Biodiversity Loss and Stochastic Technological Processes: a Sustainable Growth Analysis

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Abstract

This paper aims to show how a continuous effort in R&D activities can relax the availability constraint of renewable resources simultaneously affected by harvesting and pollution degradation. We also study the possibility for the economy to reach an expected sustainable growth path if the impatience of society is balanced by the positive effects of three different types of new innovations. The necessary condition of sustainability requires the social discount rate to be small enough compared to the marginal probability of innovation.[1]

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1 Introduction

The two most critical environmental challenges that our society faces nowadays are two human-induced global changes: the greenhouse effect and the biodiversity loss. We develop a stochastic endogenous growth model to investigate the biodiversity loss challenge for the purpose of sustainability. In fact we investigate the conditions for an optimal growth path to be sustainable.

There are four motivations for developing such a model. First, the problem of biodiversity loss has not received attention from the growth literature yet (on the contrary to the greenhouse effect). Second, the analysis of the biodiversity loss requires an approach that is fundamentally different from that one used for assessing the greenhouse effect. Third, an optimal growth model is the best way to study sustainability, since we can use such a model to derive conditions under which sustained growth can go hand in hand with environmental improvement. Finally, a stochastic approach seems well suited to describe the inherent uncertainty technological innovations.

The importance of biodiversity loss as an indicator of environmental sustainability has only recently come to the limelight of research. For instance, the current observed rate of extinction per century just for birds and mammals is 100-1000 times the "natural" background rate, based on fossil records (Townsend et al. 2003). Furthermore, the tropical moist forest clearance and burning due to land conversion (one of the major causes of biodiversity loss) will increase over the next 50 years by 30 percent and 20 percent, respectively (Tilman et al. 2001).

Why do we need a different approach to account for biodiversity loss? This can be seen immediately from recalling that the two major reasons for biodiversity loss (at the level of species, communities, and ecosystems) are overexploitation of renewable resources and habitat disruption, all of which is mainly due to various forms of pollution. The consideration of biodiversity loss requires viewing renewable resources from a broader perspective (including biological populations and water, soil and atmosphere). Hence, it is insufficient to analyze either the optimal use of renewable resources or the lasting effects of stock pollution problems resulting from that use. In fact both aspects are simultaneously relevant for biodiversity loss. This is inherently different from the greenhouse effect whose cause is the cumulation of pollutants rather than the exhaustion of non-renewable resources per se.

There is a notable disconnection between the "70s growth models" being interested in the optimal use of non-renewable resources (Dasgupta and Heal 1974, Siglitz 1974, Solow 1974) and the present pollution-induced awareness intrinsic in the "90s growth models" on environmental quality (Bovenberg and Smulders 1995, Stokey 1998, Aghion and Howitt 1998, Brock and Taylor 2004a). Brock and Taylor (2004b) provide a critical discussion on this ‘unbundling’ of interests in environmental issues. Obviously, the case of renewable resources calls for a simultaneous treatment of ”optimal use” and ”quality degradation” issues.

This paper combines the ideas of the standard environmental quality growth literature (which investigates pollution awareness) with that of the ”corn-eating” framework (used in the analysis of optimal use of renewable resources; Pezzi 1992). We investigate the joint effect of harvesting and induced pollution degradation on renewable resources. Thereby, two standard shortcomings will be overcome: (i) in contrast to previous research, the recruitment curve will not treat the environment as invariant (Townsend 2003) or, said in other words, the regenerative capacity (Clark 1990) will not be exogenous and fixed but ”conditional on the particular environment circumstances that happen to prevail, and [it will] change if any of those circumstances change” (Perman 2003); (ii) the regeneration function will be affected by the scientific and technological advancements, unlike in previous research. We introduce three possible types of environmental friendly technologies: techniques that affect the productivity of harvested resources (e.g., to avoid clear-felling in the forest harvest), techniques that reduce pollution damages (e.g., to restore water quality), and techniques that reduce the production of pollution (e.g., to increase the efficiency of the practices used to prevent soil degradation). The technologies are generated by a non-stationary Poisson process whose arrival rate depends on both time and R&D investments (Lafforge 2004). Only in steady state, this process will be stationary. If these R&D investments produced an infinite number of innovations, the three lasting effects on renewable resources would potentially disappear.

The paper is organized in five sections. We present the model in the following one. In section 3 we examine the new Hotelling rule, taking into account the negative effects of pollutants and the positive ones of new innovations. In section 4 analytical optimal solutions are discussed. Then, in section 5, we study the optimal trajectories of extraction and consumption and we determine the necessary conditions for the optimal growth paths to be sustainable. A summary and some concluding remarks are given in section 6.

2 Structure of the model

Assume an economy that produces a homogeneous final good Y_t, using labor (L) and renewable resources according to

$$Y_t = F(R_t, L_t) = R_t^{\theta} L_t^{1-\theta} \tag{1}$$

where R_t is the harvested resources (i.e., the flow) at period t. L is constant over time and inelastically supplied at each point of time at a perfect labor market. Furthermore, there are constant returns to scale, $\theta \in (0, 1)$.

The stock of resources is modelled as a stochastic process. It evolves over time by the following stochastic equation

$$dS_t = (\mu P_t^{-\zeta} S_t^{\kappa} dt + \sigma_1 \mu P_t^{-\zeta} S_t^{\kappa} dq_{1,t}) - (R_t dt - \sigma_2 R_t dq_{2,t}) - (P_t^{\xi} S_t^{\rho} dt - \sigma_3 P_t^{\xi} S_t^{\rho} dq_{3,t}) \tag{2}$$

where $\mu, \kappa, \zeta, \rho, \xi \geq 0$ and $\sigma_1, \sigma_2, \sigma_3 \geq 0$ and are parameters. The parameters $\sigma_i (i = 1, 2, 3)$ denote the expected rate of growth in the availability of the resources due to technological innovations. The components on the right-hand side of (2) can be interpreted as follows. μ is the regeneration rate of resources that is endogenously determined in equilibrium. Then, $\mu P_t^{-\zeta} S_t^{\kappa} dt$ is a modified regeneration function of resources. Note that this differs from the one used in existing sustainable growth models (Aghion and Howitt, 1998), where the regeneration rate is constant. Here, this rate inter alia depends on the stage of pollution of the economy (P). There is a direct negative effect on the change in resource stocks from the harvesting ($R_t dt$) and the change in the stage of pollution ($P_t^{\xi} S_t^{\rho} dt$). However, there is a threefold role to play technological innovations. Among those, there are two ways of how technological innovations affect the impact of the stage of pollution on the availability of resources. First, σ_1 is the rate at which polluted resources can be used in production beyond their original level. Second, σ_3 reflects the rate at which the direct negative effect of pollution on resources is reduced. Note that σ_1 and σ_3 are due to innovations affecting resource abatement. In contrast, σ_2 is due to innovations in the efficiency of resources usage. Therefore, there is a direct positive effect on dS_t from the latter.

Note that the law of motion for the renewable resources takes into account all the considerations given in the introduction. 1) The regeneration rate of resources μ is not exogenously given but negatively influenced by the lasting effects of pollutants, especially, of cumulative ones (Gariup 2004). 2) The regeneration function is reduced not only by either harvesting $R_t dt$ or environmental degradation induced by the emission of pollutants $P_t^{\xi} S_t^{\rho} dt$, but by both of them simultaneously. 3) These three lasting effects on the regeneration function can be reduced by three possible types of environmental friendly innovations: $\sigma_1 \mu P_t^{-\zeta} S_t^{\kappa} dq_{1,t}$, $\sigma_2 R_t dq_{2,t}$, and $\sigma_3 P_t^{\xi} S_t^{\rho} dq_{3,t}$. 4) The variations in the random cumulated number of innovations ($dq_{1,t}$,$dq_{2,t}$, $dq_{3,t}$)

follow a non stationary Poisson process[2] with arrival rates $\lambda_i(N_t)(i = 1, 2, 3)$, where N_t denotes the fraction of labor devoted to R&D (Lafforgue 2004).

$$\mathbb{P}(q_t - q_s = k) = \frac{(\lambda_i(N_t)(t - s))^k}{k!} e^{-\lambda_i(N_t)(t-s)} \tag{3}$$

where $0 \leq s \leq t$. The instantaneous probability function $\lambda_i(.)$ is assumed to be increasing and concave in N_t, such that $\lambda_i(N_t) \in [0, 1], \forall N_t \in [0, 1]$. In the time intervals between the innovation jumps of the technological sector, the resource evolves deterministically.

Labor can be devoted to either production or R&D activities. Therefore, the following constraints must hold:

$$N_t \geq 0$$

$$L_t \geq 0$$

$$1 - L_t - N_t \geq 0$$

We consider pollutants as an inevitable consequence of human activity. Following the argument of the law of thermodynamics and the considerations of (Common 1995) about the environmental impact, in the very long run, we define

$$\dot{P} = \Gamma P$$

Note that this specification is consistent with an inverted U shape of the environmental Kuznets curve. However, it also covers the case where no such inverted U shaped environmental Kuznets curve prevails.[3] What is important here is that the environmental impact per unit of income does not converge to zero but to a maybe even small level $\Gamma > 0$ as income grows.

Equation (3) tells us that the current probability of a new successful innovation increases with the effort devoted to R&D activities. As long as no such effort is undertaken, the probability of success is zero ($\lambda_i(0) = 0$). In each point of time either a new innovation is developed ($dq_{i,t} = 1$) with

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[2] A Poisson process (q_t) is a time dependent family of identically and independent distributed (iid) random variables with integer values $q_0 = 0$. The increments $q_t - q_s$ and $q_v - q_u$ are stochastically independent and stationary.

[3] Although the empirical finding of an inverted U-shaped environmental Kuznets curve is uncontroversial for some particular forms of pollution (see Grossman and Krueger 1995). However, recent papers cast doubt on an application of the concept to pollution in general (see Bertinelli and Strobl 2005, Stern 2004).

probability $\lambda_i(N_t)dt$, or is not $(dq_{i,t} = 0)$ with probability $[1 - \lambda_i(N_t)dt]$. In this last case equation (2) is reduced to its deterministic components and becomes

$$dS_t = \mu P_t^{-\zeta} S_t^{\kappa} dt - R_t dt - P_t^{\xi} S_t^{\rho} dt$$

where the technological progress does not mitigate the negative effects on the regenerative function. Assume that the consequences of successful innovations are instantaneous. Then, each time a new innovation occurs, q_i is instantaneously increased by one unit and $dt = 0$. Thus, the availability of resources instantaneously grows but in a discontinuous manner since the stock trajectory jumps upward at each new success of the technological process. The size of such a jump is given

$$\Delta S_t = S(t, q_i + 1) - S(t, q_i) = \sigma_i \digamma_i(P, S, R, t, q_i)$$

where $S(t, q_i + 1)$ indicates the availability of resources following innovations in either one or several of the environmental friendly technologies. The discrete changes in the availability of resources are assumed to be proportional to the size of the lasting effects, to maintain the notion that the R&D activity is proportional to the severity of the lasting effects. Each of the three possible types of innovations happens independently of each other. On average, the positive effects of innovations may balance the lasting pressures of harvesting and pollutants.

The instantaneous utility function of the infinitely lived representative agent is characterized by

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma};\ \gamma > 0;\ \gamma \neq 1$$

where C_t is the consumption quantity of the final good at date t and $\frac{1}{\gamma}$ is the elasticity of intertemporal substitution of consumption.

3 The new Hotelling Rule

The program (Ω) of the social planner is to maximize the expected present value of the utility

$$V(S) = \max_{R_t, N_t, t \geq 0} E \int_0^{\infty} u(C_t) e^{-\delta t} dt$$

subject to

$$C_t = F(R_t, (1 - N_t))$$

$$dS_t = (\mu P_t^{-\zeta} S_t^{\kappa} dt + \sigma_1 \mu P_t^{-\zeta} S_t^{\kappa} dq_{1,t}) - (R_t dt - \sigma_2 R_t dq_{2,t}) - (P_t^{\xi} S_t^{\rho} dt - \sigma_3 P_t^{\xi} S_t^{\rho} dq_{3,t})$$

$$R_t, N_t, (1 - N_t), S_t \geq 0 \quad \forall t \geq 0$$

where future utility flows are discounted at rate δ ($\delta > 0$) and one control variable is redundant through $L_t = 1 - N_t$. Using the dynamic programming technique (Merton, 1990) and the results of Sennewald/Waelde (2005) and Sennewald (2005)[4] we find the Hamilton-Bellman-Jacobi equation associated with the value function of (Ω), $V(S_t)$.

$$\delta V(S_t) = \max_{R_t, N_t \geq 0} \{u(C_t) + \frac{1}{dt} EdV(S_t)\} \tag{4}$$

If we expand the stochastic differential $dV(S_t)$, equation (4) becomes

$$\begin{aligned} \delta V(S) \quad = \quad & \max_{R_t, N_t \geq 0} \{u(C_t) + V'(S)[\mu P^{-\zeta} S^{\kappa} - R_t - \ P_t^{\xi} S^{\rho}] + \\ & + \lambda_1(N_t)\Delta_1 V(\hat{S}) + \lambda_2(N_t)\Delta_2 V(\hat{S}) + \lambda_3(N_t)\Delta_3 V(\hat{S})\} \end{aligned} \tag{5}$$

where $\Delta_1 V(\hat{S})$, $\Delta_2 V(\hat{S})$, and $\Delta_3 V(\hat{S})$ are the respective instantaneous increases in social welfare due to the development of a new environmental friendly innovation, reducing the permanently negative effects of pollutants on the regenerative capacity μ, the harvesting pressure, and the temporary emission intensity:

$$\begin{aligned} \Delta_1 V(\hat{S}) &= \underbrace{V(S + \sigma_1 \mu P^{-\zeta} S_t^{\kappa})}_{V_1(\hat{S})} - V(S) \\ \Delta_2 V(\hat{S}) &= \underbrace{V(S + \sigma_2 R_t)}_{V_2(\hat{S})} - V(S) \\ \Delta_3 V(\hat{S}) &= \underbrace{V(S + \sigma_3 P_t^{\xi} S^{\rho})}_{V_3(\hat{S})} - V(S) \end{aligned}$$

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[4] The Hamilton-Jacobi-Belman equation is derived for unbounded utility functions and for a larger class of argument functions in the underlying stochastic differential equations of types like (3). Since we do not bother at the beginning of necessary conditions for the sufficiency this restriction derived in Sennewald (2005) do not apply for us. But after deriving the optimal paths including the discussion afterwards, we see that for the cases we are interested in necessary conditions are fulfilled authomatically.

The first order conditions for (Ω) are

$$\frac{\partial u}{\partial C}(C_t)\frac{\partial F}{\partial R}(R_t, L_t) = V'(S) + \lambda_2(N_t)\sigma_2 V'(S + \sigma_2 R_t) \tag{6}$$

$$\frac{\partial u}{\partial C}(C_t)\frac{\partial F}{\partial L}(R_t, L_t) = \lambda_1'(N_t)\Delta_1 V(\hat{S}) + \lambda_2'(N_t)\Delta_2 V(\hat{S}) + \lambda_3'(N_t)\Delta_3 V(\hat{S}) \tag{7}$$

As usual equation (6) indicates that along any optimal path the marginal benefit of using (harvesting) the resource in terms of instantaneous utility must be equal to the resource rent $V'(S)$, corrected for the second type of innovations – in the efficiency of resources usage (σ_2) – that are not proportional to the stock of resources. Equation (7) assures that along any optimal path the marginal cost of the R&D activity in terms of instantaneous utility must equal the expected utility gain due to the development of a new type of innovation, being the marginal probability of success in the development of a new innovation $\lambda_i'(N_t)$, and the associated instantaneous increment of social welfare, $\Delta_i V(\hat{S})$. If we replace the $u(.)$ and $F(.)$ functions by their analytical analogues and divide equation (6) by equation (7), then

$$\frac{\theta(1 - N_t)}{(1-\theta)R_t} = \frac{V'(S) + \lambda_2(N_t)\sigma_2 V'(S_t + \sigma_2 R_t)}{\lambda_1'(N_t)\Delta_1 V(\hat{S}_t) + \lambda_2'(N_t)\Delta_2 V(\hat{S}_t) + \lambda_3'(N_t)\Delta_3 V(\hat{S}_t)} \tag{8}$$

In models of optimal use of renewable resources the standard result is that the resource rent (or shadow price of the resource) grows at the difference between the interest rate and the regeneration rate of the resource. This result is known as the Hotelling Rule extended to renewable resources and describes the permanent intertemporal trade-off resource users are faced with. But in our model, with the lasting effects of pollutants and the R&D activities, the standard Hotelling Rule needs to be modifiedtakes a new form.

Proposition 1 *If the stock of the renewable resources (S_t) satisfies (2), then the resource rent in average grows as following*

$$\begin{aligned} \frac{\frac{1}{dt}E(dV'(S_t))}{V'(S_t)} &= \delta - (\mu P_t^{-\zeta}\kappa S^{\kappa-1} - P_t^{\xi}\rho S_t^{\rho-1}) + \qquad (9) \\ &\quad -[\lambda_1(N_t)\sigma_1\mu P_t^{-\zeta}\kappa S_t^{\kappa-1}\frac{V_1'(\hat{S}_t)}{V'(S_t)} + \lambda_3(N_t)\sigma_3 P_t^{\xi}\rho S_t^{\rho-1}\frac{V_3'(\hat{S}_t)}{V'(S_t)}] \end{aligned}$$

where $V_1'(\hat{S}_t) = V'(S_t + \sigma_1\mu P_t^{-\zeta} S_t^{\kappa})$ is the value of the resource rent following an innovation for the mitigation of the lasting effects of pollutants on the regeneration rate and $V_3'(\hat{S}_t) = V'(S_t + \sigma_3 P_t^{\xi} S_t^{\rho})$ is the value of the resource rent following an innovation for reducing the production of pollutants.

Proof 1 *See Appendix 1*

In this model, the rate of growth of the resource rent is not simply equal to the social discount rate net of the regeneration rate of the resource. But rather, the latter has to be corrected for two crucial factors of influence: (i) the effect on the growth rate through technological progress (σ_1 and σ_3), and (ii) the sum of the indirect effects on the regeneration rate μ and the direct one on the stock of resources. The corrections connected with μ increase the growth rate due to the fact that lasting effects of pollutants increase the physical scarcity of the resource. The effect of technology as such reduces the growth rate because at the time a new innovation is developed, the physical scarcity of resources is instantaneously reduced. As a result, the adverse effects of pollutants on the regeneration rate may slow down the use of the resource, but uncertainty on the return of innovations may speed up the harvesting.

Equation (9) is also important to verify the transversality condition of (Ω). Being

$$\frac{1}{dt}E[\frac{d(e^{-\delta t}V'(S))}{V'(S)}] < 0$$

a necessary condition for the transversality condition to hold, from equation (9) it becomes

$$\begin{aligned}
\frac{1}{dt}E[\frac{d(e^{-\delta t}V'(S_t))}{V'(S_t)}] &= e^{-\delta t}[-\delta + \frac{\frac{1}{dt}E(dV'(S_t))}{V'(S_t)}] \\
&= -e^{-\delta t}[\lambda_1(N_t)\sigma_1\mu P_t^{-\zeta}\kappa S_t^{\kappa-1}\frac{V_1'(\hat{S}_t)}{V'(S_t)} \\
&\quad +\lambda_3(N_t)\sigma_3 P_t^{\xi}\rho S_t^{\rho-1}\frac{V_3'(\hat{S}_t)}{V'(S_t)} \\
&\quad +(\mu\kappa P_t^{-\zeta}S^{\kappa-1} - \rho P_t^{\xi}S_t^{\rho-1})] < 0
\end{aligned}$$

Hence, the transversality condition holds if the expression in the parenthesis on the right hand side is positive.

Note that $(\mu\kappa P_t^{-\zeta}S^{\kappa-1} - \rho\ P_t^{\xi}S_t^{\rho-1}) > 0$ is a sufficient condition for this. However, even if the effect on the regeneration rate and the direct effect of pollutants on S is negative, $(\mu\kappa P_t^{-\zeta}S^{\kappa-1} - \rho P_t^{\xi}S_t^{\rho-1}) < 0$, the transversality condition holds as long as it is smaller in absolute value than the positive technological effect, $\lambda_1(N_t)\sigma_1\mu P_t^{-\zeta}\kappa S_t^{\kappa-1}\frac{V_1'(\hat{S}_t)}{V'(S_t)} + \lambda_3(N_t)\sigma_3 P_t^{\xi}\rho S_t^{\rho-1}\frac{V_3'(\hat{S}_t)}{V'(S_t)}$.

4 The optimal paths

For finding an analytical solution of the optimal policy functions of harvesting and R&D effort, we set $\lambda_i(N_t) = \lambda_i N_t$, with $\lambda_i \in [0,1]$. In fact only a linear functional form for the probability allows to solve (Ω) analytically. So, an increase in the R&D effort leads to a higher probability of technological innovation, but leaving the marginal probability unchanged. For analytical simplicity we also restrict ourself to the case where $\sigma_2 = \sigma_3 = 0$ because the main result of the positive effect of new developed innovations will not be affected and no additional qualitative insight will be gained.

Proposition 2 *The optimal paths of harvesting, R&D effort, and consumption are unique and regular. They are*

$$\begin{aligned} 1 - N_t^* = L_t^* &= \frac{(1-\theta)(1-\gamma)m_t}{\lambda\eta_t} && (10)\\ R_t^* &= m_t S_t && (11)\\ C^*(S_t) &= m_t[\frac{(1-\gamma)(1-\theta)}{\lambda\eta_t}]^{1-\theta} S_t^{\theta} && (12) \end{aligned}$$

where

$$\begin{aligned} \eta_t &= (1 + \sigma_1 \mu P_t^{-\zeta} S_t^{\kappa-1})^{\theta(1-\gamma)} - 1\\ m_t &= \frac{1}{\gamma}[\delta - \lambda\eta_t - \theta(1-\gamma)(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1})] \end{aligned}$$

Proof 2 *See Appendix 2*

Since η_t and μ_t are time dependent, the optimal allocation of labor in the production sector and in the R&D sector are not constant. But since we have to meet the constraints $0 \leq N^*, L^* \leq 1$, we have to find necessary conditions for the existence of the equations (10), (11) and (12). This means to characterize a feasible set of parameters that guarantees the existence of an interior optimal solution.

Proposition 3 *An interior optimal solution exists if and only if*

$$\delta \in \{\underline{\delta}, \overline{\delta}\}$$

where

$$\underline{\delta}_t = \begin{cases} \theta(1-\gamma)(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) + \lambda\eta_t & if \gamma < 1\\ 0 & if \gamma > 1 \end{cases}$$

$$\overline{\delta}_t = \theta(1-\gamma)(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) + \lambda(\eta_t + \frac{\eta_t \gamma}{(1-\gamma)(1-\theta)}) \ \ \forall \gamma$$

and

$$(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) \geq 0 \ \forall t$$

Proof 3 *See appendix 3*

The lower and the upper constraints delimitate our feasible set and are time dependent. But since we are interested in balanced steady state solutions, we use now the definition given to the evolution of pollutants $g_P = \Gamma = \frac{\dot{P}}{P}$ and assume that also $g_S = \frac{\dot{S}}{S}$ is a constant. Nine combinations are possible, but only one is feasible. This means that only when $\xi g_P = (1-\rho) g_S$ and $-\zeta g_P = (1-\kappa) g_S$ we can find a balance steady state solution for the optimal paths, where $A_t = P_t^{-\zeta} S_t^{\kappa-1} = A$(a constant) and $B_t = P_t^{\xi} S_t^{\rho-1} = B$(a constant). Thanks that both the feasible set for the parameters and the optimal path are no more time dependent. Hence we can rewrite:

$$\underline{\delta} = \begin{cases} \theta(1-\gamma)(\mu A - B) + \lambda\eta & if \gamma < 1 \\ 0 & if \gamma > 1 \end{cases}$$

$$\overline{\delta} = \theta(1-\gamma)(\mu A - B) + \lambda(\eta + \frac{\eta\gamma}{(1-\gamma)(1-\theta)}) \ \ \forall \gamma$$

In graphical terms, when $\gamma < 1$, $\underline{\delta}$ and $\overline{\delta}$ are linear in λ with slope η and $\frac{\gamma\eta}{(1-\gamma)(1-\theta)}$, respectively and have a common intercept $(1-\gamma)\theta(\mu A - B)$. They span a cone in which all (λ, δ) -pairs are associated with feasible equilibria given the constraints. See Figure 2a),b).

The following Table 1 (as in Lafforgue 2004) summarizes the qualitative effects of parameters on the optimal paths.

	δ	λ	σ_1	S_t	θ
N^*	$-$	$+$	$+$	/	$\gamma < 1 : +$ $\gamma > 1 : -$
L^*	$+$	$-$	$-$	/	$\gamma < 1 : -$ $\gamma > 1 : +$
R^*	$+$	$\gamma < 1 : -$ $\gamma > 1 : +$	$\gamma < 1 : -$ $\gamma > 1 : +$	$+$	$\gamma < 1 : -$ $\gamma > 1 : +$
C^*	$+$	$\gamma < 1 : -$ $\gamma > 1 : \pm$	$\gamma < 1 : -$ $\gamma > 1 \pm$	$+$	$\gamma < 1 : -$ $\gamma > 1 : +$

Let us concentrate on the social discount rate (δ). An increase in δ causes the (labor and resources) input usage to increase, the R&D effort to diminish, and the consumption to increase. An increase in the marginal probability of success increases the effort in R&D $N^*$and consequently decreases the productive labor input L^*; it also increases current extraction R^* if and only if society favors its present consumption capacity, i.e. $\gamma > 1$. In that case, it raises consumption C^* if the positive effect on extraction more than compensates the negative effect on labor.

5 The optimal paths analysis

We first characterize the exact optimal paths and then find their asymptotic behavior. In balanced steady state, according to solutions (10) and (11), our stochastic differential equation (2) is

$$dS_t = \mu A S_t dt - m S_t dt - B S_t dt + \sigma_1 \mu A S_t dq_t \tag{13}$$

where we recall

$$m = \frac{1}{\gamma}(\delta - \lambda\eta - (1-\gamma)\theta(\mu A - B))$$

As long as there is not a new innovation (no jump) $dq_t = 0$, then equation (13) has the solution

$$S(t,0) = S_0 e^{(\mu A - m - B)t}$$

When a new innovation is developed $dq_t = 1$ and the availability of resources is instantaneously increased by σ_1 percent. Then the resources follow the optimal trajectory

$$S^*(t,q_t) = (1+\sigma_1\mu A)^{q_t} S(t,0) = (1+\sigma_1\mu A)^{q_t} S_0 e^{(\mu A - m - B)t} \tag{14}$$

the optimal harvesting trajectory is

$$R^*(t,q_t) = m S^*(t,q_t) = m(1+\sigma_1\mu A)^{q_t} S_0 e^{(\mu A - m - B)t} \tag{15}$$

and that one for consumption reads

$$C^*(t,q_t) = m^\theta (1+\sigma_1\mu A)^{\theta q_t} S_0^\theta e^{\theta(\mu A - m - B)t}(1-N^*)^{1-\theta} \tag{16}$$

Since the exact trajectories (14),(15) and (16) are piecewise discontinuous (they jump upwards at the instant an innovation is developed) and their asymptotic behavior is undetermined (q_t tends to infinity in probability along an infinite time horizon and $S_0 e^{(\mu A - m - B)t}$ could decline to zero if $\mu A < m + B$), we compute the smoothed trajectories. Hence, we consider the paths of the expected value of S_t^*, R_t^* and C_t^*.

Integrating (13)and computing the expected value, we get

$$\overline{S_t} = S_0 \exp((\mu A - m - B + \lambda N^* \sigma_1 \mu A)t)$$

and therefore the solution for (15)(the expected optimal extraction rate path)is

$$\overline{R_t} = m\overline{S_t} \tag{17}$$

and in a similar way the solution for (16) (the expected optimal consumption path) is

$$\overline{C_t} = (\mu S_0)^\theta (1 - N^*)^{1-\theta} \exp(\{\theta(\mu A - m - B) + \lambda N^*[(1 + \sigma_1 \mu A)^\theta - 1]\}t) \tag{18}$$

The corresponding growth rates $g_{(.)}$ are constant over time. For R and S they are

$$\begin{aligned}
g_{\overline{S}} &= g_{\overline{R}} = \\
&= \mu A - m - B + \lambda N^* \sigma_1 \mu A = \\
&= \mu A - m - B + \lambda(1 - \frac{(1-\gamma)(1-\theta)m}{\lambda\eta})\sigma_1 \mu A = \\
&= \mu A - B - \frac{1}{\gamma}(\delta - \lambda\eta - (1-\gamma)\theta(\mu A - B))(1 + \frac{(1-\gamma)(1-\theta)}{\eta}\sigma_1 \mu A) + \lambda \sigma_1 \mu A
\end{aligned} \tag{19}$$

and for C the growth rate is

$$\begin{aligned}
g_{\overline{C}} &= \\
&= \theta\mu A - \theta B - \theta m + \lambda N^*(1 + \sigma_1 \mu A)^\theta - \lambda N^* = \\
&= \theta(\mu A - B) - \frac{\theta}{\gamma}(\delta - \lambda\eta - (1-\gamma)\theta(\mu A - B)) + \\
&\quad + \frac{\lambda\eta - (1-\gamma)(1-\theta)\frac{1}{\gamma}(\delta - \lambda\eta - (1-\gamma)\theta(\mu A - B)}{\eta}(1 + \sigma_1 \mu A)^\theta \\
&\quad - \frac{\lambda\eta - (1-\gamma)(1-\theta)\frac{1}{\gamma}(\delta - \lambda\eta - (1-\gamma)\theta(\mu A - B)}{\eta} = \\
&= \theta(\mu A - B) + \lambda[(1 + \sigma_1 \mu A)^\theta - 1] + \\
&\quad - \frac{1}{\gamma}(\delta - \lambda\eta - (1-\gamma)\theta(\mu A - B))(\theta + \frac{(1-\gamma)(1-\theta)}{\eta}[(1 + \sigma_1 \mu A)^\theta - 1]
\end{aligned} \tag{20}$$

We are interested in the signs of those rates. In particular, the aim of this paper is to determine the necessary conditions for sustainable growth, i.e., positive consumption growth and non-decreasing resources over time. Therefore we find the two functions that guarantee $g_{\overline{S}} \geq 0$ and $g_{\overline{C}} \geq 0$

using (19)

$$(\delta - \lambda\eta - (1-\gamma)\theta(\mu A - B))[1 + \frac{(1-\gamma)(1-\theta)}{\eta}\sigma_1 \mu A] \leq \gamma(\mu A - B) + \gamma\lambda\sigma_1\mu A$$

if and only if

$$\delta \leq \delta_S = (\mu A - B)((1-\gamma)\theta + \frac{\gamma}{Q}) + \lambda(\eta + \frac{\gamma\sigma_1\mu A}{Q}) \qquad (21)$$

where $Q = 1 + \frac{(1-\gamma)(1-\theta)}{\eta}\sigma_1\mu A$

Using (20)

$$(\delta - \lambda\eta - (1-\gamma)\theta(\mu A - B))[1 + \frac{(1-\gamma)(1-\theta)}{\eta}[(1+\sigma_1\mu A)^\theta - 1]] \leq \theta\gamma(\mu A - B) + \gamma\lambda[(1+\sigma_1\mu A)^\theta - 1]$$

if and only if

$$\delta \leq \delta_C = \theta(\mu A - B)((1-\gamma) + \frac{\gamma}{Z}) + \lambda(\eta + \frac{\gamma[(1+\sigma_1\mu A)^\theta - 1]}{Z}) \qquad (22)$$

where $Z = \theta + \frac{(1-\gamma)(1-\theta)}{\eta}[(1+\sigma_1\mu A)^\theta - 1]$.

Now, recall the original constraints $\underline{\delta}(\lambda)$ and $\overline{\delta}(\lambda)$ assure a feasible set of parameters. Together with the new non-negativity constraints $\delta_S(\lambda)$ and $\delta_C(\lambda)$ given in (21) and (22), we can study the sustainability issue in the δ and λ space. For the sake of better readability, let us give this system of four equations again:

$$\begin{aligned}
\underline{\delta}(\lambda) &= (\mu A - B)\theta(1-\gamma) + \lambda\eta \\
\overline{\delta}(\lambda) &= (\mu A - B)\theta(1-\gamma) + \lambda(\eta + \frac{\eta\gamma}{(1-\gamma)(1-\theta)}) \\
\delta_S(\lambda) &= (\mu A - B)(\theta(1-\gamma) + \frac{\gamma}{Q}) + \lambda(\eta + \frac{\gamma\sigma_1\mu A}{Q}) \\
\delta_C(\lambda) &= \theta(\mu A - B)((1-\gamma) + \frac{\gamma}{Z}) + \lambda(\eta + \frac{\gamma[(1+\sigma_1\mu A)^\theta - 1]}{Z})
\end{aligned}$$

When $\gamma < 1$, $\underline{\delta}(\lambda)$ and $\overline{\delta}(\lambda)$ have the same intercept at $\lambda = 0$, namely $\theta(1-\gamma)(\mu A - B)$. Since $Q, Z > 0$ and $\gamma > 0$ the intercept of $\delta_S(\lambda)$ and $\delta_C(\lambda)$ is higher than that one of $\underline{\delta}(\lambda), \overline{\delta}(\lambda)$. Which lines intersect? Certainly, the slope of $\overline{\delta}(\lambda)$ is bigger than that of $\underline{\delta}(\lambda)$. Also, the slope of both $\delta_S(\lambda)$ and $\delta_C(\lambda)$ are bigger than that one of $\underline{\delta}(\lambda)$, since $\frac{\gamma\sigma_1\mu A}{Q} > 0$ and $\frac{\gamma[(1+\sigma_1\mu A)^\theta - 1]}{Z} > 0$. They exhibit a bigger intercept than $\underline{\delta}(\lambda)$ so that they can not intersect with $\underline{\delta}(\lambda)$. Now, compare the slopes of $\delta_S(\lambda)$ and $\overline{\delta}(\lambda)$:

$$\begin{aligned}
\frac{\gamma\eta}{(1-\gamma)(1-\theta)} &\geq \frac{\gamma\sigma_1\mu A}{Q} \Leftrightarrow \\
\frac{\eta}{(1-\gamma)(1-\theta)} &\geq \frac{\sigma_1\mu A}{1 + \frac{(1-\gamma)(1-\theta)}{\eta}\sigma_1\mu A} \Leftrightarrow \\
\frac{\eta + (1-\gamma)(1-\theta)\sigma_1\mu A}{(1-\gamma)(1-\theta)} &\geq \sigma_1\mu A \Leftrightarrow \\
\frac{\eta}{(1-\gamma)(1-\theta)} + \sigma_1\mu A &\geq \sigma_1\mu A
\end{aligned} \tag{23}$$

The last condition always holds with strict inequality, since $\eta(1-\gamma) > 0$. Thus, the slope of $\overline{\delta}(\lambda)$ is strictly bigger than that of $\delta_S(\lambda)$. Together with our knowledge about their intercepts, this ensures an intersection between these two functions.

Now, let us compare $\delta_C(\lambda)$ and $\overline{\delta}(\lambda)$:

$$\begin{aligned}
\frac{\gamma\eta}{(1-\gamma)(1-\theta)} &\geq \frac{\gamma[(1+\sigma_1\mu A)^\theta - 1]}{Z} \Leftrightarrow \\
\frac{\eta}{(1-\gamma)(1-\theta)} &\geq \frac{(1+\sigma_1\mu A)^\theta - 1}{\theta + \frac{(1-\gamma)(1-\theta)}{\eta}[(1+\sigma_1\mu A)^\theta - 1]} \Leftrightarrow \\
\frac{\theta\eta + (1-\gamma)(1-\theta)[(1+\sigma_1\mu A)^\theta - 1]}{(1-\gamma)(1-\theta)} &\geq (1+\sigma_1\mu A)^\theta - 1 \Leftrightarrow \\
\frac{\eta}{(1-\gamma)(1-\theta)} + (1+\sigma_1\mu A)^\theta - 1 &\geq (1+\sigma_1\mu A)^\theta - 1
\end{aligned} \tag{24}$$

Again the last condition holds with strict inequality for the same reasons as above. Hence, the slope of $\overline{\delta}(\lambda)$ is strictly bigger than that of $\delta_C(\lambda)$.

Finally, let us compare the slopes of $\delta_S(\lambda)$ and $\delta_C(\lambda)$:

$$
\begin{aligned}
\frac{\gamma\sigma_1\mu A}{Q} &\geq \frac{\gamma[(1+\sigma_1\mu A)^\theta-1]}{Z} \Leftrightarrow \\
\sigma_1\mu A Z &\geq Q[(1+\sigma_1\mu A)^\theta-1] \Leftrightarrow \\
&\quad \frac{(1-\gamma)(1-\theta)}{\eta}[(1+\sigma_1\mu A)^\theta-1]\sigma_1\mu A+ \\
+\theta\sigma_1\mu A &\geq [(1+\sigma_1\mu A)^\theta-1]+ \\
&\quad +\frac{(1-\gamma)(1-\theta)}{\eta}[(1+\sigma_1\mu A)^\theta-1]\sigma_1\mu A \Leftrightarrow \\
\theta\sigma_1\mu A &\geq [(1+\sigma_1\mu A)^\theta-1]
\end{aligned}
\tag{25}
$$

The last condition holds with strict inequality throughout, since the auxiliary function θx and $[(1+x)^\theta-1]$ start both in 0 but the first one has a bigger derivative thus, lies always above the second one. Hence, the slope of $\delta_S(\lambda)$ is strictly bigger than that of $\delta_C(\lambda)$.

When $\gamma < 1$ we can summarize the above analysis as following, see Figure 1a):

1. Since neither of the other constraints intersects with $\underline{\delta}(\lambda)$, factor market clearing is consistent with the other constraints, hence, also with sustainable growth.
2. Given that $\delta_C(\lambda)$ exhibits a bigger intercept but a smaller slope than $\delta_S(\lambda)$, there has to be an intersection point (I_1). Hence, the region below $\delta_C(\lambda)$ and to the right of this intersection point always implies not only positive consumption growth but also positive growth of the resources.
3. Both straight lines $\delta_S(\lambda)$ and $\delta_C(\lambda)$ have bigger intercepts but smaller slopes than $\overline{\delta}(\lambda)$. Thus, these lines will necessarily intersect (say at points I_2 and I_3, respectively).

The intersection points are: I_1 from $\delta_C(\lambda) = \delta_S(\lambda)$

$$
(\mu A - B)(\theta Q - Z) = \lambda(P\sigma_1\mu A - Q[(1+\sigma_1\mu A)^\theta - 1])
$$

hence,

$$
\lambda_1^* = \frac{(\mu A - B)(\theta Q - Z)}{\theta\sigma_1\mu A - [(1+\sigma_1\mu A)^\theta - 1]} = \frac{(\mu A - B)(1-\gamma)(1-\theta)}{\eta} \tag{26}
$$

The second intersection point I_2 follows from posing $\overline{\delta}(\lambda) = \delta_S(\lambda)$

$$\frac{\mu A - B}{Q} = \lambda(\frac{\eta}{(1-\gamma)(1-\theta)} - \frac{\sigma_1 \mu A)}{Q})$$

hence,

$$\lambda_2^* = \frac{(\mu A - B)(1-\gamma)(1-\theta)}{\eta Q - (1-\gamma)(1-\theta)\sigma_1 \mu A} = \frac{(\mu A - B)(1-\gamma)(1-\theta)}{\eta} \tag{27}$$

The third intersection point I_3 follows from posing $\overline{\delta}(\lambda) = \delta_C(\lambda)$

$$\frac{(\mu A - B)\theta}{Z} = \lambda(\frac{\eta}{(1-\gamma)(1-\theta)} - \frac{[(1+\sigma_1 \mu A)^\theta - 1]}{Z})$$

hence,

$$\lambda_3^* = \frac{(\mu A - B)\theta(1-\gamma)(1-\theta)}{\eta Z - (1-\gamma)(1-\theta)[(1+\sigma_1 \mu A)^\theta - 1]} = \frac{(\mu A - B)(1-\gamma)(1-\theta)}{\eta} \tag{28}$$

We see that the $\lambda_1^* = \lambda_2^* = \lambda_3^*$ and, hence, I_1, I_2, and I_3 are identical and the three linear constraints $\overline{\delta}(\lambda)$, $\delta_S(\lambda)$, $\delta_C(\lambda)$ intersects in the same pointI^*. This implies that, positive consumption growth is always consistent with growth of the resources, whenever we are within the feasible parameter range constrained by $C(\lambda)$ and $\overline{\delta}(\lambda)$until I^* and by $\underline{\delta}(\lambda)$ and $\delta_C(\lambda)$ after I^*.

After the intersection point I^* we can therefore distinguish three areas

$\delta_\lambda \in [\underline{\delta}, \delta_C]$	$\delta_\lambda \in (\delta_C, \delta_S]$	$\delta_\lambda \in (\delta_S, \overline{\delta}]$
C non-decreasing S non-decreasing	C decreasing S non-decreasing	C decreasing S decreasing

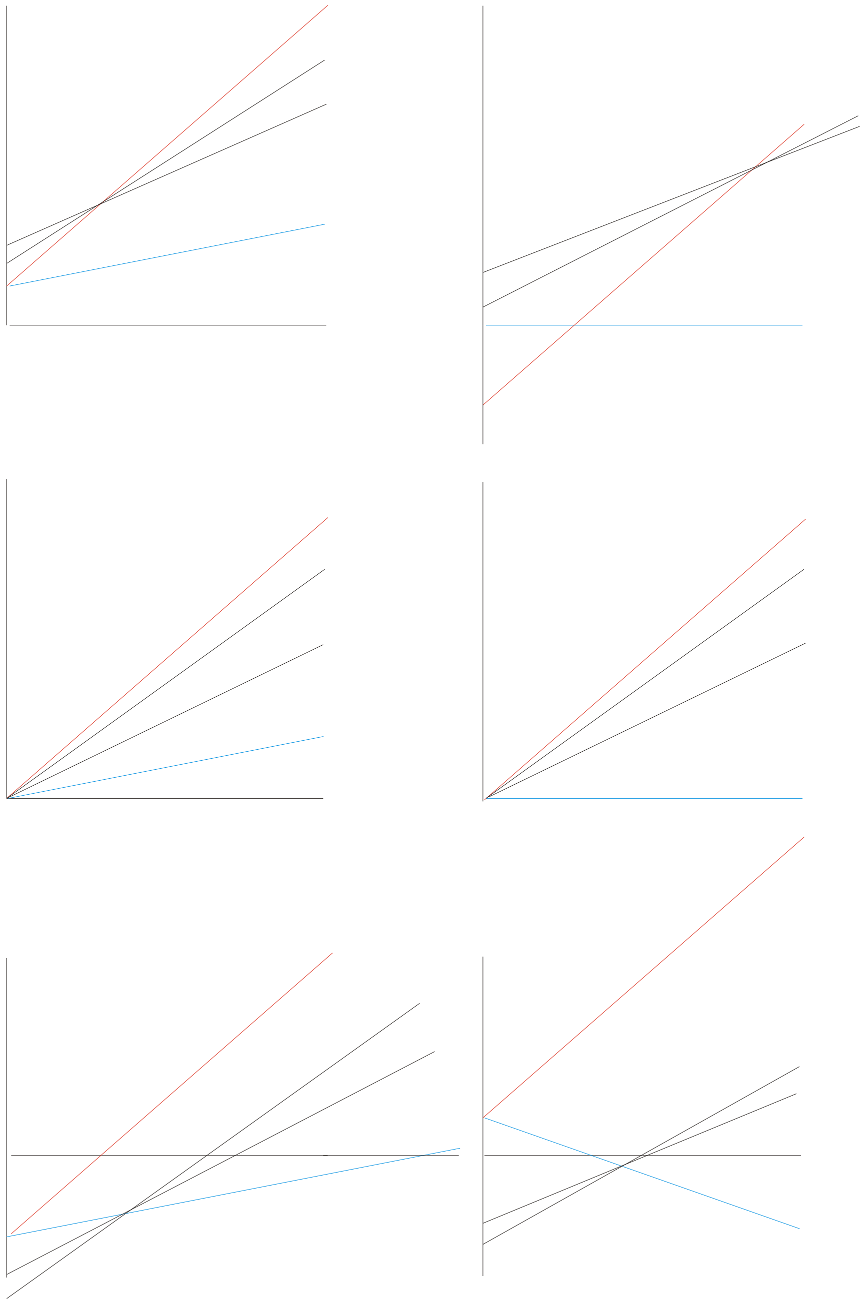

Figure 1: Sustainability areas for $\gamma <, > 1$ and $\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) >, =, < 0$

6 Conclusion

Thanks to all the three new considerations in the regeneration function (connection between harvesting and quality degradation of renewable resources; non-constancy of the regenerative capacity; stochastic technological processes that directly affect the availability of resources), we obtain a new Hotelling rule. In fact, in this model, the rate of growth of the resource rent is not simply equal to the social discount rate net of the regeneration rate of the resource. But rather, the latter has to be corrected for two crucial factors of influence: (i) the effect on the growth rate through technological progress, and (ii) the sum of the indirect effects of pollutants on the regeneration rate and the direct one on the stock of resources. The correction connected with the regeneration rate increases the growth rate due to the fact that lasting effects of pollutants increase the physical scarcity of the resource. The effect of technology as such reduces the growth rate because at the time a new innovation is developed, the physical scarcity of resources is instantaneously reduced. As a result, the adverse effects of pollutants on the regeneration rate may slow down the use of the resource, but uncertainty on the return of innovations may speed up the harvesting.

We have also determined the necessary conditions for sustainable growth i.e., positive consumption growth and non-decreasing resources over time. For that reason, we have firstly found an analytical solution of the optimal policy functions of harvesting, R&D effort and consumption; secondly characterized the smoothed optimal paths (being the exacted ones only piecewise continuous and asymptotically undetermined); thirdly computed the growth rates. As final result we have that, if the marginal probability of innovations is high enough compared with the degree of impatience of society, the expected positive effect of R&D activities overrides expected negative effects of harvesting and environmental degradation (caused by the direct and indirect impacts of pollutants) so that the smoothed trajectory of renewable resources and consumption increases over time.

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7 Appendix 1

We start with the description of the eight possible states according to the three independent Poisson process $(q_{1,t}), (q_{2,t}), (q_{3,t})$. For this we consider a given time t and a later time $t+dt$.

Event in "dt"	State	Probability
Only jump of $(q_{1,t})$	I	$\lambda_1(N_t)dt(1-\lambda_2(N_t))dt(1-\lambda_3(N_t))dt$
Only jump of $(q_{2,t})$	II	$\lambda_2(N_t)dt(1-\lambda_1(N_t))dt(1-\lambda_3(N_t))dt$
Only jump of $(q_{3,t})$	III	$\lambda_3(N_t)dt(1-\lambda_1(N_t))dt(1-\lambda_2(N_t))dt$
Only jump of $(q_{1,t})$ and $(q_{2,t})$	IV	$\lambda_1(N_t)dt\lambda_2(N_t)dt(1-\lambda_3(N_t))dt$
Only jump of $(q_{1,t})$ and $(q_{3,t})$	V	$\lambda_1(N_t)dt\lambda_3(N_t)dt(1-\lambda_2(N_t))dt$
Only jump of $(q_{2,t})$ and $(q_{3,t})$	VI	$\lambda_2(N_t)dt\lambda_3(N_t)dt(1-\lambda_1(N_t))dt$
Jump of $(q_{1,t})$, $(q_{2,t})$, $(q_{3,t})$	VII	$\lambda_1(N_t)dt\lambda_2(N_t)dt\lambda_3(N_t)dt$
No jump at all	VIII	$(1-\lambda_1(N_t))dt(1-\lambda_2(N_t))dt(1-\lambda_3(N_t))dt$

For the derivative $V'(S)$ at time $t+dt$ we have

$$V'(S)|_{t+dt} = V'(S)|_t + V''(S)dS_t.$$

This gives according to the above table the following representation of $V'(S)|_{t+dt}$ depending of the state

$$V'(S)|_{t+dt} = \begin{cases} V'(S)|_t + \mu P_t^{-\zeta} S_t^{\kappa} V''(S)dt - R_t V''(S)dt - & \\ P_t^{\xi} S_t^{\rho} V''(S)dt & \text{state VIII} \\ V'_{1,2,3} = V'(S_t + \sigma_1 \mu P_t^{-\zeta} S_t^{\kappa} + \sigma_3 P_t^{\xi} S_t^{\rho} + \sigma_2 R_t) & \text{state VII} \\ V'_{2,3} = V'(S_t + \sigma_3 P_t^{\xi} S_t^{\rho} + \sigma_2 R_t) & \text{state VI} \\ V'_{1,3} = V'(S_t + \sigma_1 \mu P_t^{-\zeta} S_t^{\kappa} + \sigma_3 P_t^{\xi} S_t^{\rho}) & \text{state V} \\ V'_{1,2} = V'(S_t + \sigma_1 \mu P_t^{-\zeta} S_t^{\kappa} + \sigma_2 R_t) & \text{state IV} \\ V'_3(\hat{S}) = V'(S_t + \sigma_3 P_t^{\xi} S_t^{\rho}) & \text{state III} \\ V'_2(\hat{S}) = V'(S_t + \sigma_2 R_t) & \text{state II} \\ V'_1(\hat{S}) = V'(S_t + \sigma_1 \mu P_t^{-\zeta} S_t^{\kappa}) & \text{state I} \end{cases}$$

This implies

$$\frac{V'(S)|_{t+dt}-V'(S)|_t}{dt} = \begin{cases} [\mu P_t^{-\zeta}S_t^{\kappa} - R_t - \; P_t^{\xi}S_t^{\rho}]V''(S) & \\ = M_{V''(S)} & \text{state VIII} \\ \frac{1}{dt}(V'_{1,2,3} - V'(S)|_t) & \text{state VII} \\ \frac{1}{dt}(V'_{2,3} - V'(S)|_t) & \text{state VI} \\ \frac{1}{dt}(V'_{1,3} - V'(S)|_t) & \text{state V} \\ \frac{1}{dt}(V'_{1,2} - V'(S)|_t) & \text{state IV} \\ \frac{1}{dt}(V'_3(\hat{S}) - V'(S)|_t) & \text{state III} \\ \frac{1}{dt}(V'_2(\hat{S}) - V'(S)|_t) & \text{state II} \\ \frac{1}{dt}(V'_1(\hat{S}) - V'(S)|_t) & \text{state I} \end{cases}$$

We apply the expectation

$$\begin{aligned} E \quad & (\frac{V'(S)|_{t+dt}-V'(S)|_t}{dt}) = ([1-(\lambda_1(N_t)+\lambda_2(N_t)+\lambda_3(N_t))dt] + \\ + \quad & [\lambda_1(N_t)\lambda_2(N_t)+\lambda_2(N_t)\lambda_3(N_t)+\lambda_1(N_t)\lambda_3(N_t)](dt)^2 + \lambda_1(N_t)\lambda_2(N_t)\lambda_3(N_t)(dt)^3)M_{V''(S)} \\ + \quad & \lambda_1(N_t)\lambda_2(N_t)\lambda_3(N_t)(dt)^2(V'_{1,2,3} - V'(S)|_t) \\ + \quad & \lambda_2(N_t)\lambda_3(N_t)dt(1-\lambda_1(N_t)dt)(V'_{2,3} - V'(S)|_t) \\ + \quad & \lambda_1(N_t)\lambda_3(N_t)dt(1-\lambda_2(N_t)dt)(V'_{1,3} - V'(S)|_t) \\ + \quad & \lambda_1(N_t)\lambda_2(N_t)dt(1-\lambda_3(N_t)dt)(V'_{1,2} - V'(S)|_t) \\ + \quad & \lambda_3(N_t)(1-\lambda_1(N_t)dt)(1-\lambda_2(N_t)dt)(V'_3(\hat{S}) - V(S)|_t) \\ + \quad & \lambda_2(N_t)(1-\lambda_1(N_t)dt)(1-\lambda_3(N_t)dt)(V'_2(\hat{S}) - V(S)|_t) \\ + \quad & \lambda_1(N_t)(1-\lambda_2(N_t)dt)(1-\lambda_3(N_t)dt)(V'_1(\hat{S}) - V(S)|_t) \end{aligned} \tag{29}$$

We partially differentiate (5) with respect to S

$$\begin{aligned} \delta V'(S) &= V''(S)(\mu P_t^{-\zeta}S_t^{\kappa} - R_t - P_t^{\xi}S_t^{\rho}) + V'(S)(\kappa\mu P_t^{-\zeta}S_t^{\kappa-1} - \rho P_t^{\xi}S_t^{\rho-1}) \\ &+ \lambda_1(N_t)\Delta_1 V'(\hat{S}) + \lambda_1(N_t)\sigma_1\mu\kappa P_t^{-\zeta}S_t^{\kappa-1}V'_1(\hat{S}) \\ &+ \lambda_2(N_t)\Delta_2 V'(\hat{S}) \\ &+ \lambda_3(N_t)\Delta_3 V'(\hat{S}) + \lambda_3(N_t)\sigma_3\rho P_t^{\xi}S_t^{\rho-1}V'_3(\hat{S}) \end{aligned}$$

After rearranging terms, we derive

$$\begin{aligned} M_{V''(S)} + \sum_{i=1}^{3}\lambda_i(N_t)\Delta_i V'(\widehat{S}) &= \delta V'(S) - \lambda_1(N_t)\sigma_1\mu\kappa P_t^{-\zeta}S_t^{\kappa-1}V'_1(\hat{S}) \\ &- \lambda_3(N_t)\sigma_3\rho P_t^{\xi}S_t^{\rho-1}V'_3(\hat{S}) - V'(S)(\kappa\mu P_t^{-\zeta}S_t^{\kappa-1} - \rho P_t^{\xi}S_t^{\rho-1}) \end{aligned} \tag{30}$$

We now divide (29) by "$V'(S)$"and let dt tend to zero so that (29)becomes

$$\frac{\frac{1}{dt}E(dV'(S_t))}{V'(S_t)} = \frac{M_{V''(S)} + \sum_{i=1}^{3}\lambda_i(N_t)\Delta_i V'(\hat{S})}{V'(S_t)}$$

With (30) we conclude the proof

$$\begin{aligned}\frac{\frac{1}{dt}E(dV'(S_t))}{V'(S_t)} = \delta \quad &- \quad [\lambda_1(N_t)\sigma_1\mu\kappa P_t^{-\zeta}S_t^{\kappa-1}\frac{V_1'(\hat{S}_t)}{V'(S_t)} \\ &+ \quad \lambda_3(N_t)\sigma_3\rho P_t^{\xi}S_t^{\rho-1}\frac{V_3'(\hat{S}_t)}{V'(S_t)} \\ &+ \quad (\mu\kappa P_t^{-\zeta}S_t^{\kappa-1} - \ \rho P_t^{\xi}S_t^{\rho-1})]\end{aligned}$$

8 Appendix 2

We follow the standard technic and use the first order conditions in (6) and (7) to compute the optimal solution for the extraction rate and labour input rate

$$\begin{aligned}\frac{\partial u}{\partial C}(C_t) &= C_t^{-\gamma} \\ \frac{\partial F}{\partial R}(R_t, L_t) &= \theta R_t^{\theta-1}L_t^{1-\theta} \\ \frac{\partial F}{\partial L}(R_t, L_t) &= (1-\theta)R_t^{\theta}L_t^{-\theta}\end{aligned}$$

Using $C_t = F(R_t, L_t)$ and $\sigma_2 = \sigma_3 = 0$, the conditions (6) and (7) read

$$\frac{\partial u}{\partial C}\frac{\partial F}{\partial R} = \theta R_t^{\theta(1-\gamma)-1}L_t^{(1-\gamma)(1-\theta)} = V'(S) \tag{31}$$

and

$$\frac{\partial u}{\partial C}\frac{\partial F}{\partial L} = (1-\theta)R_t^{\theta(1-\gamma)}L_t^{-\gamma-(1-\gamma)\theta} = \lambda\Delta_1 V(\hat{S}) \tag{32}$$

Dividing (32) by (31) and omitting the time index reveals

$$R = \frac{\theta}{1-\theta}\frac{\lambda\Delta_1 V(\hat{S})}{V'(S)}L \tag{33}$$

Inserting back into (32) gives

$$(1-\theta)\big(\frac{\theta}{1-\theta}\big)^{\theta(1-\gamma)}\frac{(\lambda\Delta_1 V(\widehat{S}))^{\theta(1-\gamma)}}{V'(S)^{\theta(1-\gamma)}}\cdot L^{\theta(1-\gamma)}\cdot L^{-\gamma-(1-\gamma)\theta}=\lambda\Delta_1 V(\hat{S})$$

After rearrangements we get

$$L^{\gamma}=(\frac{1-\theta}{\lambda\Delta_1 V(\hat{S})})^{1-\theta(1-\gamma)}(\frac{\theta}{V'(S)})^{\theta(1-\gamma)}$$

And finally

$$L=(\frac{1-\theta}{\lambda\Delta_1 V(\hat{S})})^{\frac{1-\theta(1-\gamma)}{\gamma}}(\frac{\theta}{V'(S)})^{\frac{\theta(1-\gamma)}{\gamma}} \tag{34}$$

We insert (34) into (33)

$$R=(\frac{1-\theta}{\lambda\Delta_1 V(\hat{S})})^{\frac{(1-\theta)(1-\gamma)}{\gamma}}(\frac{\theta}{V'(S)})^{\frac{1-(1-\theta)(1-\gamma)}{\gamma}} \tag{35}$$

We now insert this into the HJB equation (5) and, using the abbreviations $G_1=\mu P_t^{-\zeta}S_t^{\kappa}$ and $G_3=P_t^{\xi}S_t^{\rho}$, we get

$$\delta V(S)=\frac{R^{\theta(1-\gamma)}L^{(1-\theta)(1-\gamma)}}{1-\gamma}+V'(S)G_1-V'(S)R-V'(S)G_3+\lambda(1-L)\Delta_1 V(\hat{S})$$

With (34) and (35) we realize after rearrangements that

$$\begin{aligned}\delta V(S) &= (\frac{\gamma}{1-\gamma})[\frac{1-\theta}{\lambda\Delta_1 V(\hat{S})}]^{\frac{(1-\theta)(1-\gamma)}{\gamma}}[\frac{\theta}{V'(S)}]^{\frac{\theta(1-\gamma)}{\gamma}} \\ &+ \lambda\Delta_1 V(\hat{S})+\frac{G_1}{S}SV'(S)-\frac{G_3}{S}SV'(S)\end{aligned} \tag{36}$$

Thus,

$$\begin{aligned}\delta V(S) &= (\frac{\gamma}{1-\gamma})[\frac{1-\theta}{\lambda\Delta_1 V(\hat{S})}]^{\frac{(1-\theta)(1-\gamma)}{\gamma}}[\frac{\theta}{V'(S)}]^{\frac{\theta(1-\gamma)}{\gamma}} \\ &+ \lambda\Delta_1 V(\hat{S})+\mu P_t^{-\zeta}S_t^{\kappa-1}[SV'(S)]-P_t^{\xi}S_t^{\rho-1}[SV'(S)]\end{aligned} \tag{37}$$

Assuming that for all $t\geq 0$

$$P_t^{-\zeta}S_t^{\kappa} = A_t S_t \tag{38}$$

$$P_t^{\xi}S_t^{\rho} = B_t S_t, \tag{39}$$

with A_t, B_t continuous. Condition (38) guarantees that the HJB equation (5) defines a necessary condition for finding an optimal path. For the sufficient condition, as we already remarked, no additional assumption is required (see note 3). Then equation (37)turns into

$$\begin{aligned} \delta V(S) &= (\frac{\gamma}{1-\gamma})[\frac{1-\theta}{\lambda\Delta_1 V(\hat{S})}]^{\frac{(1-\theta)(1-\gamma)}{\gamma}}[\frac{\theta}{V'(S)}]^{\frac{\theta(1-\gamma)}{\gamma}} \\ &+ \lambda\Delta_1 V(S) + \mu A_t[SV'(\hat{S})] - B_t[SV'(S)] \end{aligned} \tag{40}$$

Equation (40) shows an ordinary differential equation in ”$V(S)$”. For solving it we use the approach

$$V(S) = \Psi S^{\theta(1-\gamma)} \tag{41}$$

where $\Psi \in \mathbb{R}$ is unknown, and needs to be determined below. We compute the derivative and $\Delta_1 V(\hat{S})$

$$V'(S) = \theta(1-\gamma)\Psi S^{\theta(1-\gamma)-1} \tag{42}$$

$$\Delta_1 V(\hat{S}) = \Psi[(1+\sigma_1\mu A_t)^{\theta(1-\gamma)} - 1]S^{\theta(1-\gamma)} \tag{43}$$

Insertion (41),(42) and (43) into (40) gives

$$\begin{aligned} \delta\Psi S^{\theta(1-\gamma)} &= (\frac{\gamma}{1-\gamma})(\frac{1-\theta}{\lambda})^{\frac{(1-\theta)(1-\gamma)}{\gamma}}[\Psi[(1+\sigma_1\mu A_t)^{\theta(1-\gamma)} - 1]]^{-\frac{(1-\theta)(1-\gamma)}{\gamma}} \\ &\cdot S^{-\frac{(1-\theta)(1-\gamma)}{\gamma}\theta(1-\gamma)}\theta^{\frac{\theta(1-\gamma)}{\gamma}}(\theta(1-\gamma)\Psi)^{-\frac{\theta(1-\gamma)}{\gamma}}S^{\frac{\theta(1-\gamma)}{\gamma}(1-\theta(1-\gamma))} \\ &+ \lambda\Psi[(1+\sigma_1\mu A_t)^{\theta(1-\gamma)} - 1]S^{\theta(1-\gamma)} \\ &+ \mu A_t\theta(1-\gamma)\Psi[S \cdot S^{\theta(1-\gamma)-1}] - B_t\theta(1-\gamma)\Psi[S \cdot S^{\theta(1-\gamma)-1}] \end{aligned}$$

Collecting terms that involve S, we obtain in a first step

$$\begin{aligned} \delta\Psi S^{\theta(1-\gamma)} &= (\frac{1-\theta}{\lambda})^{\frac{(1-\theta)(1-\gamma)}{\gamma}}[\Psi[(1+\sigma_1\mu A_t)^{\theta(1-\gamma)} - 1]]^{-\frac{(1-\theta)(1-\gamma)}{\gamma}} \\ &\cdot (\frac{\gamma}{1-\gamma})\theta^{\frac{\theta(1-\gamma)}{\gamma}}(\theta(1-\gamma)\Psi)^{-\frac{\theta(1-\gamma)}{\gamma}}S^{\theta(1-\gamma)} \\ &+ \lambda\Psi[(1+\sigma_1\mu A_t)^{\theta(1-\gamma)} - 1]S^{\theta(1-\gamma)} \\ &+ \mu A_t\theta(1-\gamma)\Psi S^{\theta(1-\gamma)} - B_t\theta(1-\gamma)\Psi S^{\theta(1-\gamma)} \end{aligned} \tag{44}$$

In (44) each term contains ”$S^{\theta(1-\gamma)}$” by which we divide to derive

$$\begin{aligned}
\delta\Psi &= (\frac{1-\theta}{\lambda})^{\frac{(1-\theta)(1-\gamma)}{\gamma}}[(1+\sigma_1\mu A_t)^{\theta(1-\gamma)}-1]^{-\frac{(1-\theta)(1-\gamma)}{\gamma}} \\
&\cdot\ (\frac{\gamma}{1-\gamma})\theta^{\frac{\theta(1-\gamma)}{\gamma}}(\theta(1-\gamma))^{-\frac{\theta(1-\gamma)}{\gamma}}\Psi^{-\frac{\theta(1-\gamma)}{\gamma}}\Psi^{-\frac{(1-\theta)(1-\gamma)}{\gamma}} \qquad (45)\\
&+\ \lambda[(1+\sigma_1\mu A_t)^{\theta(1-\gamma)}-1]\Psi \\
&+\ \mu A_t\theta(1-\gamma)\Psi - B_t\theta(1-\gamma)\Psi
\end{aligned}$$

Now note that $\Psi^{-\frac{\theta(1-\gamma)}{\gamma}-\frac{(1-\theta)(1-\gamma)}{\gamma}} = \Psi\cdot\Psi^{-\frac{1}{\gamma}}$. Hence, each term in (45) contains ”Ψ”. We divide again and use the abbreviation $x=\sigma_1\mu A_t$, $y=\mu A_t$ and $z=B_t$ to derive

$$\begin{aligned}
&\delta-\lambda[(1+x)^{\theta(1-\gamma)}-1]-(y-z)\theta(1-\gamma) = \qquad (46)\\
&\Psi^{-\frac{1}{\gamma}}(\frac{\gamma}{1-\gamma})(\frac{1-\theta}{\lambda})^{\frac{(1-\theta)(1-\gamma)}{\gamma}}[(1+x)^{\theta(1-\gamma)}-1]^{-\frac{(1-\theta)(1-\gamma)}{\gamma}}\theta^{\frac{\theta(1-\gamma)}{\gamma}}(\theta(1-\gamma))^{-\frac{\theta(1-\gamma)}{\gamma}}
\end{aligned}$$

This implies

$$\Psi = [\frac{\gamma(1-\gamma)^{-\frac{\gamma+\theta(1-\gamma)}{\gamma}}[\frac{1-\theta}{\lambda[(1+x)^{\theta(1-\gamma)}-1]}]^{\frac{(1-\theta)(1-\gamma)}{\gamma}}}{\delta-\lambda[(1+x)^{\theta(1-\gamma)}-1]-(y-z)\theta(1-\gamma)}]^{\gamma} \qquad (47)$$

Now we use the expression $V(S)=\Psi S^{\theta(1-\gamma)}$ and insert (47) into the expressions for L and R given in (34) and (35), respectively

$$\begin{aligned}
R &= (\frac{1-\theta}{\lambda\Psi[(1+x)^{\theta(1-\gamma)}-1]S^{\theta(1-\gamma)}})^{\frac{(1-\theta)(1-\gamma)}{\gamma}}(\frac{\theta}{\theta(1-\gamma)\Psi S^{\theta(1-\gamma)-1}})^{\frac{1-(1-\theta)(1-\gamma)}{\gamma}} = \\
&= (\frac{1}{1-\gamma})^{\frac{1}{\gamma}}(\frac{(1-\theta)(1-\gamma)}{\lambda[(1+x)^{\theta(1-\gamma)}-1]})^{\frac{(1-\theta)(1-\gamma)}{\gamma}}\Psi^{-\frac{1}{\gamma}}S \qquad (48)
\end{aligned}$$

Collecting terms, this can be rewritten as

$$R = \frac{\delta-\lambda[(1+x)^{\theta(1-\gamma)}-1]-(y-z)\theta(1-\gamma)}{\gamma}S \qquad (49)$$

And for L we compute

$$\begin{aligned}
L &= (\frac{1-\theta}{\lambda\Psi[(1+x)^{\theta(1-\gamma)}-1]S^{\theta(1-\gamma)}})^{\frac{1-\theta(1-\gamma)}{\gamma}}(\frac{\theta}{\theta(1-\gamma)\Psi S^{\theta(1-\gamma)-1}})^{\frac{\theta(1-\gamma)}{\gamma}} = \\
&= (\frac{1-\theta}{\lambda[(1+x)^{\theta(1-\gamma)}-1]})^{\frac{1}{\gamma}}(\frac{\lambda[(1+x)^{\theta(1-\gamma)}-1]}{(1-\gamma)(1-\theta)})^{\frac{\theta(1-\gamma)}{\gamma}}\Psi^{-\frac{1}{\gamma}} \qquad (50)
\end{aligned}$$

Collecting terms we derive

$$L = \frac{(1-\theta)(1-\gamma)}{\lambda[(1+x)^{\theta(1-\gamma)}-1]\gamma}(\delta - \lambda[(1+x)^{\theta(1-\gamma)}-1] - (y-z)\theta(1-\gamma)) \quad (51)$$

Recalling the definition of x and y and setting

$$\begin{aligned}
\eta_t &= [(1+x)^{\theta(1-\gamma)}-1] = [(1+\sigma_1 \mu P^{-\zeta} S_t^{\kappa-1})^{\theta(1-\gamma)}-1] \\
m_t &= \frac{\delta - \lambda[(1+x)^{\theta(1-\gamma)}-1] - (y-z)\theta(1-\gamma)}{\gamma} \\
&= \frac{\delta - \lambda[(1+\sigma_1 \mu P^{-\zeta} S_t^{\kappa-1})^{\theta(1-\gamma)}-1] - \theta(1-\gamma)[\mu P^{-\zeta} S_t^{\kappa-1} - P^{\xi} S_t^{\rho-1}]}{\gamma}
\end{aligned}$$

we complete the proof.

9 Appendix 3

When $\gamma < 1$ then $\eta_t > 0$.

According to (10) $L_t^* \geq 0 \Leftrightarrow m_t \geq 0$

Hence when $(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) > 0$

$$m_t = \delta - \lambda \eta_t - \theta(1-\gamma)(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) \geq 0$$

the lower bound becomes

$$\underline{\delta} = \theta(1-\gamma)(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) + \lambda \eta_t \quad (52)$$

According to (10) again,$L_t^* \leq 1 \Leftrightarrow$

$$1 - \frac{(1-\gamma)(1-\theta)m_t}{\lambda \eta_t} \geq 0$$

hence when$(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) > 0$

$$\lambda \eta_t - (1-\gamma)(1-\theta)\frac{1}{\gamma}(\delta - \lambda \eta_t - \theta(1-\gamma)(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1})) \geq 0$$

and the upper bound becomes

$$\overline{\delta} = \theta(1-\gamma)(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) + \lambda(\eta_t + \frac{\eta_t \gamma}{(1-\gamma)(1-\theta)}) \quad (53)$$

When $\gamma > 1$ then $\eta_t < 0$.

According to (10)and when $(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) > 0$ $L_t^* \geq 0 \Leftrightarrow$ the lower bound is

$$\underline{\delta} = 0 \tag{54}$$

and $L_t^* \leq 1 \Leftrightarrow$ as before the identical upper bound holds

$$\overline{\delta} = \theta(1-\gamma)(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) + \lambda(\eta_t + \frac{\eta_t \gamma}{(1-\gamma)(1-\theta)}) \tag{55}$$

When $(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) = 0$ we fall in the identical results of Lafforgue.

When $(\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) < 0$ the lower and upper bounds for $\gamma < 1$ are identical as for $\gamma > 1$, but then it is necessary to study their intersections with the λ ax to constraint the (λ, θ) space to the positive quadrant.

See Figure 2

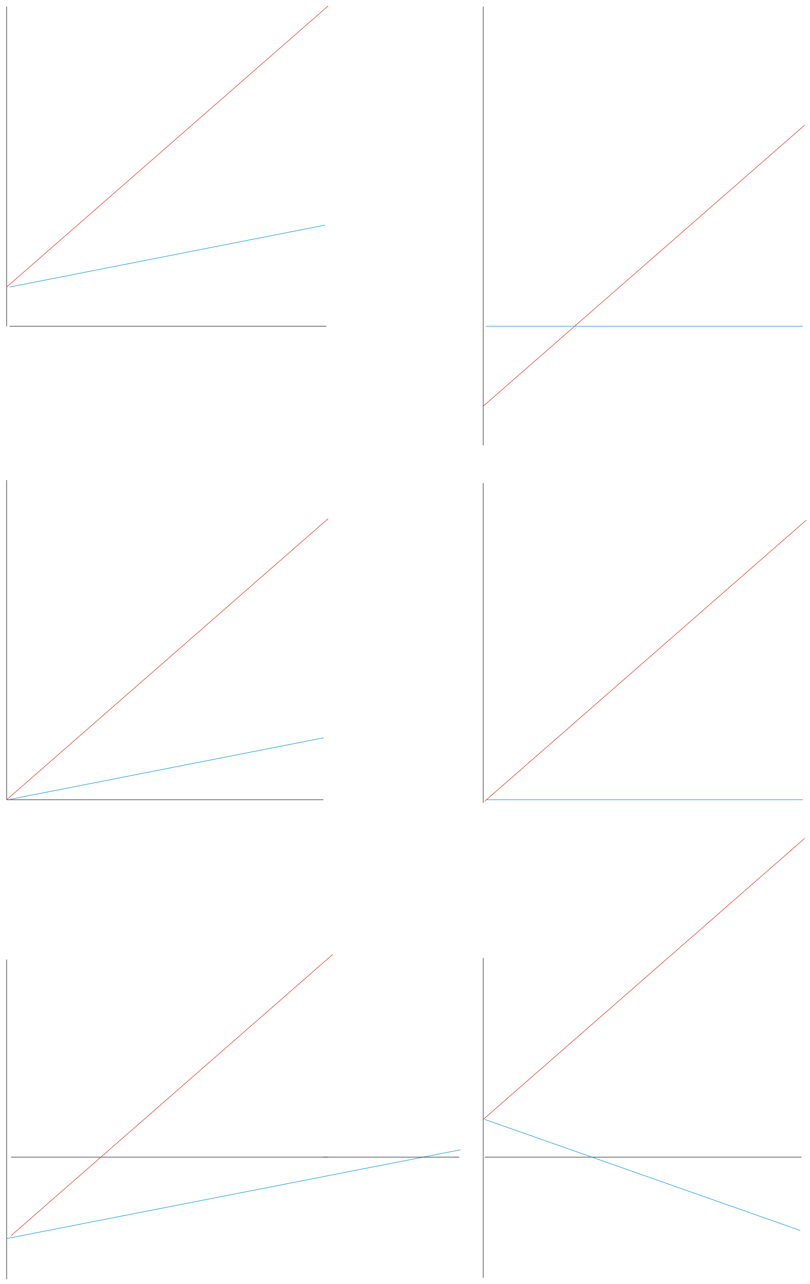

Figure 2: Necessary conditions for $\gamma <,> 1$ and $\mu P_t^{-\zeta} S_t^{\kappa-1} - P_t^{\xi} S_t^{\rho-1}) >,=,< 0$