

# Spatial Analysis in Descriptive Models of Renewable Resource Management

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JEL Classification Q2, C6

Keywords: Spatial analysis, Pattern formation, Turing mechanism, bioeconomic problems.

## 1. Introduction

Economic and ecological systems evolve in time and space, while their interactions take place among units occupying distinct spatial points. Thus geographical patterns of production activities, urban concentrations, and species or resource concentrations occur. Despite the obvious empirical relevance of spatial analysis, since spatial patterns in human economies are profound, the emergence of spatial patterns in economics received relatively little systematic analysis until the early 1990s, with the notable exception of the *new economic geography* (KRUGMAN, 1993, 1998; FUJITA, KRUGMAN and VENABLES, 1999).<sup>1</sup>

Pattern formation and the emergence of spatial patterns have received relatively more attention in ecology. A focal point of this line of approach is morphogenesis which is the study of patterns and form, (e.g. mammalian coat patterns,

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1 As indicated by KRUGMAN (1998), one of the main reasons was the inability of earlier research to work with tractable models of imperfect competition which is implied by unexhausted economies of scale. Since in a homogeneous environment with transportation costs but no returns to scale, spatial patterns of economic activity cannot emerge, and hence economic activity should spread evenly across space to minimize transportation costs, there is a need for increasing returns to generate spatial patterns. The ability to model imperfect competition and increasing returns, using mainly the Dixit/Stiglitz model (DIXIT and STIGLITZ, 1977), made possible the development of New Economic Geography.

butterfly wing patterns), along with analysis of the emergence of spatial patterns in resources and species concentrations. To explain spatial pattern formation in ecological systems, the concept of diffusion has been used in ecological modelling, as well as the development of metapopulation models. Metapopulation models which basically analyze species movements in discrete patchy environments with dispersal among patches have been used to analyze the management of renewable resources (e.g. SANCHIRICO and WILEN, 1999, 2001; SANCHIRICO, 2004; BROCK and XEPAPADEAS, 2002).

In this context diffusion is defined as a process whereby the microscopic irregular movement of particles, such as cells, bacteria, chemicals, animals, commodities, or economic activities, results in some macroscopic regular motion of the group (OKUBO and LEVIN, 2001; MURRAY, 1993, 2003). Diffusion defined in this way is based on random walk models, which when coupled with population growth equations, or capital accumulation equations could lead to general reaction-diffusion systems.

The purpose of this paper is to examine the impact of diffusion in models of renewable resource management. Diffusion takes place both with respect to the movement of the resource in a given spatial domain, but also with respect to the movement of the harvesting effort in the same domain. In this framework we seek to explore the emergence of spatial heterogeneity with respect to the resource and effort concentration, where spatial heterogeneity means that the resource concentration and the harvesting effort take different values at different spatial points. This spatial heterogeneity can be persistent in the long run, so that a spatially heterogeneous steady state is reached.

In our paper the mechanism creating this spatial heterogeneity is the so-called *Turing mechanism* (TURING, 1952) in reaction-diffusion systems, which generates *diffusion induced instability*, or *diffusive instability* (OKUBO et al., 2001). Diffusive instability in resource management models can generate spatially heterogeneous patterns of resource and harvesting effort concentration. This approach for studying the emergence of spatial patterns in ecological/economic models is different from the one most commonly used to address spatial issues, which is the use of metapopulation models in discrete patchy environments with dispersal among patches. We believe that the use of the Turing mechanism allows us to analyze in detail the conditions under which diffusion could produce spatial heterogeneity and generation of spatial patterns, or spatial homogeneity. Thus the Turing mechanism can be used to uncover conditions which generate spatial heterogeneity in models where ecological variables interact with economic

variables. Once the mechanism is uncovered, the impact of regulation in promoting or eliminating spatial heterogeneity can also be analyzed.

In this context, we present a descriptive model in which the biomass of a renewable resource (e.g. fish) diffuses in a finite one-dimensional spatial domain, and harvesting effort diffuses in the same domain, attracted to locations where profits per boat are higher. We examine conditions under which: (i) open-access equilibrium generates travelling waves for the resource biomass, and (ii) the Turing mechanism can induce spatial heterogeneity, in the sense that the steady-state fishing stock and fishing effort are different at different points of the spatial domain. We also show how regulation can promote or eliminate spatial heterogeneity.

## 2. Modelling Diffusion<sup>2</sup>

Let  $x(z, t)$  denote the concentration of a biological or economic variable at time  $t \geq 0$  at the spatial point  $z \in Z$ , where space is assumed to be one dimensional and modelled by a line segment. Let  $\phi(z, t)$  denote the flow of 'material' such as animals, commodities, or capital, past  $z$  at time  $t$ . We assume that this flux is proportional to the gradient of the concentration of the material, or

$$\phi(z, t) = -D_x \frac{\partial x(z, t)}{\partial z} \quad (1)$$

where  $D_x$  is the diffusion coefficient and the minus sign indicates that material moves from high levels of concentration to low levels of concentration. Under this diffusion assumption, the evolution of the material's stock in a small interval  $\Delta z$  is defined as:

$$\frac{d}{dt} \int_z^{z+\Delta z} x(s, t) ds = \phi(z, t) - \phi(z + \Delta z, t) + \int_z^{z+\Delta z} F(s, t) ds \quad (2)$$

2 For a detailed analysis, see MURRAY (1993, 2003).

where  $F(z,t)$  is a source or a growth function for the material in question. Dividing (2) by  $\Delta z$  and taking limits as  $\Delta z \rightarrow 0$  the evolution of the material is determined as:

$$\frac{\partial x(z,t)}{\partial t} = -\frac{\partial \phi(z,t)}{\partial z} + F(z,t).$$

Using the definition of diffusion in (1) we obtain the basic diffusion equation

$$\frac{\partial x(z,t)}{\partial t} = F(z,t) + D_x \nabla^2 x(z,t), \text{ where } \nabla^2 x(z,t) = \frac{\partial^2 x(z,t)}{\partial z^2}. \quad (3)$$

If we assume that the source term is zero,  $F(z,t) \equiv 0$  and an amount  $Q$  of particles is released per unit area at  $z=0$  at  $t=0$ , then the solution of (3) is (Murray 1993):

$$x(z,t) = \frac{Q}{2(\pi D_x t)^{1/2}} e^{-z^2/(4D_x t)}, \quad t > 0. \quad (4)$$

If the source term represents logistic population growth  $F(z,t) = x(z,t)(s - rx(z,t))$ , where  $s$  is intrinsic growth rate and  $s/r$  is the environment's carrying capacity, then we obtain the *Fisher* equation:

$$\frac{\partial x(z,t)}{\partial t} = sx(z,t) \left( 1 - \frac{rx(z,t)}{s} \right) + D_x \nabla^2 x(z,t). \quad (5)$$

The Fisher equation can be generalized to several interacting species or activities. With two interacting species  $(x,y)$  and no cross diffusion, we obtain:

$$\frac{\partial x}{\partial t} = F_1(x,y) + D_x \nabla^2 x \quad (6)$$

$$\frac{\partial y}{\partial t} = F_2(x,y) + D_y \nabla^2 y \quad (7)$$

3 In general  $\nabla^d y = \partial^d y / \partial z^d$ ,  $d=1,2$ .

System (6)–(7) is referred to as a *reaction diffusion system* or as an *interacting population diffusion system*.<sup>4</sup> We use the diffusion concepts described above to analyze, in the following sections, equilibrium in a fishery under biomass and harvesting effort diffusion.

### 3. Spatial Open Access Equilibrium with Resource Diffusion

We start by considering the case of an open access fishery, where the resource's biomass diffuses in a spatial domain and harvesting takes place in an open access way. Let  $x(z,t)$  denote the concentration of the biomass at spatial point  $z \in Z$ , at time  $t$ . Following the discussion in the previous section, fish biomass evolves in time and space following the Fisher equation with a constant diffusion coefficient  $D_x$ . That is,

$$\frac{\partial x(z,t)}{\partial t} = F(x(z,t)) - H(z,t) + D_x \nabla^2 x(z,t) \quad (8)$$

where  $F(x(z,t))$  can be modelled by the standard logistic equation and harvesting  $H(z,t)$  of the resource is determined as  $H(z,t) = qx(z,t)E(z,t)$  where  $E(z,t)$  denotes the concentration of harvesting effort (e.g. boats) at spatial point  $z$  and time  $t$  and  $q$  is catchability coefficient. The state space  $X \times E$ , where  $(x,E) \in X \times E$ , is assumed to be a compact subset of the nonnegative quadrant. Assuming further that the harvest is sold at a fixed world price, profits accruing at location  $z$  are defined as

$$pqx(z,t)E(z,t) - C(E(z,t)) \quad (9)$$

where  $C(E(z,t))$  is the total cost of applying effort  $E(z,t)$  at location  $z$ . We assume that effort is attracted by profits per boat and that effort (boats) diffuses in the spatial domain infinitely fast so that profits are equated in every site. Then in

<sup>4</sup> Generalization to  $n$  species is straightforward.

open access equilibrium with boats allowed to enter from “outside”, profits are driven to zero at each site, or

$$pqxE - C(E) = 0 \text{ or } (pqx - AC(E))E = 0 \text{ for all } z \quad (10)$$

where  $AC(E)$  denotes average costs. The shape of the average cost curve reflects different assumptions regarding the structure of the fishery and the impact that this structure has on the emergence of spatial patterns. In standard analysis of fisheries (e.g. CLARK, 1990), total costs are assumed to be linear in effort or,  $TC = cE$ , in which case average costs are constant, or  $AC(E) = c$ . If we assume that average costs are constant and independent of the spatial location, we obtain from (10):

$$x(z) = \frac{c}{pq} \text{ the same for all } z. \quad (11)$$

This is a typical spatially homogeneous bionomic equilibrium. Spatial heterogeneity in open access equilibrium can emerge in this case only if average cost is site dependent (SANCHIRICO and WILEN, 1999). Then  $x(z) = c(z)/pq$ .<sup>5</sup>

A richer solution regarding the emergence of spatial patterns occurs if we allow for increasing average cost. Increasing average costs are compatible with a more general formulation of harvesting costs (SMITH, 1969).<sup>6</sup> Assuming therefore linear average costs, or  $AC(E) = c_0 + (c_1/2)E$ , profit dissipation implies, using (10), that effort is determined as

$$\begin{aligned} \bar{E}(t, z) &= \frac{2(pqx(t, z) - c_0)}{c_1} > 0 \text{ if } pqx - c_0 > 0 \\ \bar{E}(t, z) &= 0 \text{ otherwise.} \end{aligned} \quad (12)$$

- 5 Spatial heterogeneity could also emerge if other economic or technological parameters such as  $p$  or  $q$  are site dependent.
- 6 Smith (1969) uses a general cost function that allows for stock externalities, congestion externalities and mesh externalities, and where costs increase with harvesting effort, given the fish stock, the number of vessels and the mesh size. In our simplified model we assume that these effects, although not explicitly modelled, result in a non linear cost function that gives rise to increasing average costs.

Thus with harvesting, logistic growth  $F(x) = x(s - rx)$ , and open access equilibrium at all sites, and effort determined by (12) biomass evolves according to the following Fisher equation:

$$\frac{\partial x}{\partial t} = s'x(1 - ax) + D_x \nabla^2 x(z, t) \quad (13)$$

$$s' = \left( s + \frac{2qc_0}{c_1} \right), \quad r' = \left( r + \frac{2q^2 p}{c_1} \right), \quad a = \frac{r'}{s'}.$$

Following MURRAY (1993), rescaling (13) by writing

$$t^* = s't, \quad z^* = z \left( \frac{s'}{D_x} \right)^{1/2},$$

and omitting asterisks, we obtain the Fisher-Kolmogorov equation:

$$\frac{\partial x}{\partial t} = x(1 - ax) + \frac{\partial^2 x}{\partial z^2}$$

with spatially homogeneous states 0 and  $1/a$ , obtained by setting  $\partial^2 x / \partial z^2 \equiv 0$  which are unstable and stable respectively. In this case the positive bionomic equilibrium biomass is defined as  $K = 1/a$ . As shown by MURRAY (1993), (13) has a travelling wave solution which can be written as

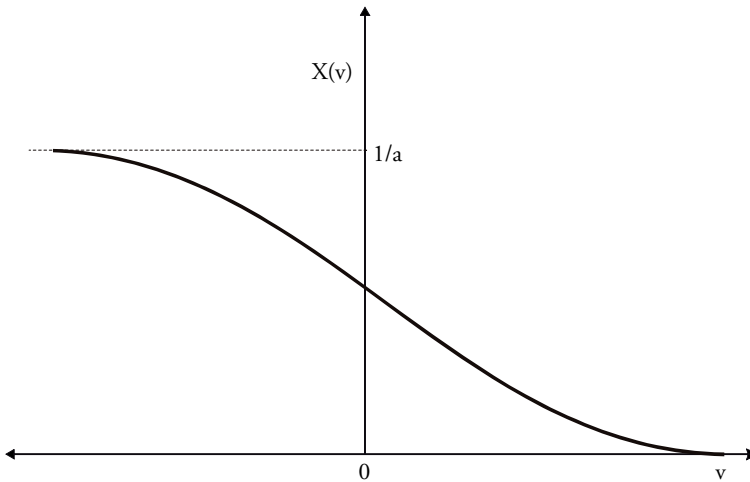
$$x(z, t) = X(v), \quad v = z - \gamma t$$

where  $\gamma$  is the speed of the wave. For a travelling wave to exist, the speed  $\gamma$  must exceed the *minimum* wave speed, which under Kolmogorov initial conditions is determined for the dimensional equation (13) by

$$\gamma \geq \gamma_{\min} = 2(s'D_x)^{1/2} = 2 \left[ \left( s + \frac{2qc_0}{c_1} \right) D_x \right]^{1/2}. \quad (14)$$

The wave front solution is depicted in figure 1.

Figure 1: Wave Front Solution



These results can be summarized in the following proposition.

*Proposition 1. When fish biomass disperses in space according to (8), then open access harvesting under increasing average costs, with harvesting effort diffusing fast and resulting in zero profit spatial equilibrium, induces convergence to a travelling wave solution for the biomass  $x(z,t) = X(v)$ ,  $v = z - \gamma t$ , with corresponding effort*

$$\bar{E}(v) = \frac{2(pqX(v) - c_0)}{c_1}.$$

From (14) it can be seen that the wave speed depends on both ecological and economic parameters. In particular it is increasing in  $s$ , the catchability coefficient  $q$ , and the initial marginal effort cost  $c_0$ , but declining in the slope of marginal effort cost  $c_1$ .

Our model can be used to analyze the impact of regulation. Assume that regulation involves linear spatially homogeneous taxes on effort (e.g. number or size of boats) or harvesting. Under an effort tax, zero profit condition and open access effort become

$$pqxE - \tau E - C(E) = 0 \text{ or } [pqx - \tau - AC(E)]E = 0 \forall z, \text{ and}$$

$$\bar{E}(t, z; \tau) = \frac{2(pqx(t, z) - c_0 - \tau)}{c_1}$$

respectively. Under a linear spatially homogeneous harvesting tax they become

$$pqxE - \tau qEx - C(E) = 0 \text{ or } [(p - \tau)qx - AC(E)]E = 0 \forall z, \text{ and}$$

$$\bar{E}(t, z; \tau) = \frac{2[(p - \tau)qx(t, z) - c_0]}{c_1}$$

respectively. Given the above equations, the effects of regulation are obtained in the following proposition.

*Proposition 2. Under the assumptions ensuring the existence of a travelling wave solution for the fish biomass, a spatially homogeneous linear tax on effort will increase the wave speed  $\gamma$  and the bionomic equilibrium biomass  $K$ , while a spatially homogeneous linear tax on harvesting will increase the bionomic equilibrium biomass  $K$  but leave the wave speed  $\gamma$  unchanged.*

For Proof see Appendix.

#### 4. Biomass-Effort Reaction Diffusion and Pattern Formation

In the previous section we assumed that in an unbounded spatial domain effort diffuses fast to dissipate profits under open access across all sites. In this section we consider a bounded spatial domain  $Z = [0, \alpha]$  and we assume that effort does not diffuse infinitely fast in search of profits. This assumption allows us to study the interactions between biomass and effort diffusion and the generation of spatial patterns where biomass and effort exhibit different concentrations at different spatial points. These assumptions imply that the movement of biomass and effort in time and space can be described by the following reaction diffusion system:

$$\frac{\partial x}{\partial t} = x(s - rx) - qEx + D_x \nabla^2 x \tag{15}$$

$$\frac{\partial E}{\partial t} = \delta E(pqx - AC(E)) + D_E \nabla^2 E, \delta > 0 \quad (16)$$

$$x(z, 0), E(z, 0) \text{ given, } \nabla x = \nabla E = 0 \text{ for } z = \{0, \alpha\} \quad (17)$$

where the terms  $x(s - rx) - qEx$  and  $\delta E(pqx - AC(E))$  are the kinetics of the problem,  $AC(E)$  is the average cost curve, and by (17), it is assumed that there is no external biomass or effort input on the boundary of the spatial domain.<sup>7</sup>

In the above model the main assumption is that effort (that is boats or fishermen) is attracted by profits per boat and that this effort disperses in the spatial domain with a constant diffusion coefficient  $D_E$ . The sluggish movement of effort across space along the effort gradient can be regarded along different lines. Although boats could move fast in open access property regimes, movements could be restricted in communal property regimes (e.g., BERKES, 1996), where due to institutional arrangements, there is exclusion of boats from certain areas and general frictions in the movement of boats towards the biomass. Along the same lines, if congestion externalities are present and fishermen realize these externalities, then they will tend to move away from each other in order to reduce costs associated with congestion. This type of reasoning could be used to justify (16). Given therefore, the system of (15)–(17), we seek to examine conditions under which the Turing mechanism induces diffusive driven instability and creates a heterogeneous spatial pattern of resource biomass and harvesting effort.

#### 4.1. Biomass-Effort Spatial Patterns

In analyzing diffusion induced instability we start from a system which, in the absence of diffusion, exhibits stable spatially homogeneous steady states. The spatially homogeneous system of (15) and (16), with  $D_x = D_E = 0$ , is defined as:

$$\begin{aligned} \dot{x} &= x(s - rx) - qEx \\ \dot{E} &= \delta E(pqx - AC(E)), \delta > 0 \end{aligned}$$

7 This is a zero flux boundary condition which is imposed so that the organizing pattern between biomass and effort is emerging as a result of their interactions, is self-organizing and not driven by boundary conditions (MURRAY, 2003, Vol II, p. 82).

where a steady state  $(x^*, E^*) > 0$  for the spatial homogeneous is determined as the solution of  $\dot{x} = \dot{E} = 0$ . A homogeneous steady state is defined by the intersection of the isoclines

$$x|_{\dot{x}=0} = \psi_1(E) = \frac{s}{r} - \frac{q}{r}E \quad (18)$$

$$x|_{\dot{E}=0} = \psi_2(E) = \frac{AC(E)}{pq} \quad (19)$$

where (18) is linear with a negative slope, while the slope of (19) depends on the slope of the average cost curve.

The idea behind the Turing mechanism for diffusive instability is that a spatially homogeneous steady state that is stable in the absence of diffusion can be destabilized locally by perturbations induced by diffusion. The result of this instability could be the emergence of a regular stable patterned distribution of biomass and effort across the spatial domain. To follow this approach<sup>8</sup> consider the linearization around a spatially homogeneous steady state  $(x^*, E^*)$

$$\dot{\mathbf{w}} = J\mathbf{w}, \quad \mathbf{w} = \begin{pmatrix} x - x^* \\ E - E^* \end{pmatrix}$$

where the linearization matrix  $J$  around a steady state is defined as

$$J = \begin{bmatrix} -rx^* & -qx^* \\ \delta pqE^* & -\delta E^* AC'(E^*) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

At a stable spatially homogeneous steady state:

$$\begin{aligned} \text{tr}(J) &= (-rx^* - \delta E^* AC'(E^*)) < 0 \\ \text{Det}(J) &= \delta E^* x^* (rAC'(E^*) + pq^2) > 0 \end{aligned}$$

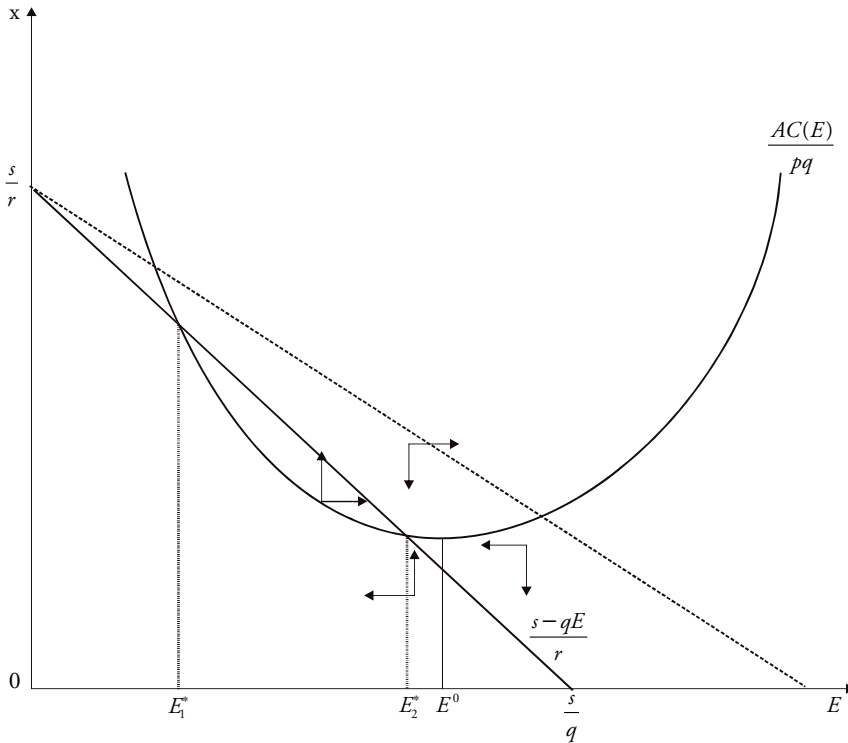
8 We follow MURRAY (2003, Vol II, Ch. 2.3).

Turing’s mechanism for diffusive instability implies that diffusion will destabilize the spatially homogeneous steady state. As it has been shown (OKUBO et al., 2001), for this destabilization to occur opposite signs between  $a_{11}$  and  $a_{22}$  and between  $a_{12}$  and  $a_{21}$  are required. To examine whether this is possible in our model consider a general U-shaped average cost curve corresponding to a cubic cost function. Let  $E^0 = \operatorname{argmin} AC(E)$  be the effort minimizing average cost. Assume that two steady states  $E_1^*$  and  $E_2^*$  exist. As shown in figure 2:

$$0 < E_1^* < E_2^* \text{ where } AC'(E_1^*) < 0, AC'(E_2^*) \leq 0.$$

Furthermore, as indicated by the flows of the phase diagram, the high effort steady state is stable while the low effort steady state is unstable.

Figure 2: Spatially Homogeneous Steady States



In general, if the stable steady state is at the increasing part of the average cost, then  $a_{11} a_{22} > 0$  while  $a_{12} a_{21} < 0$ . If the stable steady state is at the decreasing part of the average cost, then  $a_{11} a_{22} < 0$ ,  $a_{12} a_{21} < 0$ , and according to the previous statement, destabilization may occur. Therefore in order to examine the emergence of diffusion-driven instability or Turing instability, we consider the high effort steady state occurring at the declining part of the U-shaped average cost curve  $AC(E)$ . In this case the sign pattern for  $J$  is  $(a_{11}, a_{12}) < 0$ ,  $(a_{21}, a_{22}) > 0$ . Thus a preliminary remark can be stated:

*Remark: In the biomass-effort dispersal model described above, the possibility of spatial patterns emerging as a result of diffusion-driven instability or Turing instability requires a cost structure inducing a U-shaped average cost curve.*

To derive conditions for diffusion driven instability we linearize the full system (15)–(17) to obtain

$$\mathbf{w}_t = J\mathbf{w} + D\nabla^2 \mathbf{w}, \tag{20}$$

$$\mathbf{w}_t = \begin{pmatrix} \partial x / \partial t \\ \partial E / \partial t \end{pmatrix}, D = \begin{pmatrix} D_x & 0 \\ 0 & D_E \end{pmatrix}.$$

Following MURRAY (2003) we consider the time-independent solution of the spatial eigenvalue problem

$$\nabla^2 \mathbf{W} + k^2 \mathbf{W} = 0, \nabla \mathbf{W} = 0, \text{ for } z = \{0, a\} \tag{21}$$

where  $k$  is the eigenvalue. For the one-dimensional domain  $[0, a]$  we have solutions for (23) which are of the form

$$\mathbf{W}_k(z) = \mathbf{A}_n \cos\left(\frac{n\pi z}{a}\right), n = \pm 1, \pm 2, \dots, \tag{22}$$

where  $\mathbf{A}_n$  are arbitrary constants. Solution (24) satisfies the zero flux condition at  $z = 0$  and  $z = a$ . The eigenvalue is  $k = n\pi/a$  and  $1/k = a/n\pi$  is a measure of the

wave-like pattern. The eigenvalue  $k$  is called the *wavenumber* and  $1/k$  is proportional to the wavelength  $\omega : \omega = 2\pi/k = 2\alpha/n$ .

Let  $\mathbb{W}_k(z)$  be the eigenfunction corresponding to the wavenumber  $k$ . Then we look for solutions of (20) of the form

$$\mathbf{w}(z,t) = \sum_k c_k e^{\lambda t} \mathbb{W}_k(z). \quad (23)$$

Substituting (25) into (20), using (23) and canceling  $e^{\lambda t}$ , we obtain for each  $k$  or equivalently each  $n$ , the relationship  $\lambda \mathbb{W}_k = J \mathbb{W}_k - D k^2 \mathbb{W}_k$ . Since we require non trivial solutions for  $\mathbb{W}_k$ ,  $\lambda$  is determined by  $|\lambda I - J + D k^2| = 0$ . Then the eigenvalue  $\lambda(k)$ , as a function of the wavenumber, is obtained as the roots of

$$\lambda^2 + [(D_x + D_E)k^2 - (a_{11} + a_{22})]\lambda + h(k^2) = 0 \quad (24)$$

$$h(k^2) = D_x D_E k^4 - (D_x a_{22} + D_E a_{11})k^2 + \text{Det}(J). \quad (25)$$

Since the spatially homogeneous steady state  $(x^*, E^*)$  is stable, it holds that  $\text{Re}\lambda(k^2 = 0) < 0$ . For the steady state to be unstable in spatial disturbances, it is required that  $\text{Re}\lambda(k) > 0$  for some  $k \neq 0$ . But  $\text{Re}\lambda(k^2) > 0$  only if  $h(k^2) < 0$ . The minimum of  $h(k^2)$  occurs at  $k_c^2$  obtained after differentiating (25) as

$$k_m^2 = \frac{(D_x a_{22} + D_E a_{11})}{2D_x D_E},$$

which implies that for diffusive instability we need  $h(k_m^2) < 0$ . Then, the final condition for diffusive instability (or Turing instability) becomes (Okubo et al. 2001)<sup>9</sup>

9 The assumption of friction in the boat movements because of institutional reasons, or because of congestion costs, implies that  $\delta$  in (16) is sufficiently low to sustain the spatial pattern.

$$a_{11}D_E + a_{22}D_x > 2(a_{11}a_{22} - a_{12}a_{21})^{1/2}(D_E D_x)^{1/2} > 0. \tag{26}$$

Assuming that this condition is satisfied at the spatially homogeneous steady state, then the spatially heterogeneous solution is the sum of the unstable modes with  $Re\lambda(k^2) > 0$ , or

$$w(z,t) \sim \sum_{n_1}^{n_2} C_n \exp \left[ \lambda \left( \frac{n^2 \pi^2}{a^2} \right) t \right] \cos \frac{n\pi z}{a}, k^2 = \left( \frac{n\pi}{a} \right)^2 \tag{27}$$

where  $\lambda$  are the positive solutions of the quadratic (24),  $n_1$  is the smallest integer greater or equal to  $ak_1/\pi$ , and  $n_2$  is the largest integer less than or equal to  $ak_2/\pi$ . The wavenumbers  $k_1$  and  $k_2$  are such that  $k_1^2 < k_m^2 < k_2^2$ , with  $h(k_1^2) = h(k_2^2) = 0$  and  $h(k^2) < 0$  for  $k^2 \in (k_1^2, k_2^2)$ . That is,  $(k_1, k_2)$  is the range of unstable wavenumbers for which  $Re\lambda(k^2) > 0$ .

To obtain an idea of the solution described by (29), we follow MURRAY (2003) and assume that the range of unstable wave numbers  $(k_1^2, k_2^2)$  is such that there exists only one unstable wave number, which means that  $n = 1$ . Then the only unstable mode is  $\cos(\pi z/a)$  and

$$w(z,t) \sim C_1 \exp \left[ \lambda \left( \frac{\pi^2}{a^2} \right) t \right] \cos \frac{\pi z}{a}, k^2 = \left( \frac{\pi}{a} \right)^2.$$

The solution for the biomass and effort, assuming small positive  $C_1 = (\varepsilon_x, \varepsilon_E)'$ , takes the form

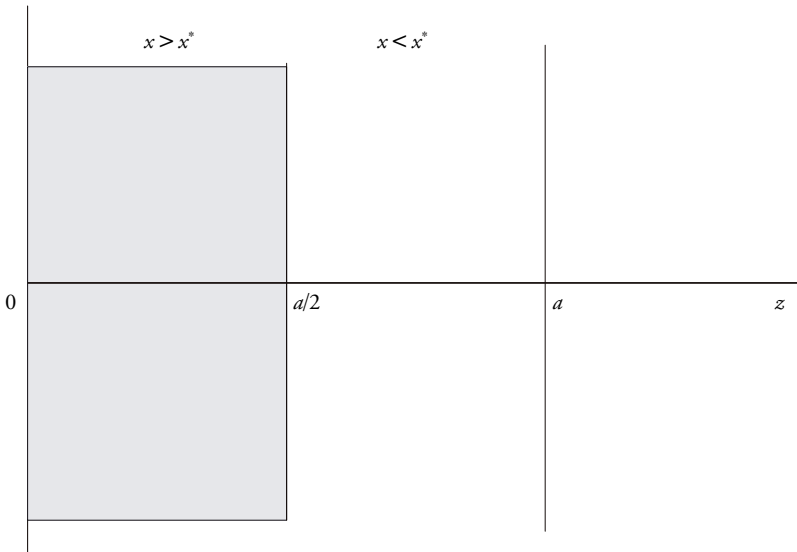
$$x(z,t) \sim x^* + \varepsilon_x \exp \left[ \lambda \left( \frac{\pi^2}{a^2} \right) t \right] \cos \frac{\pi z}{a} \tag{28}$$

$$E(z,t) \sim E^* + \varepsilon_E \exp \left[ \lambda \left( \frac{\pi^2}{a^2} \right) t \right] \cos \frac{\pi z}{a}. \tag{29}$$

Since  $\lambda(\pi^2/a^2) > 0$ , as  $t$  increases the deviation from the spatial homogeneous solution does not die out and could eventually be transformed into a spatial pattern which is like a single cosine mode. If the domain is sufficiently large to include a larger number of unstable wavenumbers, then the spatial pattern is more

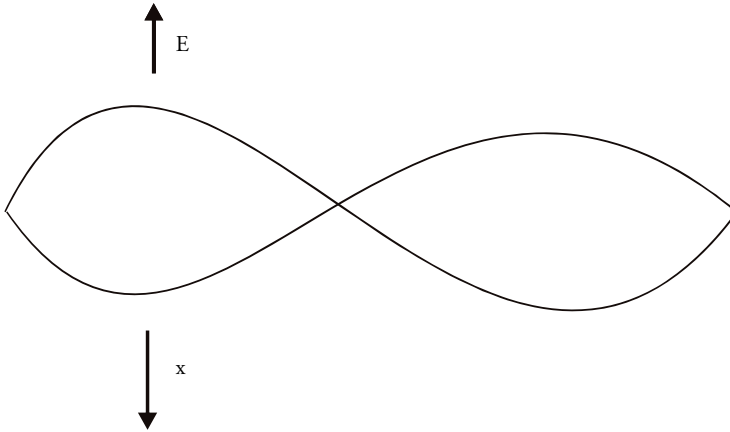
complex. With exponentially growing solutions for all time for (30) and (31), it might seem that  $x \rightarrow \infty$  and  $E \rightarrow \infty$  as  $t \rightarrow \infty$  would be implied. However this is not possible given the compactness of the state space. Furthermore, if the linear unstable eigenfunctions are bounded by the nonlinear terms then a spatially heterogeneous steady state will emerge. The main assumption here is the existence of a bounding domain for the kinetics of (15) and (16) in the positive quadrant (MURRAY, 2003, Vol II p. 87). Thus the bounding set that constrains the kinetics will also contain the solutions (30) and (31) when diffusion is present. Then the growing solution approaches, as  $t \rightarrow \infty$ , a cosine like spatial pattern, which implies spatial heterogeneity of the steady state. Figure 3, draws on MURRAY (2003, Vol II, pp. 94–95) to represent one possible spatial pattern for  $x(z, t)$ . Shaded areas represent spatial biomass concentration above  $x^*$ , while non shaded areas represent spatial biomass concentration below  $x^*$ .

Figure 3: Spatial Heterogeneity



The interactions between effort and biomass are shown in figure 4. Assume that effort increases and reduces biomass below the steady state  $x^*$ . This would result in a flux of biomass from neighboring regions which would reduce the effort in these regions, causing fish biomass to increase and so on until a spatial pattern is reached.

Figure 4: Interactions between Effort and Biomass



A measure of spatial heterogeneity at the steady state is given by the heterogeneity function which is defined as

$$G = \int_0^a (x'^2 + E'^2) dz \geq 0 \tag{30}$$

The spatially heterogeneous steady state is defined by using (15) and (16) for  $(\delta x / \delta t) = (\delta E / \delta t) = 0$  and denoting  $\nabla^2 x = x''(z)$ ,  $\nabla^2 E = E''(z)$ , since at the steady state the solution does not depend on time. Integrating by parts (30) and using the zero flux condition (17), we obtain

$$G = -\int_0^a (xx'' + EE'') dz. \tag{31}$$

Using (15) and (16) at the steady state the heterogeneity function is further defined as:

$$G = \int_0^a \left[ \frac{x^2}{D_x} (s - rx - qE) + \frac{\delta E}{D_E} (pqx - AC(E)) \right] dz$$

If there is no spatial patterning,  $s - rx - qE = 0$  and  $pqx - AC(E) = 0$ , which are the spatially homogeneous solutions, and  $G = 0$ .

## 5. Spatial Heterogeneity and Regulation

As we showed in the previous section, the adaptive biomass-effort system is likely to create spatial heterogeneity under an appropriate institutional regime and a cost structure associated with a U-shaped average cost curve for a certain parameter constellation. This implies, for example, that in the case presented in figures 3 and 4, the biomass concentration, effort and profits will be different at different locations of our spatial domain. This can emerge in situations where, because of institutional allocation of the “rights to fish” which restricts boats from certain patches, or because of congestion effects causing boats to disperse, fish biomass and boat movements are compatible in speed for the Turing mechanism to create spatial patterns and potential spatial inequalities. The measure of inequality can be given, for example, by the heterogeneity function (30), then social justice would require regulation to support spatial homogeneity. The problem then is reduced to that of finding instruments that will prevent diffusive instability.

As indicated in the previous section, diffusive instability cannot occur if the sign pattern of the linearization matrix  $J$  does not show opposition of signs between  $a_{11}$  and  $a_{22}$  and between  $a_{12}$  and  $a_{21}$ . Thus given  $J$ , the target is to change the sign structure, through a regulatory instrument, in a way that will prevent diffusive instability. An instrument affecting harvesting behavior will affect profits and consequently the second row of  $J$ .

We consider feedback control instruments in the general form of a non linear tax on effort (e.g. boat size or boat numbers) or on harvesting  $\tau(x, E)$ , with the property that when the tax is applied, then either  $a_{21}$  or  $a_{22}$  will change sign so that diffusive instability is not supported.

*Proposition 3. A spatially homogeneous nonlinear tax on effort of the feedback form  $\tau(E)$  with  $\tau'(E) > 0$  and  $\tau'(E^+) + AC'(E^+) > 0$ , where  $E^+$  is the regulated spatially homogeneous steady state for effort, will prevent the emergence of spatial heterogeneity.*

For proof see Appendix.

The effect of the nonlinear tax on effort is to shift the average cost curve, or equivalently the  $x|_{\dot{E}=0}$  curve, so that the intersection with the  $x|_{\dot{x}=0}$  curve takes

place at the increasing part of the average cost curve as shown in figure 5, where the new curve is  $AC_{reg}$ .

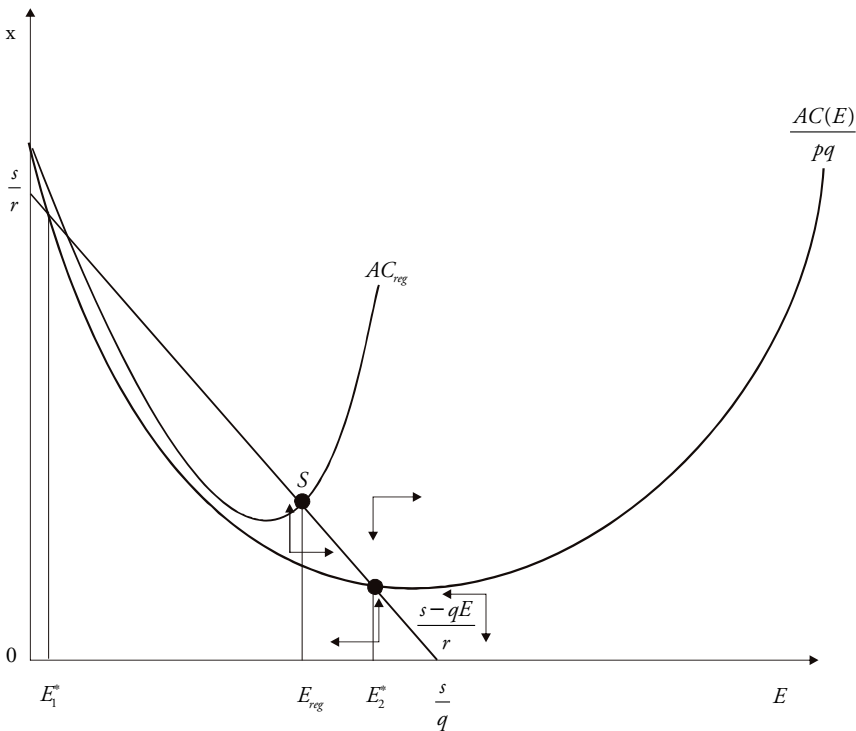
A feedback tax on harvesting can also be used as a regulatory instrument.

*Proposition 4. A spatially homogeneous nonlinear tax on harvesting, of the feedback form  $\tau(E,x)$  with  $p - \tau(E,x) > 0$ ,  $\partial\tau/\partial E > 0$ ,  $\partial\tau/\partial x < 0$ , and  $qx^+ + AC(E^+) > 0$ , where  $(E^+, x^+)$  is the regulated spatially homogeneous steady state for effort, will prevent the emergence of spatial heterogeneity.*

For proof see Appendix.

The effect of the nonlinear tax on effort is to shift the  $x|_{\dot{E}=0}$  curve so that the intersection with the  $x|_{\dot{x}=0}$  curve takes place at the increasing part of the average cost curve as shown in figure 5.

Figure 5: Regulation



It is interesting to note from these two propositions that a feedback tax on harvesting which depends on biomass alone, that is a tax  $\tau(x)$ , cannot exclude diffusive instability under the appropriate cost structure, because in this case the  $a_{21}$  element is positive, but the  $a_{21}$  element is now  $-\delta E^+ AC'(E^+)$ . Thus intersections at the decreasing part of the average cost curve cannot be excluded.

On the other hand, consider the introduction of a new technology, say because of subsidization, that increases the catchability coefficient  $q$ , and assume that with the old technology the  $x|_{\dot{x}=0}$  isocline was intersecting the  $x|_{\dot{E}=0}$  at the increasing part of the average cost curve, (e.g. figure 2), so that diffusive instability was not possible. The increase in  $q$  rotates the  $x|_{\dot{x}=0}$  isocline towards the origin so that the new steady state could take place at the decreasing part of the average cost curve. Then, as has been shown above, diffusive instability is possible. Thus, we have the following proposition.

*Proposition 5. In the model of biomass and effort diffusion described above, an increase in the catchability coefficient might generate spatial heterogeneity.*

## 6. Concluding Remarks

This paper adapts Turing analysis for diffusion-driven instability to bio-economic problems and explores conditions under which spatial patterns can emerge and persist at a steady state, along with a possible approach for regulating spatially heterogeneous systems.

We formulate a spatial harvesting model in continuous space-time of the CLARK (1990), SANCHIRICO and WILEN (1999, 2001) type, in order to illustrate how the interaction of the Turing mechanism with economic forces can produce travelling wave solutions and spatial heterogeneity in an analytically tractable descriptive framework. We show how this framework could be used to study the interaction of various tax and regulatory policies with the economic dynamics and the biomass dynamics over space-time, to produce or to moderate emergent spatial heterogeneity. We use this framework to expose the key role of the dispersion relation in the study of emergent spatial heterogeneity.

We believe that the analytical method developed in this paper could provide some useful insights in the analysis in time-space of some important bio-economic problems, such as fishery management and control of bioinvasions, in a way that is different from the metapopulation approach and has the advantage of reducing the potential number of state variables of metapopulation models. The analysis based on the Turing mechanism might also provide a solid basis

for the analysis of a variety of economics problems where spatial considerations are important.

### Appendix

Proof of Proposition 2: From the definition of  $s'$  and  $r'$  we have the following conditions under regulation:

$$\begin{aligned} \text{Effort tax : } s' &= \left( s + \frac{2q(c_0 + \tau)}{c_1} \right), \quad r' = \left( r + \frac{2q^2 p}{c_1} \right), \quad K = \frac{1}{a} = \frac{s'}{r'} \\ \text{Harvesting tax : } s' &= \left( s + \frac{2qc_0}{c_1} \right), \quad r' = \left( r + \frac{2q^2(p - \tau)}{c_1} \right), \quad K = \frac{1}{a} = \frac{s'}{r'}. \end{aligned}$$

Thus under the effort tax  $s'$ ,  $\gamma_{min}$  in (14) and  $K$  increase, while under the harvesting tax, only  $r'$  declines and thus  $K$  increases. ■

Proof of Proposition 3: Under the tax, the evolution of effort for the spatially homogeneous system is described by

$$\dot{E} = \delta E(pqx - \tau(E) - AC(E)).$$

The regulated homogeneous steady state is defined by the intersection of the isocines

$$\begin{aligned} x|_{\dot{x}=0} &= \frac{s}{r} - \frac{q}{r} E \\ x|_{\dot{E}=0} &= \frac{\tau(E) + AC(E)}{pq}. \end{aligned}$$

Then the linearization matrix at the spatially homogeneous steady state becomes

$$J^+ = \begin{bmatrix} -rx^+ & -qx^+ \\ \delta pqE^+ & -\delta E^*[\tau'(E^+) + AC'(E^+)] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : \begin{bmatrix} - & - \\ + & - \end{bmatrix}.$$

It is clear that

$$\text{tr}(J^+) < 0 \text{ and } \text{Det}(J^+) > 0,$$

so the regulated steady state is stable, and because of the sign order at the steady state no diffusive instability is possible. ■

Proof of Proposition 4: Under the tax, the evolution of effort for the spatially homogeneous system is described by

$$\dot{E} = \delta E(pqx - \tau(x, E)qx - AC(E)).$$

The regulated homogeneous steady state is defined by the intersection of the isocines

$$\begin{aligned} x|_{\dot{x}=0} &= \frac{s}{r} - \frac{q}{r} E \\ x|_{\dot{E}=0} &= \frac{\tau(E) + AC(E)}{(p - \tau(x, E))q}. \end{aligned}$$

Then the linearization matrix at a spatially homogeneous steady state becomes

$$J^+ = \begin{bmatrix} -rx^+ & -qx^+ \\ \delta \left( p - \tau - \frac{\partial \tau}{\partial x} x^+ \right) qE^+ & -\delta E^* \left[ \frac{\partial \tau}{\partial E} qx^+ + AC'(E^+) \right] \end{bmatrix}, \text{ with } \begin{bmatrix} - & - \\ + & - \end{bmatrix}.$$

It is clear that

$$\text{tr}(J^+) < 0 \text{ and } \text{Det}(J^+) > 0,$$

so the regulated steady state is stable, and because of the sign order at the steady state, no diffusive instability is possible. ■

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### SUMMARY

This paper adapts Turing analysis and applies it to dynamic bioeconomic problems where the interaction of coupled economic and ecological dynamics over space endogenously creates (or destroys) spatial heterogeneity. We examine the emergence of traveling wave solutions in open access fishery models, as well as the emergence of persistent spatial patterns in biomass/effort space caused by Turing instability. We also examine regulatory approaches for restoring spatial homogeneity.

### ZUSAMMENFASSUNG

Dieser Beitrag wendet die Turing-Analyse auf dynamische bioökonomische Probleme an, in welchen die Interaktion von ökonomischer und ökologischer Raumdynamik endogen räumliche Heterogenität erzeugt (oder zerstört). Wir untersuchen die Entwicklung von Lösungen für sich bewegende Wellen in Fischerei-Modellen mit offenem Eintritt, sowie die Entwicklung von anhaltenden räumlichen Mustern im Biomasse/Anstrengungs-Raum, welche durch Turing-Instabilität hervorgerufen werden. Wir betrachten zudem regulatorische Ansätze zur Wiederherstellung räumlicher Homogenität.

### RÉSUMÉ

Ce papier adapte l'analyse de Turing pour les problèmes dynamiques bio-économiques, où l'interaction de dynamiques économiques et écologiques dans l'espace crée (ou détruit) l'hétérogénéité spatiale de manière endogène. Nous examinons l'émergence de solutions d'ondes en mouvement dans des modèles de pêche à libre accès, ainsi que l'émergence de dessins spatiaux persistants dans l'espace biomasse/effort causés par l'instabilité de Turing. Nous regardons aussi les approches réglementaires pour rétablir l'homogénéité spatiale.