

Precautionary Decision Rules under Risk - Needed or Redundant? A general representation result.

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Abstract: I give a general class of preference representations conforming with the von Neumann-Morgenstern axioms, time consistency and additivity of welfare over time on certain outcomes (certainty additivity). I discuss whether precautionary decision rules are needed and in what sense. By explicitly taking care of the gauge freedom allowed for Bernoulli utility my representation theorem allows to relate different setups in the literature. This way a general relation between intertemporal substitutability, risk aversion and a preference for the timing of uncertainty resolution is derived. I point out that only the difference between intertemporal substitutability and risk aversion is gauge invariant. This invariant is shown to characterize precautionarity.

Keywords: uncertainty, welfare, precautionary principle, expected utility, recursive utility, risk aversion, intertemporal substitutability, certainty additivity, temporal lotteries, gauge-freedom

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1 Introduction

There is an ongoing debate between (and among) economists, environmental scientists and policy makers about the necessity and meaning of the precautionary principle. From a technical point of view the ambiguity representations by Gilboa & Schmeidler (1989) or Ghirardato, Maccheroni & Marinacci (2004) imply that in settings of Knightian uncertainty there seems to be some need for precautionary aspects in the evaluation rules to represent general preferences. But the question of whether precautionary decision rules are needed or obsolete when intersubjective (or scientific) probabilities prevail seems to be more disputable. This matter has recently been treated by Gollier & Treich (2003). While they focus on irreversibility, arriving information, market distortions and time inconsistency, I want to address the question from the mere preference representation perspective.

In this paper I give a representation for the most general time consistent model that falls back to expected utility in the atemporal (or one period) setting and to additivity over time when restricted to certain outcomes. Both of which are the predominantly used specifications for the respective framing scenarios. While this can be seen as an interest of its own the derived representation theorem is also new in the sense that it allows for different gauges. These relate different settings in the literature and shed new light on known results and prevailing questions. First, my paper allows to discuss soundly the relation between expected value maximization and precautionary decision rules in the presence of risk. Second, the question raised by Epstein & Zin (1989, 952 et seq.) on the interlacement of preference for the timing of risk resolution with risk aversion and intertemporal substitutability can be answered. Third, it sheds new light on the relation between risk aversion, intertemporal substitutability and precautionarity.

My paper relates to the seminal work of Kreps & Porteus (1978) who extend the atemporal von Neumann-Morgenstern setting for choice under uncertainty to a temporal structure. Under the assumption of intertemporal consistency they obtain a recursive representation that uses expected utility evaluation within each period and a generally nonlinear time aggregation from one period to the next. Kreps and Porteus show that an agent behaving in accordance with their axioms exhibits a preference for the timing of uncertainty resolution. Rendering a related model this paper derives a general representation for preferences complying with the von Neumann-Morgenstern axioms, time consistency and certainty additivity. *Certainty additivity* assumes that welfare evaluation over certain consumption paths is additively separable over time. However in difference to their approach

my model explicitly considers the different gauges that can be applied to Bernoulli utility¹ focusing on the interpretability of the value function as welfare.

A similar axiomatic setup is followed by Chew & Epstein (1990). Assuming certainty additivity, ordinal dominance, homotheticity, time consistency² and some technicalities they show that preferences can be represented by the standard additively time-separable expected utility function. With the goal of separating intertemporal substitutability from risk aversion, which is not possible within the additively time-separable expected utility representation, they feel pressed to weaken one of their axioms. Chew & Epstein (1990, 65) identify ordinal dominance and time consistency "as the two 'culprit' axioms" for the restrictedness of their result. They decide to weaken time consistency and allow for time inconsistent planning. However, the restrictedness of Chew & Epstein's (1990) representation is due not only to the explicit axioms they state, but also hinges on an implicit assumption in their setup. Remaining in a time consistent framework I develop a more general representation for certainty additive preferences that allows to distinguish between intertemporal substitution and risk aversion. In this sense my representation is similar to that of Epstein & Zin (1989). However they just assume a recursive utility representation with specified aggregator functions, for which they prove the existence of a solution.³ Moreover their work relies crucially on the assumption of a single consumption commodity. Compared with my representation, Epstein & Zin (1989) assume a less general (constant elasticity of substitution) intertemporal aggregation and a more general uncertainty evaluation rule than supported by my axioms.

My model is also intended as a normative decision support model. This requirement is a cornerstone for my axioms. First, planning should not be time inconsistent. Second, I design my model to apply in a case where intersubjective probabilities exist.⁴ Therefore I apply the von Neumann-Morgenstern axioms. And third, certainty additivity will allow for a gauge that renders the represen-

¹A value function on outcomes is called a Bernoulli utility function if it represents choice over certain first period outcomes. For a precise definition see pages 2.2 and 2.3.

²Chew & Epstein (1990) refer in their axiomatization to the usual time consistency axiom as recursivity axiom.

³Epstein & Zin (1989, 946) explicitly encourage an axiomatization specifying the forms of the aggregator functions in their model.

⁴I acknowledge the fact that purely objective probabilities are rarely met if they exist at all. But let me point out that purely subjective probabilities might exist but are useless for many applications. Gilboa, Postlewaite & Schmeidler (2004) give a neat discussion of the problems with using subjective priors.

tation much more intuitive in terms of welfare evaluation than the one derived by Kreps & Porteus (1978).

The paper is structured as follows. Section 2 gives the preliminaries. First different uncertainty aggregation rules are defined in correspondence to different kinds of weighted means in section 2.1. Then section 2.2 revisits the atemporal von Neumann-Morgenstern setup paying special attention to the possibility of fixing (gauging) the originally ordinal value function over the certain outcomes in different ways. Finally section 2.3 briefly looks at the other framing scenario of additively separable preferences over certain outcomes.

Section 3 derives the general representation theorems for the intertemporal uncertain setting. First this happens in section 3.1 for the simple “certain \times uncertain” case. Then section 3.2 works out the effect of different gauges for Bernoulli utility on the representational form. In section 3.3 I give the extension to the general multiperiod framework.

Section 4 discusses the economic content of the representations. In section 4.1 I extend Kreps & Porteus’s (1978) criterion for the preference for the timing of uncertainty resolution to the general gaugeable setting. This allows to relate it to the notions of risk aversion and intertemporal substitutability which are further discussed in section 4.2. Finally in section 4.3 a notion of precautionarity is established on the sound basis developed in section 3. Its intuition and its relation to the notion of welfare are discussed. Section 5 concludes. The appendix gives sketches of the proofs.

2 Preliminaries

2.1 Uncertainty Aggregation Rules and Means

This section defines uncertainty aggregation rules corresponding in a way made precise below to weighted means. Let X be the set of outcomes x . It is assumed to be a connected compact metric space. The space of all continuous functions from outcomes into the reals is denoted $\mathcal{C}^0(X)$. An element $u \in \mathcal{C}^0, u : X \rightarrow \mathbb{R}$ is called a value function. Let P^s be the set of all simple probability measures on X , i.e. those probability measures having finite support. I refer to the elements p of P^s as lotteries. Where no confusion arises the degenerate lottery δ_x giving weight 1 to outcome x is written in the usual way as $x \in P^s$. Moreover the lottery giving probability $p(x) = \lambda$ to outcome x and probability $p(x') = 1 - \lambda$ to outcome x'

is written as $\lambda x + (1 - \lambda)x' \in P^s$.⁵ I call a functional $\mathcal{M} : P^s \times \mathcal{C}^0 \rightarrow \mathbb{R}$ an *uncertainty aggregation rule*. An obvious example of an uncertainty aggregation rule is given by the expected value operator through:

$$E(p, u) \equiv E_p u = \sum p(x)u(x).$$

However, any weighted mean $\overline{\mathcal{M}} : P^s(U^x) \rightarrow \mathbb{R}$ with $U \subseteq \mathbb{R}$ induces an uncertainty aggregation rule. The one above corresponds to the arithmetic mean. To define this precisely let p^u denote the probability distribution induced by the value function $u \in \mathcal{C}^0$ with range U . Then an uncertainty aggregation rule \mathcal{M} is said to be induced by the mean $\overline{\mathcal{M}}$ whenever:

$$\mathcal{M}(p, u) = \overline{\mathcal{M}}(p^u) \quad \forall p \in P^s.$$

Other mean-induced uncertainty aggregation rules are

$$G(p, u) = \prod_x u(x)^{p(x)}$$

for $U \subseteq \mathbb{R}^+$ ⁶ corresponding to the geometric mean and

$$\min_x u(x)$$

induced by the (maxi-) min principle. All of the above uncertainty aggregation rules are contained for $U \subseteq \mathbb{R}^+$ in the one induced by the power mean

$$\mathcal{M}^\alpha(p, u) = \left[\sum_x p(x)u(x)^\alpha \right]^{\frac{1}{\alpha}}$$

defined for $\alpha \in \overline{\mathbb{R}}$.⁷ It is $\mathcal{M}^1(p, u) = E_p u$ and defined by limit $\mathcal{M}^0(p, u) \equiv \lim_{\alpha \rightarrow 0} \mathcal{M}^\alpha(p, u) = G(p, u)$, $\mathcal{M}^\infty(p, u) \equiv \lim_{\alpha \rightarrow \infty} \mathcal{M}^\alpha(p, u) = \max_x u(x)$ and $\mathcal{M}^{-\infty}(p, u) \equiv \lim_{\alpha \rightarrow -\infty} \mathcal{M}^\alpha(p, u) = \min_x u(x)$.

Let me take \mathcal{M}^α as an example to illustrate the intuition of uncertainty aggregation rules. Assume that an exogenously given u specifies some cardinally measurable value information for the outcomes $x \in X$. It will be a major task of the following sections to render a sound basis to this value function assumed here. Now consider a lottery yielding $u^1 = u(x_1) = 100$ with probability $p^1 = 0.9$ and $u^2 = u(x_2) = 10$ with probability $p^2 = 0.1$. Then an expected value maximizer will evaluate the lottery by the certainty equivalent $u_E^c = u_{\alpha=1}^c = 91$. On the contrary a person who is extremely precautious might value the lottery only as high as the worst of its outcomes, that is $u_{\min}^c = u_{\alpha=-\infty}^c = 10$. However this is an extreme assessment and generally one would rather expect a precautions person to give the lottery a value somewhere between 10 and 91. It can be shown that

⁵Note that X is just a compact metric space. In case it is additionally equipped with the structure of a vector space, the vector addition will not coincide with the “+” used here.

⁶ $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}^{++} = \{x \in \mathbb{R} : x > 0\}$.

⁷ $\overline{\mathbb{R}}$ denotes the extended real line. For $\alpha = 0, -\infty, +\infty$ define \mathcal{M}^α by the respective limits.

the smaller α , the lower is the certainty equivalent for the respective power mean (e.g. Hardy, Littlewood & Polya 1964). Hence, within this setup, a precautionary decision-maker would be expected to choose a parameter α between $-\infty$ and 1. Note that already $\alpha = -10$ yields an evaluation that most would consider very precautionous. For the lottery discussed above it brings about a certainty equivalent of $u_{\alpha=-10}^c = 12.6$. This intuition of precautionarity in combination with the power mean ($\alpha < 1$) is an anticipation of a general result arrived at in a comprehensive preference representation framework in section 4.3. The extreme case, where the individual only looks at the worst outcome to evaluate a lottery, will be excluded in what follows for technical reasons.⁸

Finally define a more general uncertainty aggregation rule for any strictly monotonic, continuous function $f : U \rightarrow \mathbb{R}$ as

$$\mathcal{M}^f(p, u) = f^{-1} \left[\sum_x p(x) f \circ u(x) \right],$$

where $f \circ u$ denotes the usual composition of two functions. The composition sign will often be omitted. This shall not create confusion as usual multiplication of two functions does not appear within this paper. The uncertainty aggregation rule \mathcal{M}^f contains \mathcal{M}^α for $f(z) = z^\alpha$ and will appear in my representation theorem. The corresponding mean $\overline{\mathcal{M}}^f$ is sometimes known as the generalized mean.

In what follows I will work with the space of Borel probability measures on X , denoted by $P = \Delta(X)$, rather than with its subset P^s . The more general elements $p \in P$ will also be called lotteries. For this purpose extend the above definitions by defining $\mathcal{M}^f : P \times \mathcal{C}^0 \rightarrow \mathbb{R}$ through⁹

$$\mathcal{M}^f(p, u) = f^{-1} \left[\int_X f \circ u \, dp \right]. \tag{1}$$

2.2 Atemporal Uncertainty

This section treats the atemporal or one-period setting of von Neumann & Morgenstern (1944). Their well known representation theorem for choice under uncertainty is slightly extended to allow for a broader class of representations. A useful perspective on the subject matter is that von Neumann and Morgenstern use the originally ordinal character of utility on certain outcomes to single out the

⁸However note that the (maxi-) min rule can be approximated arbitrarily close with $\alpha \in \mathbb{R}$.

⁹Note that by continuity of $f \circ u$ and compactness of X Lebesgue's dominated convergence theorem (e.g. Billingsley 1995, 209) ensures integrability.

additive representation by gauge¹⁰ fixing. If, as developed in later sections, a cardinal evaluation of certain outcomes is given and hence gauge freedom no longer prevails, additive representations no longer represent all decision rules conforming with the vNM-axioms. This is explained below.

Let preferences over lotteries be represented by the binary relation \succeq . An uncertainty aggregation rule is said to represent the preference relation \succeq over lotteries if

$$p \succ q \Leftrightarrow \mathcal{M}(p, u) > \mathcal{M}(q, u) \quad \text{for all } p, q \in P \quad (2)$$

and some $u \in \mathcal{C}^0$. It is said to represent \succeq for $\bar{u} \in \mathcal{C}^0$ iff equation 2 holds with $u = \bar{u}$. For the following equip the space P of Borel probability measures with the Prohorov metric (generating the topology of weak convergence). The theorem by von Neumann & Morgenstern (1944) in the version of Grandmont (1972, 49) states the following.

Theorem (von Neumann-Morgenstern):

The axioms

A1 (weak order) \succeq is transitive and complete, i.e.:

- transitive: $\forall p, q, r \in P : p \succeq q \text{ and } q \succeq r \Rightarrow p \succeq r$
- complete: $\forall p, q \in P : p \succeq q \text{ or } q \succeq p$

A2 (independence) $\forall p, q, r \in P :$

$$p \sim q \quad \Rightarrow \quad ap + (1 - a)r \sim aq + (1 - a)r \quad \forall a \in [0, 1]$$

A3 (continuity) $\forall p \in P : \{q \in P : q \succeq p\}$ and $\{q \in P : p \succeq q\}$ are closed in P hold if and only if there exists a continuous function $u: X \rightarrow \mathbb{R}$ such that

$$\forall p, q \in P : \quad p \succeq q \Leftrightarrow E_p u \geq E_q u. \quad (3)$$

Moreover if u represents \succeq in the sense of (3) then $u': X \rightarrow \mathbb{R}$ also represents \succeq in this sense if and only if there exist $a, b \in \mathbb{R}, a > 0$ such that $u' = au + b$.

The theorem states that accepting axioms A1-A3 there *exists* a value function u on the outcomes such that the uncertainty aggregation rule is of the expected utility form.

Now I will ask what happens in the situation when a decision maker *has* a given evaluation for the certain outcomes.¹¹ The minimal requirement for a value function to express evaluation of certain outcomes is that it coincides with the

¹⁰A gauge is some degree of freedom within a theory that has no observable effect.

¹¹Where such a cardinal evaluation can come from will be subject of the following sections.

ordinal requirement that a certain outcome x is preferred over a certain outcome y if and only if the value for x is higher than that for y . I want to call this subset of value functions representing preferred choice on the certain outcomes the set of Bernoulli utility functions $B_{\succeq} = \{u \in C^0(X) : \delta_x \succeq \delta_y \Leftrightarrow u(x) \geq u(y) \forall x, y \in X\}$. Obviously the value function in von Neumann-Morgenstern's theorem is a Bernoulli utility function, as well as any strictly increasing transformation of it. Now let me specify the uncertainty aggregation rules that represent the decision makers preference over lotteries in the sense of (2) with a **given** value function $u \in B_{\succeq}$ satisfying the von Neumann-Morgenstern setup.

Proposition 1: Let there be given a binary relation \succeq on P and a Bernoulli utility function $u \in B_{\succeq}$ with range U . The relation \succeq satisfies axioms A1-A3 if and only if there exists a strictly monotonic continuous function $f : U \rightarrow \mathbb{R}$ such that for all $p, q \in P$

$$p \succeq q \Leftrightarrow \mathcal{M}^f(p, u) \geq \mathcal{M}^f(q, u). \quad (4)$$

Moreover if f represents \succeq in the above sense then $f' : U \rightarrow \mathbb{R}$ represents \succeq in this sense if and only if there exist $a, b \in \mathbb{R}, a \neq 0$ such that $f' = a f + b$.

¹²

Note that the indeterminacy of f up to affine transformations does not translate into the unique functional \mathcal{M} . A function $f' = a f + b$ with $a, b \in \mathbb{R}, a \neq 0$ renders the same uncertainty aggregation rule as f , that is $\mathcal{M}^f(\cdot, \cdot) = \mathcal{M}^{f'}(\cdot, \cdot)$, because the inverse f'^{-1} cancels out the affine displacement of f' relative to f . In what follows the group of nondegenerate affine transformations will be denoted $\mathbf{A} = \{\mathbf{a} : \mathbb{R} \rightarrow \mathbb{R} : \mathbf{a}(z) = a z + b, a, b \in \mathbb{R}, a \neq 0\}$ with elements $\mathbf{a} \in \mathbf{A}$ and the group of positive affine transformations will be denoted $\mathbf{A}^+ = \{\mathbf{a}^+ : \mathbb{R} \rightarrow \mathbb{R} : \mathbf{a}^+(z) = a z + b, a, b \in \mathbb{R}, a > 0\}$.

Let me come back to the perspective given in the beginning of this section. Choice under certainty only renders ordinal information on the value function u which is represented by all members of B_{\succeq} . Proposition 1 states that this gauge freedom for Bernoulli utility u translates into the representing uncertainty aggregation rule \mathcal{M} through the form of the parametrizing function f . Taking this correspondence the other way round one gets

¹²The theorem can also be stated using only increasing versions of f . Then $\mathcal{M}^{\mathbf{a}}$ would be included in a less obvious way in \mathcal{M}^f than through $f(z) = z^{\mathbf{a}}$. Strictly decreasing functions are also allowed in the proposition due to the fact that the inverse in (1) cancels out any non-degenerate affine transformation.

Corollary 1: For any strictly monotonic, continuous function $f : U \rightarrow \mathbb{R}$ the following equivalence holds:

A binary relation \succeq on P satisfies axioms *A1-A3* if and only if \mathcal{M}^f represents \succeq . The latter is: there exists a continuous function $u : X \rightarrow U$ such that

$$\forall p, q \in P : \quad p \succeq q \Leftrightarrow \mathcal{M}^f(p, u) \geq \mathcal{M}^f(q, u). \quad (5)$$

Moreover if u represents \succeq in the sense of equation (5) then $u' : X \rightarrow \mathbb{R}$ represents \succeq in this sense if and only if there exist $\mathbf{a}^+ \in \mathbf{A}^+$ such that $u' = f^{-1}\mathbf{a}^+ f u$.

Note that obviously u will be a member of B_{\succeq} , as for any f it holds that $\mathcal{M}^f(\delta_x, u) = u(x)$. The uniqueness of the value function u is no longer up to affine transformations. Indeterminacy of the value function u corresponds to those transformations of u which result in affine transformations of f that leave the uncertainty aggregation rule unchanged. For example in the geometric mean representation the remaining gauge freedom for u after fixing f will be expressed by the group of transformations $u \rightarrow u' = c u^d$, $c \in \mathbb{R}^{++}$, $d \in \mathbb{R} \setminus \{0\}$.¹³

Corollary 1 points out how value functions and representations always come in pairs. For $f(z) = z^\alpha$, $\alpha < 1$ corollary 1 reproduces von Neumann-Morgenstern's theorem with expected value replaced by a precautionary decision rule (in the sense of section 2.1). In particular it states that **in the atemporal framework a dispute on whether to apply a precautionary decision rule or expected value cannot be distinguished from (or can be stated as) a disagreement on the valuation functions over the certain outcomes**. The meaning of choosing a aggregator-value pair will be further discussed in later sections. In this context, the time structure introduced in the next section will play an important role.

2.3 Certainty

This short section treats the other framing scenario of additively separable preference over certain outcome paths. Time is discrete with planning horizon $T \in \mathbb{N}$ such that $t \in \{1, \dots, T\}$. $x_t \in X$ specifies consumption in period t . A (planned) consumption path is denoted by $\mathbf{x} = (x_1, x_2, \dots, x_T)$. Let $x^0 \in X$ denote a benchmark consumption. It is arbitrarily fixed and serves to define the shorthand notation $[x] \equiv (x, x^0, \dots, x^0)$ for the consumption path that yields the specified

¹³The easiest way to see this is to note that $\lim_{\alpha \rightarrow 0} \frac{z^\alpha - 1}{\alpha} = \ln(z)$. For any $\alpha > 0$ the function $f_\alpha(z) = \frac{z^\alpha - 1}{\alpha}$ is an affine transformation of $f(z) = z^\alpha$ so that both parametrize the same uncertainty aggregation rule.

consumption x in the first period and the benchmark consumption ever after. In this section the binary relation \succeq depicts the preference relation over certain consumption paths. To focus the model I will not introduce time preference in the sense of a positive discount rate.¹⁴

As mentioned before the the model sought for shall be additive over time with respect to certain outcomes. Axiomatizations for this can be found for example in Koopmans (1960) and Radner (1982). I will take it as an assumption itself.

A4 (certainty additivity) There exists $u : X \rightarrow \mathbb{R}$ such that for all $x, x' \in X^T$

$$x \succeq x' \Leftrightarrow \sum_{i=1}^T u(x_i) \geq \sum_{i=1}^T u(x'_i). \quad (6)$$

Note that this axiom also includes the assumptions of stationarity¹⁵ and history independence¹⁶.

Again the different value functions representing preferred choice on certain one-period outcomes are called Bernoulli functions and the respective set of functions is defined by the straight-forward extension $B_{\succeq} = \{[x] \succeq [x'] \Leftrightarrow u(x) > u(x') \forall [x], [x'] \in X^T\}$, coinciding for $T = 1$ with the definition given previously in section 2.2. Note that axiom A4 ensures that the definition of B_{\succeq} does not depend on the choice of x^0 .¹⁷

In analogy to proposition 1 I seek for a representation of \succeq with a **given** Bernoulli utility function u .

¹⁴However footnotes 21 and 23 will indicate changes in the representation from introducing positive (exponential) discounting. However time preference concerning the timing of the the resolution of uncertainty will naturally arise within my setting.

¹⁵Stationarity implies that the mere passage of time does not have an effect on preferences. In a particular example it states that I will not (anticipate in my plans to) prefer Beck's beer over Budweiser in 2010 and Budweiser over Beck's beer in 2011. If u would be allowed to vary arbitrarily over time, it would be difficult to give it an interpretation of value which is sought for in this paper to render it helpful in a decision process. This is not the same as excluding a specified history dependence which might well change preferences over time in an anticipated way ('if I drink Beck's all through 2010 I might not like it anymore and prefer to drink Budweiser in 2011').

¹⁶While I consider it an interesting task to integrate rules for history dependence in an extended model, I do not consider it that helpful to allow for arbitrary dependence on history like in Kreps & Porteus (1978). This implies too little structure and allows in a finite horizon setting for the same preference changes as unstationarity, rendering it difficult to give a value interpretation to u .

¹⁷Nor does it depend on the fact that the defining paths $[\cdot]$ have constant future consumption streams.

Proposition 2: Given a Bernoulli utility function $u \in B_{\succeq}$ with range U , a binary relation \succeq on X^T satisfies axiom A4 if and only if there exists a strictly monotonic, continuous function $g : U \rightarrow \mathbb{R}$ such that for all $\mathbf{x}, \mathbf{x}' \in X^T$

$$\mathbf{x} \succeq \mathbf{x}' \Leftrightarrow g^{-1} \left[\frac{1}{T} \sum_{t=1}^T g \circ u(x_t) \right] \geq g^{-1} \left[\frac{1}{T} \sum_{t=1}^T g \circ u(x'_t) \right] \quad (7)$$

Moreover if $T \geq 2$ and g represents \succeq in the sense of equation (7) then $g' : U \rightarrow \mathbb{R}$ represents \succeq in this sense if and only if there exist $\mathbf{a} \in \mathbf{A}$ such that $g' = \mathbf{a}g$.

3 The Representation Theorems

3.1 Certain \times Uncertain

Combining the thoughts of sections 2.2 and 2.3 I now combine time structure and uncertainty. Section 3.1 tackles the simplest such case, a two period setting where consumption in the first period is certain and consumption in the second period is uncertain. Section 3.2 will look at the effect of choices of Bernoulli utility on the representation and section 3.3 extends the model to multiple periods.

In the terminology applied by Kreps & Porteus (1978) which I will adapt in my multiperiod setting in 3.3 the model treated right now is only a one-and-a-half periods. However I consider it useful to familiarize with the structure of the representation and the idea of gauging in this simplified framework first. It extends in a straight forward way to any finite time horizon. Anticipating later representations and avoiding notational confusion I will denote the first period in this section by $t = F$ and the second and last period by $t = T$.¹⁸

Elements $x_F \in X$ denote certain consumption in the first period. Degenerate lotteries δ_{x_T} yielding certain consumption x_T in the second period are also denoted by $x_T \in X$. General objects of choice in the second period are the lotteries $p \in P$ just as in section 2.2. The preference relation over these objects is denoted by \succeq_T . Objects of choice in the first period are combinations of certain consumption in the first period and lotteries faced in the next: $(x, p) \in X \times P$. Preferences over these objects are given by the relation \succeq_F . The set of preferences in both periods will be denoted by $\succeq = (\succeq_F, \succeq_T)$.

¹⁸Due to backward recursion in the derivation of the general representation what is now the structure of the second period representation will later in the multiperiod setting coincide with that of the last period representation. A full time-step back from T will be when uncertainty is introduced also for the preceding period.

I demand that preferences restricted to certain consumption paths satisfy certainty additivity (A4) and that lotteries are evaluated on basis of the von Neumann-Morgenstern axioms (A1-A3). In addition the preferences in period one and two should be connected by the consistency axiom

$$\mathbf{A5} \text{ (time consistency)} \quad \forall x \in X, p, p' \in P : \quad (x, p) \succ_F (x, p') \Leftrightarrow p \succ_T p'$$

This is *time consistency* in the sense of Kreps & Porteus (1978).¹⁹ It simply states that the decision maker in period one should prefer a consumption plan over another that does not differ for the first period if and only if he will prefer its uncertain entry as a lottery in the second period.

Again I am interested in finding a representation for \succeq for a given valuation on the certain one-period outcomes $u \in B_{\succeq} \equiv B_{\succeq_F}$. The latter definition is justified by the fact that certainty additivity A4 and time consistency A5 imply that $B_{\succeq_F} = B_{\succeq_T}$.²⁰ Denote with $\succ_F|_{X \times X}$ the restriction of \succ_F to the set of certain consumption paths. The following representation theorem holds.

Theorem 1: Let there be given a set of binary relations $\succeq = (\succeq_F, \succeq_T)$ on $(X \times P, P)$ and a Bernoulli utility function $u \in B_{\succeq}$ with range U . The set of relations \succeq satisfies

- i) A1-A3 for \succeq_T (vNM setting)
- ii) A4 for $\succeq_F|_{X \times X}$ (certainty additivity)

iii) A5 (time consistency)

if and only if there exist strictly monotonic continuous functions $f : U \rightarrow \mathbb{R}$ and $g : U \rightarrow \mathbb{R}$ such that

$$v) \quad (x, p) \succeq_F (x', p') \Leftrightarrow g^{-1} \left[\frac{1}{2} g \circ u(x) + \frac{1}{2} g \circ \mathcal{M}^f(p, w) \right] \geq g^{-1} \left[\frac{1}{2} g \circ u(x') + \frac{1}{2} g \circ \mathcal{M}^f(p', w) \right],$$

$$vi) \quad p \succeq_T p' \Leftrightarrow \mathcal{M}^f(p, u) \geq \mathcal{M}^f(p', u).$$

Moreover g and f are unique up to nondegenerate positive affine transformations.

In period T lotteries are evaluated just the same way as in proposition 1. In the first period these second period lottery-values are aggregated by means of the *time aggregator* $g^{-1} \left[\frac{1}{2} g(\cdot) + \frac{1}{2} g(\cdot) \right]$ with the values of the certain outcomes x_F the same way as in proposition 2. Now the interesting part will be to look again at the gauge-freedom of the representation.

¹⁹Adapted to the one-and-a-half period setting of this section.

²⁰This is shown in the proof of theorem 1. For the first period it is $[x] = (x, x^0) = (x, \delta_{x^0})$.

3.2 Gauging

Like in section 2.2 there is some gauge freedom rendered to the model by the freedom to choose the Bernoulli utility function in theorem 1. Given some $u \in B_{\succeq}$ any other Bernoulli utility function is some strictly increasing continuous transformation of u and any strictly increasing continuous transformation of u yields an element of B_{\succeq} . Moreover the following lemma holds.

Lemma 1: If the triple (u, f, g) represents the set of preferences \succeq in the sense of theorem 1, then so does the triple $(s \circ u, f \circ s^{-1}, g \circ s^{-1})$ for any $s : \mathbb{R} \rightarrow \mathbb{R}$ strictly increasing and continuous.

Now like in section 2.2 I can gauge the uncertainty aggregation rule in the representation of theorem 1 to any desired form parametrized by a strictly monotonic continuous f^* by choosing $s = f^{*-1} \circ f$ in lemma 1. This yields the following corollary.

Corollary 2 (f -gauge) :

For any strictly monotonic, continuous function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$ the following equivalence holds:

A set of binary relations \succeq satisfies

$i - iii)$ of theorem 1

if and only if there exists a continuous function $u : X \rightarrow \mathbb{R}$ with range U and a strictly monotonic, continuous function $\mathbf{g} : U \rightarrow \mathbb{R}$ such that

$v - vi)$ of theorem 1 hold

Moreover the pair (u, g) is unique up to simultaneous transformations of the form $(u', g') = (\mathbf{a} g f^{-1} \mathbf{a}^+ f, f^{-1} \mathbf{a}^{+-1} f u)$ with $\mathbf{a} \in \mathbf{A}, \mathbf{a}^+ \in \mathbf{A}^+$.

The gauge used implicitly by Kreps & Porteus (1978) is obtained for $f = \text{id}$ where $v)$ and $vi)$ become:

Kreps-Porteus-gauge ($f = \text{id}$ -gauge) :

$$v) \quad (x, p) \succeq_F (x', p') \Leftrightarrow g^{-1} \left[\frac{1}{2} g \circ u(x) + \frac{1}{2} g \circ E_p u \right] \geq g^{-1} \left[\frac{1}{2} g \circ u(x') + \frac{1}{2} g \circ E_{p'} u \right]$$

$$vi) \quad p \succeq_T p' \Leftrightarrow E_p u \geq E_{p'} u .$$

Note that Kreps & Porteus (1978) get a slightly more general intertemporal aggregation rule for they do not demand certainty additivity in the sense of axiom A4. In the notion of Johnsen & Donaldson (1985) my axiom corresponds to unconditional strong independence over time for certain outcomes while the analog in their setting would be conditional strong independence, which is slightly weaker.

However axiom A4 allows for a special gauge that will prove most helpful for discussing the meaning of welfare and precautionarity in section 4.3. This gauge is a special case of the following

Corollary 3 (*g-gauge*) :

For any strictly monotonic, continuous function $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}$ the following equivalence holds:

A set of binary relations \succeq satisfies

i – iii) of theorem 1

if and only if there exists a continuous function $u : X \rightarrow \mathbb{R}$ with range U and a strictly monotonic, continuous function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$ such that

v – vi) of theorem 1 hold

Moreover the pair (u, \mathbf{f}) is unique up to simultaneous transformations of the form $(u', \mathbf{f}') = (\mathbf{a} \mathbf{f} g^{-1} \mathbf{a}^+ g, g^{-1} \mathbf{a}^{+^{-1}} g u)$ with $\mathbf{a} \in \mathbf{A}, \mathbf{a}^+ \in \mathbf{A}^+$.

It renders the above mentioned *certainty additive gauge* for $g = \text{id}$.

Certainty-additive-gauge (*g = id-gauge*) :

$$\begin{aligned} v) \quad (x, p) \succeq_F (x', p') &\Leftrightarrow u(x) + \mathcal{M}^f(p, u) \geq u(x') + \mathcal{M}^f(p', u), \\ vi) \quad p \succeq_T p' &\Leftrightarrow \mathcal{M}^f(p, u) \geq \mathcal{M}^f(p', u). \end{aligned}$$

Another special gauge is possible if the outcome space is one-dimensional, $X \subset \mathbb{R}$, and nonsatiation in x is assumed. Then the value function of theorem 1 can be chosen as the identity, rendering immediately the

Epstein-Zin-gauge (*u = id-gauge, one commodity only*) :

$$\begin{aligned} v) \quad (x, p) \succeq_F (x', p') &\Leftrightarrow g^{-1} \left[\frac{1}{2}g(x) + \frac{1}{2}g \circ \mathcal{M}^f p \right] \geq g^{-1} \left[\frac{1}{2}g(x') + \frac{1}{2}g \circ \mathcal{M}^f p' \right] \\ vi) \quad p \succeq_T p' &\Leftrightarrow \mathcal{M}^f p \geq \mathcal{M}^f p' \end{aligned}$$

with $\mathcal{M}^f p \equiv \mathcal{M}^f(p, \text{id}) = f^{-1} [f_X f p]$.

This gauge is used by Epstein & Zin (1989). The representation they assume to hold slightly differs from the one supported by my axiomatization. With respect to the intertemporal aggregation rule Epstein & Zin (1989) assume the special case where $g(z) = z^\rho$, what renders a CES function. On the other hand they assume a more general uncertainty aggregation rule which does not comply with von Neumann & Morgenstern's (1944) independence axiom.

Before I will discuss the economic insights that can be gained from the derived results in section 4 I want to extend the representations to a multiperiod setting.

3.3 Multiperiod Extension

The objects of choice in period F of the last section were elements of $\tilde{X}_{T-1} \equiv X \times \Delta(X)$. Now the second-last period $t = T - 1$ will start before period F in the sense that uncertainty over the choice objects of the respective period will not have resolved at $t = T - 1$. Hence preferences in period $T - 1$ are expressed by a relation \succeq_{T-1} on the space of lotteries over \tilde{X} denoted $P_{T-1} \equiv \Delta(\tilde{X}_{T-1})$.

In general define $\tilde{X}_T = X$ and recursively $\tilde{X}_{t-1} = X \times \Delta(\tilde{X}_t)$ for $t \in \{1, \dots, T\}$. Equip the set of Borel probability measures on \tilde{X}_t , $P_t \equiv \Delta(\tilde{X}_t)$, with the Prohorov metric as to render \tilde{X}_{t-1} compact in the product topology. Its elements are called (period t -) lotteries. The set of degenerate lotteries in P_t is identified with the set \tilde{X}_t in the usual way. An uncertainty aggregation rule in period t is a functional $\mathcal{M} : P_t \times \mathcal{C}^0(\tilde{X}_t) \rightarrow \mathbb{R}$. As the arguments clearly indicate the time period I will not explicitly attach a time indice to \mathcal{M} . The time consistency requirement for the set of preferences $\succeq = (\succeq_1, \dots, \succeq_T)$ now writes as

A5' (time consistency) For all $t \in \{1, \dots, T\}$:

$$(x_t, p_{t+1}) \succeq_t (x_t, p'_{t+1}) \Leftrightarrow p_{t+1} \succeq_{t+1} p'_{t+1} \quad \forall x_t \in X, p_{t+1}, p'_{t+1} \in P_{t+1} .$$

Note that (x_t, p_{t+1}) stands for the degenerate lottery $\delta_{(x_t, p_{t+1})} \in P_t$. The interpretation of the axiom is the same as for axiom A5 on page 11. Again time consistency together with certainty additivity will make the set of Bernoulli functions coincide for all \succeq_t , so that it makes sense to define $B_{\succeq} \equiv B_{\succeq_1}$. The following representation holds.

Theorem 2: Let there be given a set of binary relations $\succeq = (\succeq_t)_{t \in \{1, \dots, T\}}$ on $(P_t)_{t \in \{1, \dots, T\}}$ and a Bernoulli utility function $u \in B_{\succeq}$ with range U . The set of relations \succeq satisfies

- i) A1-A3 for all $\succeq_t, t \in \{1, \dots, T\}$ (vNM setting)
- ii) A4 for $\succeq_1|_{X^T}$ (certainty additivity)
- iii) A5' (time consistency)

if and only if there exist strictly monotonic, continuous functions $f_t : U \rightarrow \mathbb{R}$ for $t \in \{1, \dots, T\}$ and $g : U \rightarrow \mathbb{R}$ such that with defining the functions $\tilde{u}_t : \tilde{X}_t \rightarrow \mathbb{R}$ for $t \in \{1, \dots, T\}$ by $\tilde{u}_T(x_T) = u(x_T)$ and recursively with $\alpha_t = \frac{1}{T+1-t}$ ²¹

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = g^{-1} \left[\alpha_t g \circ u(x_{t-1}) + (1 - \alpha_t) g \circ M^{f_t}(p_t, \tilde{u}_t) \right]$$

²¹In the case of a positive discount rate δ it is $\alpha_t = (1 + \delta + \delta^2 + \dots + \delta^{T-t})^{-1} = \frac{1-\delta}{1-\delta^{T+1-t}}$.

it holds that for all $t \in \{1, \dots, T\}$

$$p_t \succeq_t p'_t \Leftrightarrow \mathcal{M}^{f_t}(p_t, \tilde{u}_t) \geq \mathcal{M}^{f_t}(p'_t, \tilde{u}_t) \quad \forall p_t, p'_t \in P_t.$$

This representation allows the uncertainty aggregation rules to depend on absolute time.²² This is an unattractive feature of recursive models approaching a finite planning horizon like the one of Kreps & Porteus (1978). In models with infinite planning horizon this is usually avoided by the axiom of stationarity. To capture the point that also in a finite time horizon risk attitude should not depend on absolute time let me introduce the following notation. Let $\bar{x}^t = (\bar{x}, \bar{x}, \dots, \bar{x})$ denote the certain constant consumption path that gives consumption \bar{x} from t until T . Then $\frac{1}{2}\bar{x}^t + \frac{1}{2}\bar{x}''^t \in P_t$ is the lottery in period t that randomizes with $p = \frac{1}{2}$ between the constant consumption stream \bar{x} and \bar{x}' . The following axiom demands that these randomized consumption streams relate to certain consumption streams the same way in different periods.

A6 (quasi-stationarity) For all $t \in \{1, \dots, T\}$:

$$\frac{1}{2}\bar{x}^t + \frac{1}{2}\bar{x}'^t \succeq_t \bar{x}''^t \Leftrightarrow \frac{1}{2}\bar{x}^{t+1} + \frac{1}{2}\bar{x}'^{t+1} \succeq_{t+1} \bar{x}''^{t+1} \quad \forall x, x', x'' \in X.$$

Adding A6' to the assumptions of theorem 2 yields

Theorem 3: Let there be given a set of binary relations $\succeq = (\succeq_t)_{t \in \{1, \dots, T\}}$ on $(P_t)_{t \in \{1, \dots, T\}}$ and a Bernoulli utility function $u \in B_{\succeq}$ with range U . The set of relations \succeq satisfies

- i) A1-A3 for all $\succeq_t, t \in \{1, \dots, T\}$ (vNM setting)
- ii) A4 for $\succeq_1|_{X^T}$ (certainty additivity)
- iii) A5' (time consistency)
- iv) A6' (quasi-stationarity)

if and only if there exist strictly monotonic, continuous functions $f : U \rightarrow \mathbb{R}$ and $g : U \rightarrow \mathbb{R}$ such that with defining the functions $\tilde{u}_t : \tilde{X}_t \rightarrow \mathbb{R}$ for $t \in \{1, \dots, T\}$ by $\tilde{u}_T(x_T) = u(x_T)$ and recursively with $\alpha_t = \frac{1}{T+1-t}$ ²³

$$\tilde{u}_{t-1}(x_{t-1}, p_t) = g^{-1} \left[\alpha_t g \circ u(x_{t-1}) + (1 - \alpha_t) g \circ M^f(p_t, \tilde{u}_t) \right]$$

²²That is, my uncertainty evaluation might depend on the date when the uncertainty occurs, stays fixed while I approach this date, but might completely differ from the way I evaluate uncertainty for another date (compare also footnote 15).

²³In the case of a positive discount rate δ it is $\alpha_t = (1 + \delta + \delta^2 + \dots + \delta^{T-t})^{-1} = \frac{1-\delta}{1-\delta^{T+1-t}}$.

it holds that for all $t \in \{1, \dots, T\}$

$$p_t \succeq_t p'_t \Leftrightarrow \mathcal{M}^f(p_t, \tilde{u}_t) \geq \mathcal{M}^f(p'_t, \tilde{u}_t) \quad \forall p_t, p'_t \in P_t.$$

Moreover g and f are unique up to nondegenerate positive affine transformations.

As an example for the representation let me write out the first period evaluation functional for a full two period setting explicitly

\succ_1 representation for $T = 2$:

$$\mathcal{M}^f \left(p_1, g^{-1} \left[\frac{1}{2} g \circ u(x_1) + \frac{1}{2} g \circ \mathcal{M}^f(p_2, u) \right] \right). \quad (8)$$

Furthermore gauging works out the same way as in section 3.1.

Lemma 2: If quasi-stationarity A6 is assumed, then lemma 1, corollary 2 and corollary 3 also hold in the multiperiod setup. That is the respective statements hold true with theorem 1 replaced by theorem 3 and i-iii replaced by i-iv.

4 Discussion

4.1 On Temporal Lotteries

In the gauge of Kreps and Porteus the preferences can be represented using only the expected value operator corresponding to the arithmetic mean for uncertainty aggregation as seen on page 13. Moreover applying an increasing²⁴ transformation to v) in the Epstein-Zin-gauge on page 13 simplifies the representation

$$\text{from } g^{-1} \left[\frac{1}{2} g \circ u(x) + \frac{1}{2} g \circ E_p u \right] \quad \text{to} \quad g \circ u(x) + g \circ E_p u$$

Such a transformation additively separates the certain first period evaluation from the uncertain second period. Together with the applied $f = \text{id}$ -gauge this implies that such a one-and-a-half period model can always be gauged to yield an additively time separable expected utility representation. But as can be inferred from equation 8 this is not true anymore from a full two period time horizon on.

Chew & Epstein (1990) proof that any preference relation conforming with certainty additivity, ordinal dominance, homotheticity, time consistency²⁵ and some

²⁴In the case g is strictly decreasing replace it in the representation with $-g$.

²⁵Chew & Epstein (1990) refer in their axiomatization to the usual time consistency axiom as recursivity axiom.

further technicalities, all assumptions not far from my own setting, can be represented as an additively time separable expected utility representation. However the reason for this surprising result lies in an additional assumption implicit in their setup. In the definition of their choice space Chew & Epstein (1990) insert the assumption that first period consumption is certain and that there exists a single probability measure evaluating the whole future. Hence despite their use of an infinite planning horizon Chew & Epstein's (1990) model structure stays by assumption in a setting comparable to my one-and-a-half period model where the strictly monotone transformation of the representation is enough to additively separate the first term and hence yield an allover additive expected utility representation.

The best one can hope for in terms of additivizing Bernoulli utility over time is done in the following gauge:

$$\succsim_1 \text{ representation for } T = 2 : \quad (g = \text{id-gauge} / \text{certainty additive gauge})$$

$$\mathcal{M}^f \left(p_1, \frac{1}{2} u(x_1) + \frac{1}{2} \mathcal{M}^f(p_2, u) \right) \quad (9)$$

which renders $u(x_1) + u(x_2)$ when there is no uncertainty over the outcomes. However there is nothing in the axioms imposed on the decision makers behavior that assures additive time separability in general.

This more comprehensive structure allows for a preference for the timing of the resolution of uncertainty as discovered by Kreps & Porteus (1978). The following theorem will extend their result to general representations in the sense of theorem 3 and its gauges.

Theorem 4: Let the triple (u, f, g) represent the set of preferences \succeq in the sense of theorem 3. Then the decision maker prefers early [late] resolution of uncertainty in period t for x_t fixed in the sense of Kreps & Porteus (1978), that is for all $p_{t+1}, p'_{t+1} \in P_{t+1}, \lambda \in [0, 1]$

$$\lambda(x_t, p_{t+1}) + (1 - \lambda)(x_t, p'_{t+1}) \quad \succeq_t \quad [\preceq_t] \quad (x_t, \lambda p_{t+1} + (1 - \lambda)p'_{t+1}) \quad (10)$$

if and only if

$$fg^{-1} [\alpha_t g \circ u(x_t) + (1 - \alpha_t) gf^{-1}(\gamma)] \quad (11)$$

is convex [concave] in $\gamma \in f(U)$ for an increasing version of f .²⁶ Furthermore the decision maker is indifferent to the time of uncertainty resolution if $f = g$, which is the case if additively (time-)separable expected utility is

²⁶Which is equivalent to concavity [convexity] for a decreasing choice of f .

maximized.

As it should, this result corresponds to theorem 3 of Kreps & Porteus (1978, 197) for the $f = \text{id}$ -gauge.²⁷ Equation (10) states that the randomization between two degenerate choice objects of period t with probability λ is preferred to take place - and thus be resolved - in period t rather than in period $t + 1$ [respectively the other way round]. In particular the theorem answers a question posed by Epstein & Zin (1989, 952 et seq.) on the interlacement of the preference for early resolution of uncertainty with risk aversion and intertemporal substitutability in their ($u = \text{id}$)-gauge. This can be inferred from the next section where f and g are identified as characterizing risk aversion and intertemporal substitutability in the sense (and gauge) of Epstein & Zin (1989).²⁸

4.2 On Risk Aversion and Intertemporal Substitutability

Let me start out with the gauge used by Epstein & Zin (1989) to differentiate between risk aversion and intertemporal substitutability (compare section 3.2),

\succ_1 representation for $T = 2$: ($u = \text{id}$ -gauge / Epstein-Zin-gauge)

$$\mathcal{M}^f \left(p_1, g^{-1} \left[\frac{1}{2} g(x_1) + \frac{1}{2} g \circ \mathcal{M}^f p_2 \right] \right)$$

with $\mathcal{M}^f p_2 = \mathcal{M}^f(p_2, \text{id}) = f^{-1} [\int_X f dp_2]$.

With respect to the intertemporal aggregation rule Epstein & Zin (1989) assume the special case where $g(z) = z^\rho$, what renders a CES function for the time aggregator. On the other hand they assume a more general uncertainty aggregation rule which does not comply with von Neumann & Morgenstern's (1944) independence axiom. Using this gauge the authors obtain a distinction between ρ which they interpret as characterizing the elasticity of intertemporal substitution $\sigma = \frac{1}{1-\rho}$ and a notion of risk aversion associated with the form of their uncertainty aggregation rule. To parametrize risk aversion they go back to a simpler class of uncertainty aggregation rules which contains the well known case of constant relative risk aversion with $f(z) = z^\alpha$. This special case is frequently used for empirical estimations of risk aversion and intertemporal substitution elasticity,

²⁷Obviously I have more structure in the time aggregator than they do for I assumed certainty additivity. Note that when f is gauged to identity it is obviously dealt with the case where f increases. Moreover in my representation the expression (11) is always increasing in γ .

²⁸Epstein & Zin (1989, 952) note that "For more general recursive utility functions [than in the case corresponding to $f(z) = z^\alpha, g(z) = z^\rho$], we have not found a characterization in terms of [time and uncertainty aggregators] of the condition under which early or late resolution is preferred." This is achieved above.

also by Epstein & Zin (1991). For an overview of some of the results compare for example Giuliano & Turnovsky (2003). For an application of this functional form for evaluative purposes see for example in the context of climate change Ha-Duong & Treich (2004).

It should be clear from the preceding corollaries that Epstein and Zin use a very particular gauge. In their one-commodity world this gauge seems to be the natural environment for the question they try to answer. But some remarks resulting from the analysis above seem to be clarifying. First, applying their model to another consumption quantity will generally change as well the risk aversion parameter, as also the intertemporal elasticity of substitution of the same individual. In such a setting it might be more precise to talk about the risk aversion parameter for good i or the intertemporal elasticity of substitution for good j . But this would suggest that risk aversion parameters could arbitrarily vary across different goods which is obviously not what one would like to have and which is not what the representation structure (u, f, g) in theorem 3 proposes. Let me point out that a similar problem already prevails in a single commodity setting. While in the particular specification seen above, where uncertainty and time aggregation are linear homogeneous, the scaling²⁹ of the units measuring the consumption quantity will not affect the parameters of risk aversion and intertemporal substitutability, in a more general setting they will.³⁰

At least my intuition asks for a quantity that should not depend on the arbitrariness pointed out in the preceding paragraph.³¹ A quantity like this does exist. Denote for $f \in \mathcal{C}^0(\mathbb{R})$ with $\hat{f} = \{\mathbf{a}f : \mathbf{a} \in \mathbf{A}\}$ the class of all members of $\mathcal{C}^0(\mathbb{R})$ that coincide with f up to nondegenerate affine transformations. Inverting each member of \hat{f} yields the set $\hat{f}^{-1} = \{f^{-1}\mathbf{a} : \mathbf{a} \in \mathbf{A}\}$. Note that in fact $\hat{f}\hat{f}^{-1} = \text{id}$.

Lemma 3: In the representation of theorem 3 the 'quantity' $\hat{f}\hat{g}^{-1}$ is gauge

²⁹That is application of a multiplicative constant changing for example from grams to kilograms. Note that for non-materialistic goods a natural cardinal scale is generally not given, implying the problem pointed out even for homogeneous aggregators.

³⁰For $f(z) = z^\alpha, g(z) = z^\rho$ and $\mathbf{a}(z) = az$ find that $f^{-1}\mathbf{a}^+f(z)$ is proportional to z . Hence rescaling of u (and thus of x in the Epstein-Zin-gauge) to $u \rightarrow u' = \zeta u$ is allowed without changing f . Moreover the implied simultaneous change of g to $g \rightarrow g'$ only results in a multiplicative constant not changing the parameter ρ (compare section 3.2).

³¹If the reader might suggest that money as *the* reference good would resolve the stated problem, he should be aware that money is not a consumption good per se and trying to attach the quantities under discussion to money (without a well defined theory behind it) abandons the separation between preferences and market supply.

invariant, i.e. it uniquely depends on \succeq .

This permits at least a gauge-invariant characterization of the "difference" between risk aversion and intertemporal substitutability. In the case where $f(z) = z^\alpha$ and $g(z) = z^\rho$ this "difference" is uniquely characterized by the quotient $\frac{\alpha}{\rho}$. In a world where decision makers face different consumption goods I think that only this "difference" should be given economic interpretation (what will happen in the next section). However if one uses Epstein & Zin's (1989) separation it should be stressed, that it is the good under observation and its measurement that determine which part of the invariant quantity described above goes into risk aversion and which into intertemporal substitutability.

Note that if the wording above for f and g is applied to the general setting where u is not in the Epstein-Zin-gauge, moving into the certainty additive gauge results in setting the elasticity of intertemporal substitution to infinity ($g = \text{id}, \rho = 1, \sigma = \infty$) by definition. At the same time the uncertainty aggregation rule is applied to an endogenously specified cardinal Bernoulli utility function in the way anticipated in section 2.1. This will be worked out in the next section.

4.3 On Welfare and Precautionary Decision Rules

This section introduces a notion of precautionary decision rules that will make precise the intuitive approach appealed to in section 2.1. The main question set aside in that section was the origin and thus the meaning of the cardinal utility function entering the uncertainty aggregation rule. Meanwhile it was found that from a merely representational point of view any function $u \in B_{\succeq}$ can appear in the argument of the uncertainty aggregation rule.³² Now I want to discuss whether there might be some reason to single out a particular gauge. From a mathematical point of view obviously the certainty-additive gauge (where $g = \text{id}$) and the Kreps-Porteus-gauge (where $f = \text{id}$) have a special appeal as either makes one of the two aggregators additive.³³ To get to the bottom of why one would like to render either temporal or uncertainty aggregation additive let me go back to the work of von Neumann & Morgenstern (1944) in which they introduced their famous representation theorem. Before they do so the authors discuss the usefulness of yielding an additive representation. They explain that when calling two natural operations addition this "nomenclature is not intended as a claim that

³²And in periods $t < T$ respectively the derived functions \tilde{u}_t giving evaluation for the elements of \tilde{X}_t .

³³Remember that in the general multi-commodity setting making u linear by the the Epstein-Zin-gauge is no longer possible.

the two operations with the same name are identical, – this is manifestly not the case; it only expresses the opinion that they possess similar traits, and the hope that some correspondence between them will ultimately be established.” (von Neumann & Morgenstern 1944, 21). Furthermore they discuss as analogies for additive quantities the concepts of mass and energy. Let me add that additivity is a concept that emerged from the calculus of natural numbers. Mathematically this is through Peano’s axioms. From a more ”physical” perspective it is through the observation of similar rules for different classes of materialistic objects describing what happens when putting and taking away objects of the same type. This structure appears to be inherent also in von Neumann & Morgenstern’s (1944) examples of mass and energy.

Let me come back to the problem of having two aggregation-operations at hand. One, which is over time and another which is over uncertainty. It was shown that in general only one of them can be rendered additive. Now let me look at the analogies mentioned by von Neumann & Morgenstern (1944, section 3.4), namely mass and energy. I suppose that it is not only me who thinks about mass as an additive operation because it has the property that gaining one pound next week, another pound in the second and losing two pounds in the third will bring me back to my original weight. Hence my reasoning would always be in terms of pound aggregation over time rather than over uncertainty. Energy might even be a better example for this point as it is an abstract quantity introduced by physicists to describe some regularity that cannot be observed directly - just as utility. Furthermore energy is just defined the way it is in order to bring this regularity into a form that makes it possible to think about it in a similar way as about the physically observable mass, hence in terms of adding and subtracting energy at different points of time. To me it seems quite natural to introduce a quantity named *welfare* (as a special Bernoulli utility) **by** its characteristic that it should be additive over time. And for sure it seems unnatural to me to introduce a quantity that shall correspond to the meaning of the word welfare and which is constructed by having me evaluate different lotteries in a sophisticated manner. The latter procedure³⁴ would give rise to the u as gauged in the vNM-theorem in section 2.2 and in Kreps & Porteus (1978). Another evidence that it might be more natural to think about intertemporal choice in the certainty additive gauge will come from the following discussion on precautionarity.

It was observed in the preceding section that there is some ambiguity about specifying the degree of risk aversion and the degree of intertemporal substi-

³⁴Which is worked out for example in Kreps (1988) or Pratt, Raiffa & Schlaifer (1995).

tutability. However it was seen that up to affine transformations (from both sides) the 'quantity' fg^{-1} uniquely depends on the decision makers set of preference \succeq . The following theorem will yield an interpretation of this gauge invariant. To this purpose write $\mathbf{x}^t = (x_t, x_{t+1}, \dots, x_T)$ and $\bar{\mathbf{x}}^t = (\bar{x}, \bar{x}, \dots, \bar{x})$ like in section 3.3.

Theorem 5: Let the triple (u, f, g) represent the set of preferences \succeq in the sense of theorem 3. The decision maker is *precautionary* [unprecautionary] in the sense that for some $t \in \{1, \dots, T\}$ it holds that

$$\forall \bar{x}, x_t, \dots, x_T \in X : \quad \bar{\mathbf{x}}^t \sim_t \mathbf{x}^t \quad \Rightarrow \quad \bar{x} \succeq_T [\preceq_T] \alpha_t x_t + \dots + \alpha_t x_T \quad (12)$$

if and only if $fg^{-1}(\gamma)$ is concave [convex] in $\gamma \in g(U)$ for an increasing version of f .³⁵ Furthermore if equation (12) holds true for some t , it holds true for any $t \in \{1, \dots, T\}$.

Moreover the right side of equation (12) is fulfilled with indifference if and only if $\hat{f} = \hat{g}$ in which case the decision maker maximizes additively (time-)separable expected utility.

The premise of equation (12) states that a decision maker is indifferent between a certain constant consumption path delivering $(T + 1 - t)$ times \bar{x} and another certain consumption path delivering the elements x_t, \dots, x_T . Let me stress that in the decision problem on the left hand side the individual knows that in either case he gets all of the chosen goods with certainty (each at its time). Now precautionarity demands that for such consumption goods the decision maker should weakly prefer the consumption of \bar{x} with certainty over a lottery yielding any of the goods x_t, \dots, x_T with equal probability. The intuition is that if he values the different consumption commodities in x_t, \dots, x_T differently, he might be better off with the result of the lottery, but he might also be worse of.

To me this idea seems straight forward and easily accessible within the certainty additive gauge. Here Bernoulli utility is chosen in such a way, that the indifference on the left hand side of equation (12) is just the statement that the sum of utility is equal for either of the consumption paths. The lottery on the right hand side is evaluated in the certainty additive gauge by means of the uncertainty aggregation rule \mathcal{M}^f . Now if the decision maker is precautionary he rather takes the utility $u(\bar{x})$ for sure than facing the risky situation where he also 'gets $u(\bar{x})$ on average' (in fact he does not get it!) but might also be worse off. This concept of precautionarity coincides with the intuitively introduced notion facing the power mean in section 2.1. In fact for the power mean the concavity property of f

³⁵Which is equivalent to convexity [concavity] for a decreasing choice of f .

translates into $\alpha \leq 1$.³⁶ The reader who is still not convinced that the certainty additive gauge renders a more intuitive representation in terms of welfare might give it a try to interpret theorem 5 in the Kreps-Porteus-gauge.

Finally let me point out, that precautionarity is a characteristic of preferences that is independent of the preference for the timing of uncertainty resolution in the sense that different combinations can hold. For example in the situation discussed above where the uncertainty aggregation rule corresponds to the power mean and $g = \text{id}$ the following combinations arise. For $\alpha > 1$ the decision maker is unprecautionary and prefers early resolution of uncertainty. In the range of $\alpha \in (0, 1)$ the decision maker is precautionary and prefers late resolution of uncertainty, while finally for $\alpha < 0$ he still evaluates precautionary but prefers again early resolution of uncertainty.

Summarizing this section it was seen that in the (intuitive) certainty additive gauge a general representation of preferences depends crucially on the admittance of precautionary decision rules. However, also in the intertemporal case one can move into a gauge which allows for an additive uncertainty aggregation rule. Nevertheless now it comes at the price of introducing nonlinear utility aggregation over time. I made the point that the latter is inconvenient in terms of interpreting Bernoulli utility as welfare.

5 Conclusions

I have started out by showing that in an atemporal setting a dissent on the application of expected value maximization versus precautionary decision rules cannot be distinguished from a controversy over the underlying value functions. However, when time is introduced a new possibility to cardinalize welfare arises and appears more natural. A general representation for preferences allowing for different gauges of Bernoulli utility and satisfying the von Neumann-Morgenstern axioms, certainty additivity and time consistency is derived. It allows to relate different representations in the literature.

I showed that a distinction between intertemporal substitutability and risk aversion is possible without giving up the von Neumann-Morgenstern axioms or certainty additivity. The relation between risk aversion, intertemporal substitutability and aversion against late resolution of uncertainty was worked out by

³⁶With the definition of the power mean through $f(z) = z^\alpha$ the function f is decreasing for $\alpha < 0$ in which case the precautionarity criterion turns into convexity of f which is satisfied. Remember that $g = \text{id}$.

extending Kreps & Porteus's (1978) result on the preference for timing of uncertainty resolution into my gaugeable representation. Discussing the relation between risk aversion and intertemporal substitutability I showed that there is no canonical prescription what is to be attributed to risk aversion and what to intertemporal substitutability. Nevertheless a specified difference between the two has been found to be gauge invariant. Moreover this invariant is shown to capture the intuition of precautionarity.

Appendix

A Proofs for Section 2

Lemma 0: If the tuple (u, f) represents \succeq in the sense of theorem 1, then so does the tuple $(s \circ u, f \circ s^{-1})$ for any $s : \mathbb{R} \rightarrow \mathbb{R}$ strictly increasing and continuous.

Proof of lemma 0: The second tuple stands for the representation $sf^{-1}[\int_X (fs^{-1})(su) dp] = sf^{-1}[\int_X fu dp]$. The latter is an increasing transformation of the representation $\mathcal{M}^f(p, u)$ for \succeq and hence a representation for \succeq itself. Moreover $s \circ u$ and $f \circ s^{-1}$ are continuous and the latter is strictly monotonic. \square

Proof of proposition 1: " \Rightarrow ": By axioms A1-A3 the theorem of von Neumann-Morgenstern gives the existence of the representation (u^0, id) with u^0 as in the theorem. Because $u, u^0 \in B_{\succeq}$ it is $u^0(x^1) \geq u^0(x^2) \Leftrightarrow \delta_{x^1} \succeq \delta_{x^2} \Leftrightarrow u(x^1) \geq u(x^2)$ for all $x^1, x^2 \in X$. Therefore it exists a strictly increasing transformation such that $u = s \circ u^0$. Moreover s is continuous because u^0 and u are so. Hence by lemma 0 the tuple $(s \circ u^0, s^{-1})$ also represents \succeq . Let $f^+ \equiv s^{-1}$. Then f^+ is strictly increasing and continuous and (u, f^+) represent \succeq in the sense of proposition 1.

" \Leftarrow ": First let f be strictly increasing and (u, f) represent \succeq in the sense of proposition 1. Then with $s = f^{-1}$ by lemma 0 also (f^{-1}, id) represents \succeq in this sense and hence in the sense of the theorem of von Neumann-Morgenstern. Therefore the latter assures that A1-A3 are satisfied. For f strictly increasing note that $\mathcal{M}^f = \mathcal{M}^{-f}$ and hence the above reasoning can be applied to $(u, -f)$ with $-f$ strictly increasing.

Moreover part: " \Rightarrow ": Let (u, f) and (u', f') represent \succeq in the sense of proposition 1. First let f and f' be strictly increasing. Then by lemma 0 also the tuples $(f \circ u, \text{id})$ and $(f' \circ u', \text{id})$ are representations of \succeq . Hence by the "moreover-

part” of von Neumann-Morgenstern’s theorem it follows that $f' = \mathbf{a}^+ f$ with $\mathbf{a}^+ \in \mathbf{A}^+ = \{\mathbf{a}^+ : \mathbb{R} \rightarrow \mathbb{R} : \mathbf{a}^+(z) = az + b, a, b \in \mathbb{R}, a > 0\}$. For f strictly decreasing and f' strictly increasing apply the same reasoning to $(u, -f)$ and (u', f') and find that $f' = \mathbf{a}^+(-f) = -\mathbf{a}^+ f$ with $\mathbf{a}^+ \in \mathbf{A}^+$. The two remaining cases work out equivalently.

” \Leftarrow ”: Follows immediately from $\mathcal{M}^f = \mathcal{M}^{\mathbf{a}^+ f}$ with $\mathbf{a}^+ f$ being a nondegenerate affine transformation of f . \square

Proof of corollary 1: The main part applies the same reasoning as the proof of proposition 1.

Moreover part: ” \Rightarrow ”: First let f be increasing. As in the moreover part of proposition 1 it follows for two different tuples (u, f) and (u', f') representing \succeq that $f'u' = \mathbf{a}^+ fu$ and hence $u' = f'^{-1} \mathbf{a}^+ fu$. For f decreasing the same reasoning applied to $(u, -f)$ and $(u', -f')$ also representing \succeq yields that $-f'u' = \mathbf{a}^+ - fu$ and hence again $u' = f'^{-1} \mathbf{a}^+ fu$.

” \Leftarrow ”: First let f be increasing. Given that (u, f) represents \succeq than so does (fu, id) by lemma 0. Then by the theorem of von Neumann-Morgenstern for $\mathbf{a}^+ \in \mathbf{A}^+$ also $(\mathbf{a}^+ fu, \text{id})$ represents \succeq . Hence another application of lemma 0 with $s = f^{-1}$ gives that $(f^{-1} \mathbf{a}^+ fu, f)$ represents \succeq in the sense of proposition 1. For f decreasing one gets with the same reasoning respectively $(-fu, \text{id})$, $(\mathbf{a}^+(-f)u, \text{id}) = (-\mathbf{a}^+ fu, \text{id})$ and $(f^{-1} \mathbf{a}^+ fu, f)$ as representations for \succeq in the sense of proposition 1. In the last step I used lemma 0 with $s = (-f)^{-1} = f^{-1} \circ (-\text{id})$. \square

Proof of proposition 2: The proof is widely analogous to the proof of proposition 1. Instead of A1-A3 together with the theorem of von Neumann-Morgenstern use certainty additivity A4. By the latter it exists u^0 which is seen to be an increasing trafo of $u \in B_{\succeq}$: $u(x) \geq u(x') \Leftrightarrow [x] \succeq [x'] \Leftrightarrow (x, x^0, \dots, x^0) \succeq (x', x^0, \dots, x^0) \Leftrightarrow u^0(x^1) + \sum_{i=2}^T u^0(x^0) \geq u^0(x^2) + \sum_{i=2}^T u^0(x^0) \Leftrightarrow u^0(x) \geq u^0(x')$. In the moreover part use a theorem by Hardy et al. (1964, p. 74 theorem 83) to assure that affine uniqueness also applies in the case of fixed weights. \square

B Proofs for Section 3

Proof of theorem 1: ” \Rightarrow ”: First note that $u \in B \succeq_F$ implies $u \in B \succeq_T$ by certainty additivity A4 and time consistency A5. Then by proposition 1 there exists f as in the proposition. For any $p \in P$ define x_p^c to be an arbitrary element of the certainty equivalence set $\{x \in X : u(x_p^c) = \mathcal{M}^f(p, u)\}$. By connectedness of X ,

continuity of u and the coincidence of range $\mathcal{M}^f(\cdot, u)$ with range u the intermediate value theorem assures that this set is not empty. Then by definition of x_p^c it is $x_p^c \sim_T p \forall p \in P$. Furthermore by time consistency *iii*) it holds that $x_p^c \sim_T p \Leftrightarrow (x, x_p^c) \sim_F (x, p) \forall x \in X, p \in P$. From certainty additivity A4 and $u \in B_\succeq$ proposition 2 brings about the existence of g as in the proposition such that $(x, x_p^c) \succeq_F (x', x_{p'}^c) \Leftrightarrow g^{-1} \left[\frac{1}{2}g \circ u(x) + \frac{1}{2}g \circ u(x_p^c) \right] \geq g^{-1} \left[\frac{1}{2}g \circ u(x') + \frac{1}{2}g \circ u(x_{p'}^c) \right]$. Combining the statements I find for any two consumption paths $(x, p), (x', p') \in X \times P$ that: $(x, p) \succeq_F (x', p') \Leftrightarrow (x, x_p^c) \succeq_F (x', x_{p'}^c) \Leftrightarrow g^{-1} \left[\frac{1}{2}g \circ u(x) + \frac{1}{2}g \circ u(x_p^c) \right] \geq g^{-1} \left[\frac{1}{2}g \circ u(x') + \frac{1}{2}g \circ u(x_{p'}^c) \right] \Leftrightarrow g^{-1} \left[\frac{1}{2}g \circ u(x) + \frac{1}{2}g \circ \mathcal{M}^f(p, u) \right] \geq g^{-1} \left[\frac{1}{2}g \circ u(x') + \frac{1}{2}g \circ \mathcal{M}^f(p', u) \right]$ which implies v).

" \Leftarrow ": *i*) follows immediately from proposition 1. The same is true for *ii*) noting that $\mathcal{M}^f(\delta_x, u) = u(x)$ and using proposition 2. To show *iii*) first replace a decreasing g by $-g$ yielding the same representation v) with an increasing g (compare proof of proposition 1). Then *iii*) can be seen to hold by $(x, p) \succeq_F (x', p') \Leftrightarrow g^{-1} \left[\frac{1}{2}g \circ u(x) + \frac{1}{2}g \circ \mathcal{M}^f(p, u) \right] \geq g^{-1} \left[\frac{1}{2}g \circ u(x') + \frac{1}{2}g \circ \mathcal{M}^f(p', u) \right] \Leftrightarrow g \circ u(x) + g \circ \mathcal{M}^f(p, u) \geq g \circ u(x') + g \circ \mathcal{M}^f(p', u) \Leftrightarrow \mathcal{M}^f(p, u) \geq \mathcal{M}^f(p', u) p \succeq_T p'$.

The moreover part for g is immediate from proposition 2. As for f notice that affine transformations of f do not change \mathcal{M}^f such that the allowed transformations to f in *vi*) by proposition 1 also do not change v). \square

Proof of lemma 1: Like in the proof of lemma 0 plug the second tuple into the representation and observe that it equals the first up to an increasing transformation. More precisely \mathcal{M} is transformed to $s \circ \mathcal{M}$ showing the same behavior as u . This additional s appearing in inner-period evaluation is absorbed by $g \circ s^{-1}$ and carried to the front of the expression by sg^{-1} . \square

Proof of corollary 2: As proof of corollary 1 using lemma 1 instead of lemma 0 and hence applying the transformations also to g . \square

Proof of corollary 3: Imitates the proof of corollary 2. \square

Proof of theorem 2: Recursively apply proposition 1. Compactness and continuity are inherited from step to step, guaranteeing the existence of certainty equivalents for any lottery in every step. Recursively these certainty equivalents construct certain consumption paths from t until T . Certainty additivity applied

to the certain consumption paths ensures together with time consistency that in $t - 1$ one can choose $g_{t-1} = g_t$. \square

Proof of theorem 3: Translate axiom A6 into the representation of theorem 2. The resulting condition yields that $f_{t+1}f_t^{-1}$ is linear and hence f_{t+1} can be chosen equal to f_t due to the uniqueness only up to affine transformations. \square

Proof of lemma 2: For lemma 1 just apply the reasoning of the proof of lemma 1 recursively. Using this lemma the proof of the corollaries stays practically the same. \square

C Proofs for Section 4

Proof of theorem 4: Translating the respective condition on the preferences into the representation of theorem 3 turns the proof into simple arithmetics. After reducing the inequality apply f to both sides. This is where the distinction of f increasing or decreasing comes in. \square

Proof of lemma 3: Take an arbitrary $u^0 \in B_{\succeq}$. By theorem 3 there exist f and g as in the theorem unique up to nondegenerate affine transformations. Any other $u \in B_{\succeq}$ is a transformation of u^0 by s as in lemma 1. By lemma 1, valid for the multiperiod setting by lemma 2, application of s to the representing triple leaves fg^{-1} unchanged. Moreover also for u the representing functions f and g are unique up to affine transformations by theorem 3. \square

Proof of theorem 5: Translate the respective condition on the preferences into the representation of theorem 3 and apply f to both sides yielding the case distinction between f increasing or decreasing. Defining $\gamma = f \circ u(x)$ yields the concavity condition. The furthermore part is obvious as the resulting condition on f and g is independent of t . \square

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