The gains associated with postponing extraction are at the right-hand side of the equation. They stem from the fact that the productivity of resource use increases over time, so that an extracted unit produces more consumption goods in future than at present. However, extracting one unit of the resource extraction more tomorrow at the cost of today's extraction, shifts consumption to the future, which is costly in terms of welfare. First, it is costly because of discounting. Consumption in future is less worth than consumption today because agents are impatient (as measured by the degree of impatience ρ, see the first term at the left-hand side).[2] Second, it is costly because of satiation in consumption (as measured by the second term at the left-hand side). Consumption in future is less worth than consumption today if there is more to be consumed in the future anyway, thanks to economic growth (i.e. $\hat{C} > 0$), and if marginal utility falls with consumption. The lower σ, the stronger the satiation effect and the higher the cost of postponing consumption in a growing economy. If the economy is shrinking ($\hat{C} < 0$), there are benefits (the second term at the left-hand side becomes negative, reflecting a negative cost) associated shifting consumption to the future, since it is shifted to period of scarce, and hence relatively valuable, consumption.

Optimal growth is now characterized by the production function (3.3) and the optimality condition (3.4). Combining the two and solving for the growth rate and the depletion rate, we find:

$$\hat{C} = \sigma(a - \rho)$$

$$\hat{R} = -[(1-\sigma)a + \sigma\rho]$$

So it is optimal to grow, if the time preference is smaller than the rate of technical change.

So far, we have solved for growth rates only, so we still have to determine the (initial) levels of output, consumption, and extraction. The initial level of extraction must be chosen such that the resource stock tends to zero if time goes to infinity (it cannot become negative, but leaving some of the resource in the ground does not contribute to welfare so must be avoided too).[3] Since R declines at a constant rate, given above, we can easily find the initial rate of extraction such that over time the entire stock is asymptotically used up:

$$S(0) = \int_0^\infty R(t)dt = \int_0^\infty R(0)e^{-\bar{u}t}dt = R(0)/\bar{u} \Rightarrow R(0) = \bar{u}S(0)$$

[2] Recall that ρ is adjusted for population growth. The larger population growth, the smaller ρ. Hence, with higher population growth, the cost of postponing consumption becomes smaller. In this case, consumption is shifted to periods in which more mouths have to be fed.

[3] Formally, the transversality condition informs us that the stock should be zero. The TVC reads $\lim_{t\to\infty} e^{-\rho t}\lambda_S(t)S(t) = 0$. Note that $e^{-\rho t}\lambda_S(t)$ is constant (since $\hat{\lambda}_S = \rho$) and positive, so that we must have $S(\infty) = 0$.

where $\bar{u} = (1-\sigma)a + \sigma\rho$ is the optimal rate of decline of extraction $\hat{R}$. Note that $u = R/S$ is constant over time, so that the growth rate of S, R, Y and C are constant and we are at a balanced growth path (there is no transitional dynamics).

Isomorphy to endogenous growth models

A remarkable characteristic of the simple cake-eating model is that it is isomorphic to the well-known "aK-model" of endogenous growth.

Define $K_S = e^{at}S$ as the stock of resources measured in efficiency units, i.e. in such units that one unit produces one unit of output. We then have

$$\hat{K}_S = a + \hat{S} = a - \frac{R}{S} = a - \frac{e^{at}R}{e^{at}S} = a - \frac{C}{K_S}$$

or

$$\dot{K}_S = aK_S - C$$

This equation shows that the cake-eating model can be reformulated as a model in which output is produced according to a production function $F(K_S) = aK_S$, and in which this output is allocated over consumption (C) and capital investment ($\dot{K}_S$). This is the simplest endogenous growth model in the literature, known as the "aK model" (Rebelo, 1991; Barro and Sala-i-Martin, 1995).

In the "cake eating model", a non-renewable resource, of which productivity grows exogenously at rate a, produces consumption goods with constant returns to scale. In the "aK model", a reproducible capital good, of which productivity is (parametrically) constant at level a, produces consumption goods and capital goods. The two models are equivalent in their structural forms.

This isomorphy is a useful insight for several reasons. It provides a useful reinterpretation of the role of technical change and resource depletion. While the cake-eating formulation is about the trade-off between resource conservation and consumption, the ak-formulation is about the trade-off between capital investment and consumption. Choosing a low rate of extraction, amounts to consuming a little now, leaving a lot of resources to future periods in which their productivity is higher. This is clearly investment: we forego consumption now in order to reap higher benefits of resource use in future. Indeed, resource conservation amounts to productive investment if productivity of resources grows over time.

I.1.2. The Decreasing-Returns Cake Eating (DRCE) model

We now generalize the cake-eating model by allowing for decreasing returns with respect to resource inputs:

$$C(t) = Y(t) = e^{at}R(t)^{\psi} \tag{3.5}$$

We can say that ψ captures "resource dependence". The larger ψ, the more a given reduction in resource use hurts production. Decreasing returns with respect to

resource inputs apply if $0 < \psi < 1$: then, a one percent increase in resource inputs increases output by less than one percent. Finally, if producers are willing to pay a price for resources that equals their the marginal product, ψ is not only the production elasticity but also the cost share of resource inputs in production: $\psi = (\partial Y / \partial R)/(Y/R) = p_R R / Y$. Typically, the share of non-renewable resources in national output is less than 5 percent. Hence, we expect $0 < \psi << 1$.

Feasibility
Let us again first examine whether it is possible to keep output at least constant. Differentiating (3.5) with respect to time, we find:

(3.6) $\hat{Y} = a + \psi \hat{R}$

Per capita output, and hence consumption, is growing along a balanced growth path if $\hat{C} - n = a - n - \psi u > 0 \Leftrightarrow u < (a-n)/\psi$. Hence, society can deplete at a faster rate, the smaller the share of resources in production is. Drag on growth is smaller since the weight of resources in production is small.

Sustainability
A constant level of per capita consumption requires $\hat{C} = a - \psi u = n$ or $u = (a-n)/\psi$. Since per capita consumption equals $C(0)/L(0) = R(0)^{\psi}/L(0)$ and since $R(0) = uS(0)$, the maximum sustainable constant level of per capita consumption reads:

$$c_{Rawls} = \max\left\{0, \left(\frac{a-n}{\psi}\frac{S(0)}{L(0)^{1/\psi}}\right)^{\psi}\right\}$$

So the key condition for sustainability is (as above) that technology improves at a rate that exceeds population growth (or technology regresses at a slower pace than population shrinks).

Optimality
Along similar lines as above, we can determine the first order condition for welfare maximization. We again find (3.4), which holds in any balanced optimal growth path with constant production elasticities. Combining production function (3.6) and the optimality condition (3.4), we find the closed forms solutions for growth and depletion:

(3.7) $$\hat{Y} = \frac{\sigma(a - \psi\rho)}{\psi + (1-\psi)\sigma}$$

(3.8) $$\hat{R} = -\frac{(1-\sigma)a + \sigma\rho}{\psi + (1-\psi)\sigma}$$

Bounded utility requires that the expression in brackets in the second equation is positive. The denominator in the first equation is positive by the assumption of

decreasing returns ($0 < \psi < 1$). Increasing returns can be allowed only if they are counteracted by a sufficiently low elasticity of intertemporal substitution, such that $D \equiv \psi + (1-\psi)\sigma > 0$. The latter condition is required for second order conditions, which can be checked by plotting welfare as a function of endogenous variable *u* only (see appendix).

Positive optimal growth requires $a/\psi > \rho$. As a result, whenever *a* > *n*, positive growth is feasible but not necessarily optimal. Only with enough patience and technological progress and/or low resource dependence is positive growth optimal.

Denoting the left-hand sides of these equations by *g* and *u* respectively, and differentiating, we find the following comparative static results.

Comparative statics (equations (3.7)-(3.8)).

Variable (*i*)	Optimal growth (*g*)	Optimal depletion (*u*)
$\partial i / \partial \rho$	$-\psi\sigma / D < 0$	$\sigma / D > 0$
$\partial i / \partial \sigma$	$\psi g / \sigma D$	$-g / \sigma D$
$\partial i / \partial a$	$\sigma / D > 0$	$(1-\sigma) / D$
$\partial i / \partial \psi$	$-\sigma u / D < 0$	$-u(1-\sigma) / D$

$D \equiv \psi + (1-\psi)\sigma > 0$

Recall that there is a basic trade-off between growth and depletion: more growth requires less depletion for given technology (see 3.6). This trade-off becomes most visible when changing a preference parameter. A more impatient society (larger ρ) optimally grows slower and depletes more (see second row in table). An increasing in the rate of intertemporal substitution affects growth and depletion in opposite directions.

The trade-off between growth and depletion is mitigated by changes in technological parameters. A lower share of resource in production increases both growth and depletion (see last line). The other way around, larger resource dependence reduces optimal growth and depletion. The smaller ψ is, the smaller the influence preference parameters have on growth, but the larger their influence on depletion. (Indeed, with the limit case $\psi = 0$, we are back in a Solow world with no influence of preference parameters).

Technological change is a prerequisite for optimal growth to be positive. More technological change unambiguously raises optimal growth, but the impact on depletion is ambiguous, since income and substitution effects work in opposite directions. Faster technological change implies higher returns to investment, which is an incentive to postpone extraction (substitution effect). It also means, however, larger income over the entire horizon, which induce more consumption and hence more extraction (income effect). If $\sigma < 1$ the income effect dominates the substitution effect.

Isomorphy

The generalized Cake eating model is no longer strictly isomorphic to AK-model, but we can reinterpret the cake-eating model in a AK-*like* model. Define $K_S = e^{at} S^{\psi}$ as the stock of resources measured in consumption equivalents. We then have:

$$\hat{K}_S = a + \psi \hat{S} = a - \psi(e^{at} R)/(e^{at} S) = a - \psi\left(C / K_S\right)^{1/\psi}$$

or

$$\dot{K}_S = aK_S - C\psi\left(\frac{C}{K_S}\right)^{(1-\psi)/\psi}$$

The basic features of the "AK-model" again show up: the production function, $F(K_S) = aK_S$, has constant returns with respect to a reproducible factor (K_S) and output can be one-to-one transformed into capital goods.
The difference with the canonical AK-model is that output cannot be transformed one-to-one into consumption goods. The larger consumption is relative to the capital stock K_S, the more investment has to be foregone for each marginal unit of consumption. The transformation curve is concave rather than a straight line (as was the case with $\psi = 1$). [The term multiplying C can be interpreted as the price of consumption in terms of output or investment goods]. The parameter ψ can be interpreted in terms of adjustment costs: the exponent $(1-\psi)/\psi$ captures the degree of adjustment cost with respect to consumption. The larger this exponent, the more concave the transformation curve is and the more investment has to be given up to increase consumption. Higher adjustment costs lead to higher growth since they disfavour consumption. [So the production possibilities frontier is concave if $\psi < 1$ and convex if $\psi > 1$. My conjecture is that the condition $D > 0$ corresponds to the condition that the PPF is less convex than the iso-utility curve so than an interior solution exists.]
However, here we want to stress that the essential structure of the AK-model is preserved. By reducing current consumption through depleting less, society invests in a productive asset, which is the resource stock measured in efficiency units (K_S). The smaller the production elasticity of resources, i.e. the smaller the role of resources in production (as measured by a smaller value of ψ), the more consumption has to be given up to save one unit of resource. Nevertheless, what is crucial is that the return to investing in the asset does not fall over time, which is accomplished by the constant rate of technological change a.

3.3. Capital accumulation

We now introduce a produced input, physical capital, which is accumulated by forgoing consumption, $\dot{K} = Y - C$. The production technology is given by the Cobb-Douglas production function $Y = A_{TFP}K^{\beta}R^{\nu}L^{\alpha}$, where β,ν, and α are the production elasticities of K, R, and L, respectively, and A_{TFP} is total factor productivity. We assume a constant exogenous rate of labour growth and total factor productivity growth ($\hat{A}_{TFP}, \hat{L}$ constant). We can write the time derivative of the production function as:

(3.9) $\hat{Y} = \beta\hat{K} + \nu\hat{R} + \alpha\hat{L} + \hat{A}_{TFP}$

Along a balanced growth path, output, capital and consumption grow at the same rate. (Proof, see Lemma 2, Groth/Schou (2002)): $\hat{Y} = \hat{K} = \hat{C}$, so $Y/K, C/K, C/Y$ are constant, and, as before, $\hat{R} = \hat{S} = -R/S$ is constant. Then the above equation boils down to:

(3.6) $\hat{Y} = a + \psi\hat{R}$

where

$$\psi = \nu/(1-\beta),$$
$$a = (\hat{A}_{TFP} + \alpha\hat{L})/(1-\beta)$$

Hence, along the balanced growth path, production possibilities are isomorphic to the decreasing-returns-cake-eating-model (compare (3.6))!

Feasibility
A balanced growth path with positive per capita consumption is feasible if $u = R/S$ can be chosen for all t such that the following conditions hold:

$$\hat{Y} - \hat{L} = \beta\hat{K} + \nu\hat{R} + \alpha\hat{L} + \hat{A}_{TFP} - \hat{L} > 0 \quad \wedge \quad u > 0$$
$$\Leftrightarrow \quad \hat{Y} - n = \frac{\hat{A}_{TFP} - (1-\alpha-\beta)\hat{L} - \nu u}{1-\beta} > 0 \quad \wedge \quad u > 0$$

First consider the "normal case" in which population is growing (or at least not decreasing, $n > 0$ or $n = 0$)[4] and diminishing returns apply with respect to capital and labour together ($\alpha+\beta < 1$). In this case, per capita income can only grow if there is enough technological progress. Capital accumulation is not a substitute[5] to technological progress in this case! That is, without technological progress no growth is possible in the long run. Note that the growth rate does not depend on the rate of investment in physical capital (savings rate). [Policy implication: savings policy does not affect growth, but depletion policy does, cf. Schou/Groth (2002b)] The reason is diminishing returns with respect to capital. Investing more in physical capital makes capital more abundant relative to the other inputs (S and L). However, because of diminishing returns, the marginal productivity of capital falls when K is becoming more abundant relatively. A given savings rate therefore produces less and less growth of output (as in the Solow model). Capital substitutes for resource inputs, which are falling over time, but the effect of this substitution on output becomes smaller and smaller because of the diminishing returns.

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[4] As above, a shrinking population ($n < 0$) in the presence of diminishing returns ($\alpha+\beta<1$) helps to sustain growth in the absence of technological change: if $\hat{A}_{TFP} = 0$ and $\beta<1$, we need $0 < u < -n(1-\alpha-\beta)/\nu$. Intuitively, the resource endowment per capita increases if depletion is kept sufficiently low relative to the rate of change of population.

[5] In contrast, capital accumulation and technological change are complements: the larger β, the more effective a given rate of technological progress is in creating growth ($dg/d\hat{A}_{TFP}$ increases with β).

Technical change is needed to offset the diminishing returns of capital accumulation. Technical change increases the productivity of capital and offsets the fall in returns due to capital-resource substitution. The presence of technical change is not sufficient; it must be large enough to sustain growth. In particular, it must be larger the faster population is growing and the quicker the resource stock is depleted, in order to offset the fall in per capita resource availability.

Next consider increasing returns: can they make growth feasible without technological progress? Yes, they can. Suppose there are "mildly" increasing returns such that $\alpha + \beta > 1$ but $\beta < 1$. Then growth is feasible provided the population grows. The increase in population over time provides more inputs and growing output such that the productivity of capital (output per unit of capital *Y/K*) tends to increase over time. This creates the room to let resource inputs fall over time (as is needed to conserve resources for the future), without necessarily lowering the productivity of capital. Note however from (3.7) that the optimal growth rate is negative in the absence of technological change).

Now suppose $\beta > 1$, which implies a stronger form of increasing returns since they now apply to capital only (note $\alpha + \beta + \nu > \beta > 1$). For this case, a balanced growth path without technological change and without population growth is sustainable, but it is unstable (Groth/Schou 2002a).

Sustainability

From the previous subsection, we can easily derive the conclusions with respect to the maximum sustainable consumption level. In particular, the expression in section 3.2 gives the maximum sustainable level of consumption. It can be reached if $a > n$, by choosing the following depletion rate (u) and savings rate ($s = 1 - C/Y$):

$$u_{rawls} = \frac{a-n}{\psi}, s_{rawls} = \frac{n \cdot K(0)^{1-\beta}}{A_{TFP}(0) \cdot [u_{rawls} \cdot S(0)]^{\nu} \cdot L(0)^{\alpha}}.$$

For initial K too big relative to the other inputs, the solution for s exceeds unity, which is infeasible. The reason is that if capital is very abundant, its marginal product is very low and savings are unproductive in producing capital, so that it is impossible to invest enough to maintain the level of output. Note that for $n = 0$ or for $n = a$, the expressions are not defined. Yet, under certain conditions, a constant level of consumption can be maintained for $n = a = 0$. From the production function, the efficiency condition and capital accumulation we may write:

$$\hat{Y} = \beta\hat{K} + \nu\hat{R}$$
$$\beta Y/K = \hat{Y} - \hat{R}$$
$$\hat{K} = sY/K$$
$$(1-\nu)\hat{Y} = (Y/K)\beta(s-\nu)$$

Combining we find:

$$(1-\nu)\hat{Y} = (Y/K)\beta(s-\nu)$$

Choosing a constant savings rate $s = \nu$ gives constant consumption (Solow/Hartwick rule). Note that this implies a non-BGP, since Y is constant and K is growing so that Y/K falls and $\hat{K} = sY/K$ also falls over time. Solow (1974) shows that consumption is only positive if $\beta > \nu$.

Optimality

The first order condition for optimal depletion of the non-renewable resource is again the same as in (3.4). As we noted above, it applies whenever utility and production functions are characterized by constant elasticities. The condition for optimal capital accumulation can be derived as (cf (2.16)):

$$\rho + (1/\sigma)\hat{C} = \beta Y/K \tag{3.10}$$

This is the Ramsey rule, equating the cost of postponing consumption to the returns of investment in physical capital.

[Note that (3.4) and (3.10) together imply $\hat{Y} - \hat{R} = \beta Y/K \Leftrightarrow \hat{F}_R = F_K$. This is the *efficiency* condition in Dasgupta/Heal, Solow and Stiglitz (1974).]

Note that (3.6) and (3.4) are isomorphic to the DRCE model and fully solve for g and u in the model with capital accumulation. Hence, all expressions derived with respect to feasibility and sustainability thus also apply to the model with capital. We only have to reinterpret the expressions to trace the role of capital and population growth, which are now hidden behind parameters ψ and a.

First, for a given production elasticity of resource inputs, ν, a larger value for β increases the role of resources, since $\psi = \nu/(1-\beta)$. The reason is as follows. Resource inputs fall over time. This directly reduces output. It also negatively affects the marginal product of capital, which reduces the incentive to capital accumulation. Through reduced capital stocks the decline in resource use indirectly reduces growth.

Second, we can now check whether per capita growth can be optimal without exogenous technological change. Calculating the per capita optimal growth rate for zero TFP growth (by setting $\hat{A}_{TFP} = 0$, subtracting n, and writing out ψ), we find:

$$g - n = \frac{n[\sigma(\alpha + \beta + \nu - 1) - \nu] - \nu\rho}{\nu + (1 - \beta - \nu)\sigma}$$

This expression reveals that positive growth is optimal only if we have population growth and increasing returns to scale in production (Groth and Schou 2002).

Transitional dynamics optimal growth

Although the BGP of the model with capital is isomorphic to the DRCE model, the dynamics of the former model is such that the BGP is, in general, reached only through transitional dynamics. See Groth and Schou (2002) for details.

Market equilibrium (CRS)

Suppose there are constant returns to scale ($\alpha + \beta + \nu = 1$). Markets are competitive. Agents have rational expectations, consumers maximize utility, firms and resource owners maximize profits. Then the market equilibrium coincides with the social optimum.

Reinterpretation: one-sector model of endogenous technology (IRS)
So far we have considered exogenous technological change. In the growth literature, two ways to endogenize technical change have become popular: learning by doing and R&D-based technical change. We now show that the one-sector model just presented can be reinterpreted as a model containing endogenous technical change through learning by doing. In the next section we turn to the R&D approach.
Assume output is produced according to a function that has constant return with respect to rival factors *K*, *L*, and *R*. This is appealing because of the so-called replication argument: doubling all rival inputs should double output (Romer 1990). Note that is argument holds for given technology. Technology might improve because of learning by doing. An economy that has invested more has solved more problems and accumulated more experience. Hence, we can relate the level of total factor produtcivity , A_{TFP}, to the capital stock. Therefore, we write:

$$Y = R^{\nu} K^{\beta_1} L^{\alpha} A_{TFP}$$
$$\alpha + \beta_1 + \nu = 1$$
$$\dot{K} = Y - C$$
$$A_{TFP} = K^{\beta_2}$$

Combining these equations, we find:

$$Y = R^{\nu} K^{\beta_1 + \beta_2} L^{\alpha}, \qquad \alpha + \beta_1 + \beta_1 + \nu > 1$$

This is exactly the production function with increasing returns we studied above, if we define $\beta_1 + \beta_2 = \beta$. Groth and Schou's (2003) conclusion that with increasing returns to scale growth can be sustained without exogenous technological progress can therefore be restated as the conclusion that endogenous technological change stemming from learning by doing driven by capital accumulation makes growth sustainable without exogenous technological change.
Of course one important aspect of learning by doing is that it creates (Marshallian) externalities. Hence market solutions and social optimum will no longer coincide. We will return to this below.

I.3. Knowledge accumulation
We now want to discuss endogenous technological change that stems from deliberate investments in new technologies. This R&D-driven technological change should be distinguished from learning-by-doing, since it is an activity separate from production. Investments in technologies should also be distinguished from investment in the "physical capital stock" as discussed above. Physical capital consists of rival pieces of equipment, while technologies should be interpreted as ideas or blueprints for new ways to produce.
New technologies (ideas) are modeled as a non-rival input in production, denoted by *A*, that complements the rival inputs *K, L, R*. [CHECK JONES 2004] We again choose iso-elastic form to start with, so that we we have the following production function:

(3.11) $Y = R^{\nu} K^{\beta} L^{\alpha} A^{\eta}$

As explained above, the replication argument makes it reasonable to assume constant returns with respect to rival inputs (*K,L,R*), so that increasing returns to scale arise with respect to all inputs (*K,L,R,A*). In the absence of learning-by-doing it therefore seems reasonable to assume $\alpha + \beta + \nu = 1$.

The production of new ideas or knowledge, which will be labeled research and development, requires effort in terms of labour, denoted by L_A. [It seems reasonable that production of knowledge is less capital intensive than production of goods: the difference in production technology is another reason why knowledge and physical capital should be distinguished. The assumption that neither resources nor capital is used is not innocuous. See Bretschger (2004) and Groth and Schou (2002).] Furthermore, current research builds on the existing stock of ideas *A*. The iso-elastic formulation of the knowledge production function, or the R&D technology, reads (cf. Jones 1995):

(3.12) $\dot{A} = \xi A^{\varphi} L_A{}^{\lambda}$

where L_A is the part of total labour supply not used in production (*L*) but in research activities:

(3.13) $L + L_A = L^s$

Parameter $\varphi < 1$ captures intertemporal knowledge spillovers, $0 < \lambda \leq 1$ captures congestion (or duplication) in research. This function is the generalization that Jones (1995) put forward of Romer's (1990) specification, which assumed strong spillovers ($\varphi = 1$) and had constant returns to labour effort ($\lambda = 1$). The model with diminishing returns with respect to the stock of ideas ($\varphi < 1$) is sometimes coined the "semi-endogenous growth model", to distinguish it from the "endogenous growth model" with $\varphi = 1$. Semi-endogenous refers to the fact that technological change is endogenous (depends on the deeper parameters of the model), but long-run growth is independent of preference parameters or economic policies. Eicher and Turnovsky (1999) have generalized the semi-endogenous approach and refer to "non-scale models", since the long-run growth rate is independent of the scale of the economy (*L*).

In an alternative formulation, the cost of R&D is in terms of goods rather than labour (the so-called lab equipment formulation, see Romer and River-Batiz 1991). A further generalization of the knowledge production function is therefore $\dot{A} = \xi A^{\varphi} L_A{}^{\lambda} I_A{}^{\iota}$, which we will not discuss now.

A balanced growth path requires a constant rate of change of *A* and a constant ratio of total labour allocated to research (L_A/L constant). Hence L_A grows at rate *n*. The balanced growth rate of *A* then follows directly from (3.12):

(3.14) $\dfrac{\eta\lambda}{1-\varphi} n = \eta \hat{A} \quad (= \hat{A}_{TFP})$

which can only hold if $\varphi < 1$. We now have an expression for TFP growth which replaces the exogenous rate of TFP growth in the capital accumulation model above.

That is, after adding equation (3.12) and the labour market constraint (3.12) to the model above, all previous results go through, but now we see how long run TFP growth is determined. Population growth drives technological change (cf. Simon 1996). The reason is that a constant rate of technological change requires a constant rate of growth of the number of ideas. Every period in time the number of new ideas should growth such that a constant fraction of the total stock of ideas is added every period. However, developing a growing number of new ideas becomes harder and harder and can only be accomplished if the amount of labour effort in research grows, which requires population growth [Footnote: in alternative models, it is human capital rather than "raw labour" that matters in the knowledge production function; then *n* should be replace by the growth rate of human capital, which might be endogenous itself. See Arnold (1998).] Furthermore λ, η and φ play a role: more intertemporal spillovers (high φ) and less congestion (high λ) and more productive ideas (high η) are conducive to TFP growth.

A special case is φ = 1, explored in the resource context by Schou (1996, 1999), Scholz/Ziemes (1999), and Aghion and Howitt (1998 chapter 5). This case corresponds of the Romer (1990) model of "endogenous growth". It generates the following growth rate of total factor productivity (from (3.12)):

$$\eta\xi(L^S - L)^\lambda = \eta\hat{A} \quad (= \hat{A}_{TFP})$$

A given number of researchers ($L^S - L$) produces a constant rate of total factor productivity growth. Hence, with a constant population and strong enough spillovers ($\varphi = 1, n = 0$), a whole range of rates of technological change can be maintained along a balanced growth path, depending on the amount of labour allocated to research.

An unrealistic feature of this approach is that the growth rate depends on the scale of the economy: a bigger economy (as measured by its labour force *L*) grows faster, which is against empirical evidence. See Jones (1999). [Furthermore, economic growth should be positively related with population growth, which also runs against the evidence.] Several solutions have been proposed to eliminate the scale effect, without necessarily ending up at the semi-endogenous growth framework (Smulders/Van de Klundert 1995, Young 1998, Howitt 1998, Peretto Smulders 2002). What is important to note here is that the balanced growth path of the endogenous growth model is an analytically more convenient way to study the medium-term growth effects that would arise in the semi-endogenous growth framework (Smulders 1998, Groth and Smulders 2004). The endogenous growth framework provides an analytically useful workhorse model to say something about the effect of policies and preferences on growth and economic dynamics.

Sustainability

With endogenous growth and zero population growth ($\varphi = 1, n = 0$), per capita consumption is constant if output is constant. Along a balanced growth path, *K* and *Y* are constant and the decline in R is exactly offset by the increase in total factor productivity. $0 = \nu R + \hat{A}_{TFP} \Rightarrow u = (\eta\xi/\nu)(L^s - L)^\lambda$. This implies a continuum of combinations of depletion and labour allocation that constant consumption. The associated level of consumption is hump-shaped in *L*, since $C(0) = A(0)^\eta K(0)^\beta (uS(0))^\nu L^\alpha = A(0)^\eta K(0)^\beta (S(0)\eta\xi/\nu)^\nu (L^S - L)^\lambda L^\alpha$. Hence, the

maximum level of sustainable consumption is reached when the following fraction of labour supply is allocated to research:

$$(L_A / L^S)_{Rawls} = \frac{\lambda}{\lambda + \alpha}$$

With semi-endogenous growth ($\varphi < 1$), the maximum constant level of consumption is more challenging to determine. Since the rate of total factor productivity changes over time, we cannot directly apply the sustainability expressions from above.

Optimality

Optimal growth requires an optimal allocation of labour across production and research in addition to the allocation of production over consumption and investment and the optimal depletion of the resource over time. The social planner maximizes welfare (2.7) subject to the resource constraint (2.2) and $Y = R^{\nu} K^{\beta} L^{\alpha} A^{\eta}$, $\dot{K} = Y - C$, $\dot{A} = \xi A^{\varphi} (L^S - L)^{\lambda}$. Combining the first order conditions, we find along the balanced growth path:

$$\rho + (1/\sigma) g = g - \hat{R} = \beta Y / K = \frac{\eta\lambda}{\alpha} \hat{A} \frac{L}{L_A} + g - \lambda n$$

The first two equalities are the optimality conditions with respect to extraction and capital accumulation already discussed above. The last equality governs the optimal investment in knowledge accumulation.

Semi-endogenous growth: (3.14) gives the solution for $\hat{A}$ in balanced growth. Using this solution in the first two equalities and (3.6), we solve for *g*, *u*, and *Y/K*, which coincide with the exogenous technology case (see (3.7) and (3.8)) after replacing $\hat{A}$ by the expression in (3.14). The last equality solves for L / L_A:

$$\frac{L_A}{L} = \left(\frac{\eta\lambda}{1-\phi} \right) \left(\frac{1}{\alpha} \right) \left(\frac{\lambda n}{u + \lambda n} \right)$$

From a "resource and growth perspective", the semi-endogenous growth model does not yield big surprises or novel insights: the results are similar to the Stiglitz model, apart from a one-level-deeper-explanation of the rate of technological change, which is however independent of any resource consideration. We notice that now population growth plays a bigger role: population growth drives growth and the effect of population growth on aggregate growth is larger because of the increasing returns in the production function (with respect to rival and non-rival factors taken together) and because of the fact that labour produces non-rival knowledge inputs.

From a "pure growth perspective", the results from the model of knowledge accumulation with weak spillovers ($\varphi < 1$) and non-renewable resources ($\nu > 0$), the results might be surprising. Usually, weak spillovers imply a growth rate of output that is independent of intertemporal preferences (Jones 1995), but through the presence of non-renewables, the discount rate does affect growth. The reason is the same as why augmenting the Solow model for non-renewables (thus arriving at the Stiglitz model) makes the growth rate dependent on preferences: growth depends on

non-renewable inputs, of which the rate of decline can be chosen endogenously and involve an intertemporal trade-off.

To study the endogenous growth case, we set $\varphi = 1, n = 0, \alpha + \beta + \nu = 1, \lambda = 1$ [footnote: $\varphi = 1$ defines endogenous growth, n = 0 prevents exploding growth rates, constants returns to scale with respect to rival factors imply absence of learning by doing and the replication argument, λ = 1 allows for closed form solutions for g.] From the knowledge production function (3.12), labour market equilibrium (3.13), and the production function under balanced growth, we may write, respectively:

$$L = L^S - L_A,\ L_A = \hat{A}/\xi,\ g = \beta g + \nu \hat{R} + \eta \hat{A}$$
$$\Rightarrow \eta \xi L = \eta \xi L^S - (1-\beta) g + \nu \hat{R}$$

Hence, the optimality conditions can be written as:

$$\rho + (1/\sigma) g = g - \hat{R} = \beta Y / K = \frac{\eta \xi L^S - \nu (g - \hat{R})}{\alpha}$$

These equations solve for g, $\hat{R}$, and Y/K. The solution for the growth rate then reads:

$$g = \sigma \left(\frac{\eta}{\alpha + \nu} \xi L^S - \rho \right)$$

while depletion and output capital ratio depend on this growth rate as before according to: $\hat{R} = -[\rho + (1/\sigma - 1) g],\ Y/K = [\rho + (1/\sigma) g]/\beta$. This is the central planning solution of the model in Aghion/Howitt (1998) (they normalize $\eta = 1 - \beta = \alpha + \nu$). [Takayama (1980) already studied this case, but made a mistake. Barbier (1999) considers a version of the model with two types of labour. Scholz/Ziemes (1999) studies the market solution (see below) and Schou (2000) extends the model for polluting resources.]

Discuss efficiency condition: resource dependence ($\nu > 0$) reduces the rate of return to knowledge creation (drag on growth). Well-known element from endogenous growth: scale effect.

Discuss g: scale effect, decreasing with ν: if resource share, ν, becomes bigger at the cost of the capital share (α, rather than the labour share β), while maintaining CRS, the optimal long-run growth rate becomes smaller because of the resource drag. In particular, it is the share of non-reproducible production factors ($\alpha + \nu$) that impose the drag on growth. [Aghion/Howitt (1998) normalize $\eta = \alpha + \nu$, which is harmless, unless comparative statics with respect to production elasticities are performed.]

The solution for growth looks like the solution of the cake-eating model with constant returns: with $a = (\alpha + \nu)^{-1} \eta \xi L^S$, the constant-returns-to-scale cake-eating model and the endogenous technological change model generate the same long-run optimal growth path. Let us try to explain why the rate of return remains constant and why constant returns to resources arise in the long run. First, the return to investment, that is foregoing consumption, becomes a constant in the long run. In the cake-eating model this is exogenously so. In the endogenous growth model this arises because of arbitrage between the three assets (resource stock, capital stock and knowledge stock)

and because there are constant returns to knowledge accumulation ($\varphi = 1$), which prevents the rate of return from falling below a certain level. Second, although there are diminishing returns with respect to resource inputs for given levels of the other inputs *K*, *L*, and *A* (that is, $\nu < 1$), there appear constant returns to resource inputs along the balanced growth path at equilibrium levels of these other inputs. The reason is that an increase in resource use directly increases output with diminishing returns, but it also increases output indirectly through capital accumulation and knowledge accumulation, such that along an optimal balanced growth path a one percent increase in *R* leads to a one percent increase in *Y*. This, however, depends on the assumption 1 = α + β + ν. If we relax this (but still assume $n = 0$, $\lambda = 1$), we find

$$g = \frac{\sigma\left[\left(\frac{\eta\xi L^S}{1-\beta}\right) - \left(\frac{\alpha+\nu}{1-\beta}\right)\rho\right]}{\left(\frac{\alpha+\nu}{1-\beta}\right) - \left[1 - \left(\frac{\alpha+\nu}{1-\beta}\right)\right]\sigma}$$

which coincides with the cake-eating-model, see (3.7), if we define

$$\psi = \left(\frac{\alpha+\nu}{1-\beta}\right)$$

$$a = \frac{\eta\xi L^S}{1-\beta}$$

4. Poor substitution

4.1. Defining poor substitution

Up to now we have considered cases in which the production elasticities are constant. This is not very appealing. First, it means that, even if the amount of energy available in the economy approaches zero, a doubling of other inputs increases output by $\alpha+\beta$ percent (where α and β are the production elasticities of these other inputs). Second, it implies that the elasticity of substitution between resources and other inputs equals unity. Empirical studies find in general values different from unity. Third, a value of one seems to be a knife-edge case. Finally, the unitary elasticity of substitution rules out biases in technological change. Factor-augmenting technological change does not affect the relative marginal productivities of different factors, that is, technological change can only be neutral.

One would expect that when almost no oil is available, an increase in other inputs would not have much of an effect, compared to the situation in which there is ample of the resource. This is in fact the notion of poor substitution, which we will now formalise. Consider the following production function:

$$Y = F(R_E, K, L_E, t)$$

where the subscript E means that the input is measured in effective units. Let θ_i denote the production elasticity of factor i, that is

$$\theta_R = \frac{\partial F(R_E, K, L_E, t)}{\partial R_E}\frac{R_E}{F}$$

and similar for L and K. The growth rate of production can now be written as:

$$\hat{Y} = \theta_R \hat{R}_E + \theta_K \hat{K} + \theta_L \hat{L}_E + (\partial F / \partial t)/F \qquad \text{(II.1)}$$

We define poor substitution as follows:

Poor substitution: Factor i is a poor substitute for R if $\lim_{R_E \to 0} \theta_i(R_E, K, L_E, t) = 0 \qquad i \neq R_E$

4.2. The CES production function

The CES production function with factor augmentation is the specification most commonly used and most convenient to illustrate this. This specification can be written as:

$$Y = F(R_E, K, L_E) = \left(\nu R_E^{(\sigma_Y - 1)/\sigma_Y} + \beta K^{(\sigma_Y - 1)/\sigma_Y} + \alpha L_E^{(\sigma_Y - 1)/\sigma_Y}\right)^{\sigma_Y/(\sigma_Y - 1)}$$

(4.1) $$R_E = A_R R$$
$$L_E = A_L L$$

where σ_Y is the elasticity of substitution and A_i is the factor augmentation level for factor i. [This function has constant returns with respect to the rival factors K,R,L.

Non-CRS can be modelled by raising the RHS to a power $1+\iota$, where ι is the degree of increasing returns to scale].
The following properties applies with respect to the production elasticities:

$$\theta_R = \nu(Y/R_E)^{(1-\sigma_Y)/\sigma_Y};\ \theta_L = \alpha(Y/L_E)^{(1-\sigma_Y)/\sigma_Y};\ \theta_K = \beta(Y/K)^{(1-\sigma_Y)/\sigma_Y}$$

(4.2)

$$\frac{\partial F}{\partial R}\frac{R}{F} = \frac{\partial F}{\partial A_R}\frac{A_R}{F} = \theta_R;\quad \frac{\partial F}{\partial L}\frac{L}{F} = \frac{\partial F}{\partial A_L}\frac{A_L}{F} = \theta_L$$

The factors are poor substitutes if $\sigma_Y < 1$.

$$\begin{aligned}
&\lim_{R_E \to 0} \theta_R(R_E, K, L_E) = 1 \\
&\lim_{R_E \to 0} \theta_K(R_E, K, L_E) = 0 \\
&\lim_{R_E \to 0} \theta_L(R_E, K, L_E) = 0
\end{aligned} \tag{II.3}$$

In contrast to the Cobb Douglas case, we now need to recognize that technological change is no longer necessarily neutral. A one percent change in A_L will affect the marginal productivity of the inputs by different percentages (while in the Cobb-Douglas case, they all change by the same percentage). This implies that technological changes might be biased in favour of a particular factor by increasing its elasticity of production more than that of other factors. To make things more precise, consider the change in the relative production elasticities of labour and resources:

$$\theta_R / \theta_L = (\nu/\alpha)(L_E/R_E)^{(1-\sigma_Y)/\sigma_Y}$$
$$\Downarrow$$
$$\hat{\theta}_R - \hat{\theta}_L = \frac{1-\sigma_Y}{\sigma_Y}(\hat{L} + \hat{A}_L - \hat{A}_R - \hat{R})$$

Hence, if $\hat{A}_L > \hat{A}_R$ and $\sigma_Y < 1$, then technological change increases the production elasticity of resources relative to that of labour. In other words, production becomes more intensive in the use of resources. Technological change is resource-using and labour-saving (Burmeister and Dobell 1970, p.69). [Acemoglu (2002) uses different terminology: technological change is biased in favour of resources].
From now on, we assume the production function is CES with $\sigma_Y < 1$.

4.3. Exogenous technological change

Optimal growth
We maximize intertemporal utility (2.7) subject to the above production function (4.1) and the resource depletion equation (2.2). This gives the familiar two optimality conditions, cf (2.13)-(2.14):

$$\rho + \hat{U}_c = \hat{F}_R = F_K$$

Calculating the marginal productivity of the resource and differentiating with respect to time, we find from (4.1):

$$F_R \equiv \frac{\partial F}{\partial R} = \nu \left(\frac{Y}{A_R R} \right)^{1/\sigma_Y} A_R$$

$$\hat{F}_R = \frac{1}{\sigma_Y}(\hat{Y} - \hat{R}) - \left(\frac{1}{\sigma_Y} - 1 \right) \hat{A}_R$$

Hence, the optimality conditions (2.13)-(2.14) boil down to:

$$\rho + \frac{1}{\sigma}\hat{C} = \frac{1}{\sigma_Y}(\hat{Y} - \hat{R}) - \left(\frac{1}{\sigma_Y} - 1 \right) \hat{A}_R = \beta \left(\frac{Y}{K} \right)^{1/\sigma_Y}$$

Note that if $\sigma_Y = 1$, we find the results we derived for the Cobb Douglas case.

Balanced Growth
Growth of output can be written as:

(4.3) $\hat{Y} = \theta_R(\hat{R} + \hat{A}_R) + (1 - \theta_R - \theta_L)\hat{K} + \theta_L(\hat{L} + \hat{A}_L)$

Balanced growth requires constant growth rates, which requires constant production elasticities, which in turn requires that all inputs with asymptotically non-zero production elasticity grow at the same rate when measured in effective terms. A first possible case is therefore:

(4.4) $$(\hat{A}_R + \hat{R}) = \hat{K} = (\hat{A}_L + \hat{L}) = \hat{Y} = \hat{C} = g$$
$$\theta_L > 0,\ \theta_R > 0,\ \theta_K > 0$$

The remarkable feature of a balanced growth path is that in this case the cake-eating technology applies, since $\hat{Y} = \hat{A}_R + \hat{R}$. Substituting this equation into the optimality condition, we find:

(4.5) $$\rho + \frac{1}{\sigma} g = \hat{A}_R = \beta \left(\frac{Y}{K} \right)^{1/\sigma_Y}$$
$$-\hat{R} = \hat{A}_R - g$$

Hence, on a balanced growth path with given rate of resource augmenting technological change ($\hat{A}_R$), the production elasticity of substitution, σ_Y, does not matter for growth and not for the rate of depletion! Hence, what seems to be a knife-edge case does no longer seem that restrictive.
However, a balanced growth path needs not be feasible. (4.4) and (4.5) are consistent only if

$$g = \sigma(\hat{A}_R - \rho) = \hat{A}_L + \hat{L}$$

In the case of exogenous technological labour-augmenting and resource-augmenting technological change, the second equality in (4.5) is a condition on parameters only so it holds by coincidence only. Then two other cases become relevant.

As a second case, consider the case with $\sigma(\hat{A}_R - \rho) < (\hat{A}_L + \hat{L})$. In this case, the long-run production elasticity of labour approaches zero, while output and the other two factors grow at the same rate:

$$g = \sigma(\hat{A}_R - \rho) < (\hat{A}_L + \hat{L})$$
$$u = (1-\sigma)\hat{A}_R + \sigma\rho$$
$$\theta_L = 0, 1-\theta_R = \theta_K = \beta^{\sigma_Y}(\hat{A}_R)^{1-\sigma_Y}$$

(From the second equation we find $\hat{A}_R + \hat{R} = \sigma(\hat{A}_R - \rho) < \hat{A}_L + \hat{L}$ so that $\hat{\theta}_L < \hat{\theta}_R = 0$ and hence θ_L asymptotically tends to zero. The solution for θ_K follows from the second optimality condition (the Ramsey rule) and the definition of θ_K).

In the third case we have $\sigma(\hat{A}_R - \rho) > (\hat{A}_L + \hat{L})$, then the production elasticity of the resource approaches zero, while output, capital and effective labour grow at the same rate as in the the Solow-model. Furthermore we can calculate the long-run (assymptotic) rate of depletion:

$$g = (\hat{A}_L + \hat{L}) < \sigma(\hat{A}_R - \rho)$$
$$u = \left[(1-\sigma)\hat{A}_R + \sigma\rho\right] + \left(1 - \frac{\sigma_Y}{\sigma}\right)\left[\sigma(\hat{A}_R - \rho) - (\hat{A}_L + \hat{L})\right]$$
$$\theta_R = 0, 1-\theta_L = \theta_K = \beta^{\sigma_Y}[\rho + (\hat{A}_L + \hat{L})/\sigma]^{1-\sigma_Y}$$

This latter case seems to be what most economists would think to fit best with stylised facts (cf. Jones 2002). Indeed, US postwar experience has shown a small and decreasing resource share θ_R.

Summarizing, we conclude that with exogenous technological change, the optimum growth path either tends to the following value:

$$g = \min\left\{\sigma(\hat{A}_R - \rho), (\hat{A}_L + \hat{L})\right\} \tag{4.6}$$

In other words, depending on intertemporal preferences, population growth and technological change, the long-run optimum path is either “Cake” or “Solow”. The elasticity of substitution in production does not affect the long has growth rate, but does affect the long-run capital share and the depletion rate in the Solow-solution.

4.4. Endogenous technological change

We now allow for investment in technological change. We simply generalize (3.12) to the two factor augmentation levels in (4.1):

$$\dot{A}_R = \xi_R (A_R)^{1-\varphi_R} (L_R)^{\lambda_R}, \quad \dot{A}_L = \xi_L (A_L)^{1-\varphi_L} (L_L)^{\lambda_L}$$

If knowledge spillovers are small ($\varphi_R < 1, \varphi_L < 1$), balanced growth requires:

$$\hat{A}_R = \frac{\lambda_R}{1-\varphi_R} n, \qquad \hat{A}_L = \frac{\lambda_L}{1-\varphi_L} n$$

This implies that the long-run rates of technological change are independent of the optimality conditions. Hence, we substitute the above expressions in the solutions of the model with exogenous technological change (4.6) to find the long-run optimal semi-endogenous growth solution. This gives again a Cake or Solow path, depending now on the parameters of the knowledge production function:

$$\text{(4.7)} \quad g = \min\left\{ \sigma\left(\frac{\lambda_R n}{1-\varphi_R} - \rho \right), \left(\frac{\lambda_L + 1 - \varphi_L}{1-\varphi_L} \right) n \right\}$$

If knowledge spillovers are strong ($\varphi_R = \varphi_L = 1$) we arrive at an endogenous growth model. We need to abstract from population growth to prevent exploding growth (n = 0) and we assume $\lambda_R = \lambda_L = 1$ to simplify. The optimality conditions along the balanced growth path can then be written as:

$$\rho + \frac{1}{\sigma} g = g - \hat{R} = \theta_K \frac{Y}{K} = \xi_L L + g = \left(\frac{\theta_R}{\theta_L} \right) \xi_R L + g$$

where the last two equalities represent the investment rules for investment in A_L and A_R respectively. The solution is:

$$\begin{aligned}
&g = \sigma(\xi L^S - \rho) \\
&u = \sigma\rho + (1-\sigma)\xi L^S \\
&\xi \equiv \left(\frac{\xi_L \xi_R}{\xi_L + \xi_R} \right) \\
&\theta_R / \theta_L = \xi_L / \xi_R \\
&\theta_K = \beta^{\sigma_Y} (\xi L^S)^{1-\sigma_Y}
\end{aligned}$$

The endogenous growth solution with CES resembles most the endogenous growth solution with Cobb-Douglas. (Recall that we have assumed a CES production function with constant returns to scale; setting $\alpha + \beta + \nu = 1$ in the Cobb Douglas endogenous growth model, we find exactly the same solutions for g and u). Hence, poor substitution has no impact on the long-run optimal growth path.

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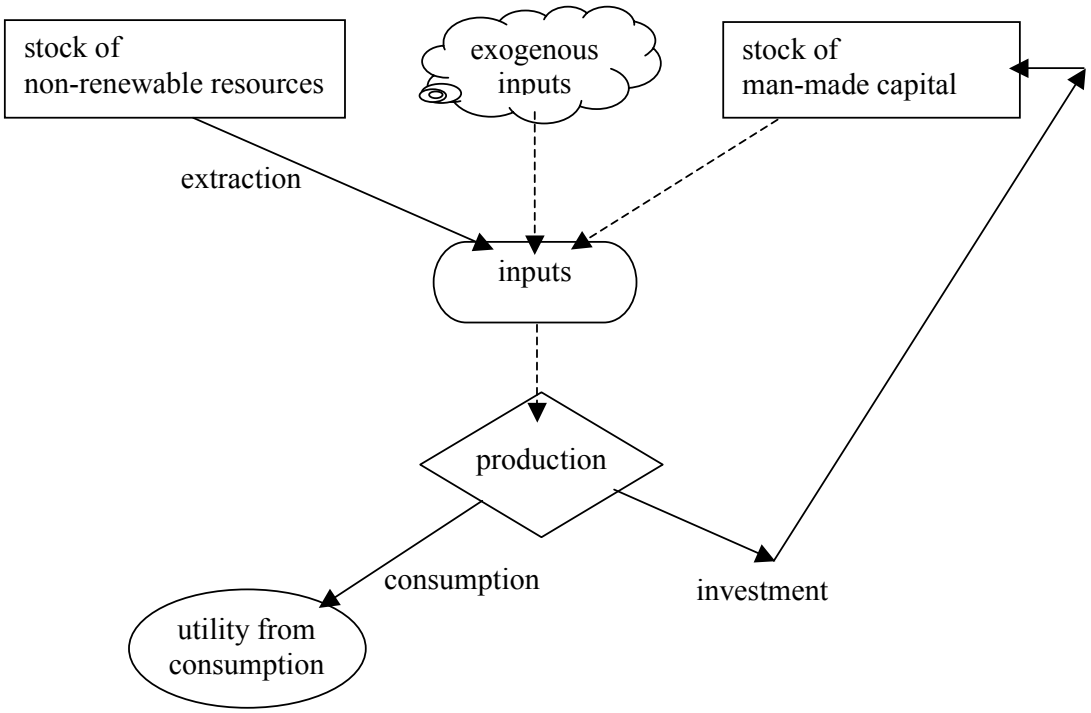
The general model