

Diffusion-Induced Instability and Pattern Formation in Infinite Horizon Recursive Optimal Control*

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Abstract

This paper develops local stability analysis for deterministic optimal control theory for recursive infinite horizon intertemporal optimization problems where there is a continuum of spatial sites and the state variable can diffuse over these sites. We identify sufficient conditions for a type of local instability which emerges from the interaction of the discount rate on the future, the curvature of the Hamiltonian, and the spatial features of the problem. We call this phenomenon *optimal diffusion-induced instability* (ODI). We illustrate our analytical methods with three stylized applications. The first application is the optimal management of spatially connected human dominated ecosystems. The second and third applications are harvesting of spatially interconnected renewable resources.

JEL Classification C6, Q2

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1 Introduction

This paper develops deterministic optimal control theory for recursive intertemporal optimization problems where there is a continuum of spatial sites and the state variable can diffuse over these sites. We develop a set of sufficient conditions on the Hamiltonian for the optimally controlled system to be locally asymptotically stable. We also develop a set of sufficient conditions on the Hamiltonian for the spatial diffusion to cause local instability of optimal steady states. This contrasts with intuition because we would expect diffusion to contribute to stability, not to instability. We show that if the discount rate on the future is small enough relative to the Hamiltonian curvature (the product of the absolute values of the second derivatives of the Hamiltonian with respect to the state variable and to the co-state variable), the usual value loss turnpike global asymptotic stability result is restored. This is so because if the discount rate on the future is zero, the optimally controlled system minimizes long-run value loss relative to the optimal steady state. The optimal steady state solves a strictly concave optimization problem over a convex set when the discount rate is zero. Intuition suggests that the same type of result should hold if the discount rate on the future is close enough to zero, provided that there is strict Hamiltonian curvature present. This intuition can be made rigorous, as we show below. Optimal diffusion-induced instability is more difficult to grasp intuitively. It turns out that diffusion-induced instability in optimal control can only occur when the discount rate is larger than Hamiltonian curvature. If the discount rate exceeds a critical value determined by the Hamiltonian curvature, the relative marginal benefits associated with the optimal control of a state variable in space-time change between a spatially homogeneous and a spatially heterogeneous solution, as the strength of diffusion across sites increases and the size of the space increases. In this case it may *not* be optimal to have a spatially homogeneous system and *optimal diffusion-induced instability (ODI)* emerges as a result of optimizing behavior.

We believe that the new contribution of this paper lies in the development and application of the theory of spatial optimal diffusion-induced stability and instability to infinite horizon recursive optimal control problems of the type that lie at the very foundation of dynamic economic theory. The literature on spatial diffusion-induced stability and instability in dynamical systems is huge, but the literature on optimally controlled spatially connected systems is much smaller. The literature on stability analysis in recursive optimal control problems with infinite dimensional state spaces is even smaller yet (e.g. Carlson et al., 1991). We believe the analysis of optimal diffusion-induced instability developed here is new and identifies a new version of

diffusion-induced instability which is different from Turing's (1952) classical case

In order to develop this theory we first present the Hamiltonian formalism for our spatial context, and then we present our stability analysis. We apply our analysis to two stylized problems of dynamic economics: (i) an optimal ecosystem management model where the ecosystems are spatially connected; and (ii) two renewable resource harvesting models where the renewable resource itself diffuses across space. The final section of the paper presents conclusions and areas for further research..

2 Optimal Control in Space-Time: The Pontryagin's Principle

In this section we state necessary and sufficient conditions for the optimal control of systems in space-time. We need this basic material in order to define precisely the notion of Hamiltonian that will be used to develop the stability results. We will use this material to study the emergence of spatial heterogeneity and pattern formation through diffusion-driven instability in an infinite time horizon optimal control problem.

We start by considering an optimal control problem defined in the spatial domain $z \in \mathcal{Z} = [z_0, z_1]$ and the time domain $t \in [0, \infty)$. Let $x(t, z), u(t, z)$ be the scalar state and control variables respectively at time t and spatial point z .¹ Let $f(x(t, z), u(t, z))$ be a net benefit function satisfying standard concavity assumptions and consider the following infinite horizon optimal control problem:

$$\max_{\{u(t,z)\}} \int_{z_0}^{z_1} \int_0^{\infty} e^{-\rho t} f(x(t, z), u(t, z)) dt dz \quad (1)$$

$$\text{s.t. } \frac{\partial x(t, z)}{\partial t} = g(x(t, z), u(t, z)) + D \frac{\partial^2 x(t, z)}{\partial z^2}, x(t_0, z) \text{ given} \quad (2)$$

with spatial boundary conditions

$$(x(t, z_0) = x(t, z_1) = \bar{x}(t), \forall t, \text{ the space is a circle} \quad (3)$$

In the above problem the transition equation (2) states that the rate of change of the state variable, which could be, for example, the concentration of an environmental stock (*e.g.*, greenhouse gasses, phosphorus in a lake) or the

¹As we discuss later, we consider state and control variables that have Fourier series expansions with respectively, piecewise continuously differentiable, and piecewise continuous, coefficients in t .

concentration of a biological resource, at a given spatial point is determined by a general growth function $g(x(t, z), u(t, z))$, which reflects the kinetics of the state variable, and by dispersion reflected by $D \frac{\partial^2 x(t, z)}{\partial z^2}$. Here the coefficient, $D > 0$, is the diffusion coefficient. We use the basic assumptions regarding diffusion, which are those of the classical approach, *i.e.*, the flux of the stock is assumed to be proportional to the gradient of the size of the stock. We assume that the movement of the stock is from high concentration towards low concentration. The boundary condition (3), which we labelled *circle*, implies that the spatial domain is a circle of given length L . In this case we have $[z_0, z_1] = [0, L]$.²

Problem (1)-(3) is an optimal control problem in a fixed spatial domain and an infinite time horizon. The circle boundary conditions (3) can be associated with a type of a “fixed endpoint problem in the spatial domain” for the state variable, since the terminal value of the state variable is fixed at terminal space. These terminal conditions in the space domain will be used to specify the appropriate transversality conditions for the problem.

Problem (1)-(3) has been analyzed in more general forms (*e.g.* Lions, 1971).³ For our problem we use the necessary and sufficient conditions obtained by Derzko et al. (1984) for the optimal control of quasilinear differential systems, appropriately modified to take into account infinite time horizon and discounting, and the corresponding transversality conditions.⁴ Pontryagin’s principle is in the spirit of the optimal control formalism used by economists, and thus can be used for other applications, as well as the applications given here. We hope this treatment will encourage economists to use this tool. Furthermore, as we shall see for recursive contexts treated here,

²When $\bar{x}(t) = 0, \forall t$, we have the so-called *hostile* boundary conditions. In a biological resource management problem, this condition indicates that the exterior of the spatial domain is completely hostile to the resource so that individuals that cross the boundary die. (*e.g.* Murray, 2003, vol. II, p. 1200, or Neubert, 2003). If $\frac{\partial x(t, z_0)}{\partial z} = \frac{\partial x(t, z_1)}{\partial z} = 0, \forall t$, we have the *zero flux* boundary condition which implies that there is no external input on the boundary of the spatial domain (Murray, 2003, vol. II, p. 82). It is well-known that hostile boundary conditions can cause heterogeneity in steady states across space, simply because heterogeneity is “forced” by the boundary conditions at the end points of the spatial domain. Our analysis is carried out under circle boundary conditions that allow homogeneity in steady states because we want to study how optimization itself can generate heterogeneity in spatial steady states.

³Similar conditions at various levels of generality have been derived or used in applied problems. See, for example, Derzko et al. (1984), Lenhart and Bhat (1992), Raymond and Zidani (1998, 1999), Lenhart et al. (1999), Carlson et al. (1991), Bhat et al. (1999), Bhat, Fisher and Lenhart (1999), Brito (2004).

⁴For a heuristic derivation of necessary and sufficient conditions for problem (1)-(3) using a variational approach based on Kamien and Schwartz (1991), see Brock and Xepapadeas (2006).

the use of Pontryagin's principle in continuous time and in space allows for a drastic reduction in the dimensionality of the dynamic system describing the phenomenon and makes the problem tractable.

We follow Derzko et al. (1984, pp. 95-96) in the precise statement of the optimal control problem. The set of admissible controls is specified by requiring that $u(\cdot, \cdot)$ be piecewise continuous in $\Omega = [0, \infty) \times [z_0, z_1]$, and that regularity conditions be assumed so that (3) is well posed for each piecewise continuous control in Ω .⁵

Maximum Principle under Diffusion: Necessary Conditions (MPD-NC) *Let $u^* = u^*(t, z)$ be a choice of instrument that solves problem (1)-(3) and let $x^* = x^*(t, z)$ be the associated path for the state variable. Then there exists a function $q(t, z)$ such that for each t and z :*

1. $u^* = u^*(t, z)$ maximizes the generalized current value Hamiltonian function

$$\begin{aligned} \tilde{H}(x(t, z), u, q(t, z)) = \\ f(x, u) + q(t, z) \left[g(x(t, z), u(t, z)) + D \frac{\partial^2 x(t, z)}{\partial z^2} \right] \end{aligned} \quad (4)$$

or for interior solutions:

$$\frac{\partial f}{\partial u} + q(t, z) \frac{\partial g}{\partial u} = 0 \quad (5)$$

- 2.

$$\begin{aligned} \frac{\partial q(t, z)}{\partial t} = \rho q(t, z) - \frac{\partial H}{\partial x} - D \frac{\partial^2 q(t, z)}{\partial z^2} = \\ \rho q(t, z) - \left(\frac{\partial f}{\partial x} + q(t, z) \frac{\partial g}{\partial x} + D \frac{\partial^2 q(t, z)}{\partial z^2} \right) \end{aligned} \quad (6)$$

$$\frac{\partial x(t, z)}{\partial t} = g(x(t, z), u^*(t, z)) + D \frac{\partial^2 x(t, z)}{\partial z^2} \quad (7)$$

evaluated at $u^* = u^*(x(t, z), q(t, z))$, $H = f(x, u) + qg(x, u)$

3. The following limiting intertemporal transversality condition holds

$$\lim_{T \rightarrow \infty} e^{-\rho T} \int_{z_0}^{z_1} q(T, z) x(T, z) dz = 0 \text{ for all } z \quad (8)$$

⁵In our application the diffusion operator in (3), combined with dx/dt , is the linear differential operator used by Derzko et al. (1984). Our setting is simpler because we are working on a circle while Derzko et al. worked on more complicated domains.

4. *The following spatial transversality condition holds for all dates t :*⁶

$$q(t, z_0) = q(t, z_1) \quad (9)$$

Conditions (5)-(7) along with the transversality conditions can be used to characterize the optimized dynamic system in continuous time space. It is interesting to note that (6)-(7) is a Modified Hamiltonian Dynamic System (MHDS) defined in continuous space-time. In this system the diffusion coefficient for the costate variable is negative, and it is the opposite of the state variable's diffusion coefficient. Since the costate variable can be interpreted as the shadow value of the state variable, negative diffusion implies that the movement in space is from low shadow values to higher shadow values. This makes economic sense because the costate variable is a price and the state variable diffuses from high concentration to low concentration. Therefore one would expect the price of the state variable to move in the opposite direction. Furthermore, the opposite signs of the diffusion coefficient for the state and the costate variable imply that time "runs backward" in the state variable and "runs forward" in the costate variable. The costate variable is equal to the capitalized value of the stream of future value flows (appropriately interpreted). That is, it is a forward capitalization type variable in capital theoretic terms.⁷

Sufficiency can be proved under the key assumption that the maximized Hamiltonian $\tilde{H}^0(x, q) = \max_u \tilde{H}(x, u, t)$ is concave for any given q (Derzko et al., 1984).⁸

3 Pattern Formation in Optimally Controlled Systems

Conditions (5)-(9) characterize the optimal paths for the state and control variables and the associated path for the costate variable in space-time. An important question in this context is whether these optimal paths are char-

⁶The corresponding transversality conditions for 'ero-fux and hostile boundary are $\frac{\partial q(t, z_1)}{\partial z} = \frac{\partial q(t, z_0)}{\partial z} = 0$ and $q(t, z_0) = q(t, z_1) = 0$, respectively.

⁷It should also be noted that because the diffusion coefficient D is a constant independent of the value of the state variable across sites, the optimal control $u^*(t, z)$ depends only on the values of the state and costate variables of its "own site". All dependencies of u^* on other sites is summarized in $q(t, z)$.

⁸This condition is along the lines of the Arrow and Kurz sufficiency theorem (Arrow and Kurz, 1970, chapter II.6) for optimal control, with the transversality condition at infinity satisfied. For a similar proof see also Brock and Xepapadeas (2006).

acterized by spatial heterogeneity. Spatial heterogeneity occurs when the values of the state, costate and control variables are different at different spatial points at the same time. Spatial homogeneity occurs when the values of these variables do not change across spatial points. We wish to study the forces that lead to spatial heterogeneity of optimal control, which persists with the passage of time, because this could lead to formation of stable spatial patterns.

We approach this problem by examining how diffusion, regarded as a perturbation, affects the steady state of an optimal control problem without spatial considerations. This is the special case of problem (1)-(2) with $D = 0$. From optimal control theory, with $D = 0$, we know that under appropriate concavity assumptions, if a steady state defined as (x^*, q^*) such that $\dot{x} = 0$, $\dot{q} = 0$ exists, then this steady state will have the local saddle point property or it will be unstable (*e.g.* Kurz, 1968; Levhari and Liviatan, 1972). Saddle point stability is a concept of conditional stability and implies that for the general n -dimensional problem there exists an n -dimensional *locally* stable manifold such that if the state-costate dynamical system starts on this manifold in the neighborhood of the steady state, it remains on it for all time and converges to the saddle point steady state. Brock and Scheinkman (1976), Cass and Shell (1976), and Rockafellar (1976) extended this local stability concept to global asymptotic stability (GAS) by introducing a *curvature assumption*. This curvature assumption is a sufficient condition which is stated in terms of the Hamiltonian of the control problem. The term, GAS, as used here means that the solution manifold of the optimal control problem is globally stable, in the sense that for any initial conditions on the solution manifold, the state-costate trajectories remain on the solution manifold and converge to the saddle point steady state.

By definition, when $D = 0$ the steady state with the saddle point property is spatially homogeneous or a Flat Optimal Steady State (FOSS). If diffusion destabilizes the stable manifold of this flat optimal steady state, then the result might be the emergence of a regular stable patterned distribution for the state, costate and control variables across the spatial domain of the optimal control problem. Pattern formation in biological applications has been studied extensively using the Turing mechanism (Turing, 1952) as discussed in Murray (2003), or Okubo and Levin (2001). This mechanism suggests that an asymptotically stable, in the absence of diffusion, spatially homogeneous steady state, can be destabilized locally by perturbations induced by diffusion. The Turing mechanism for diffusion induced instability does not however include optimal control considerations. We believe that in this paper we identify, as we explain in the discussion that follows, a new mechanism of diffusion-induced instability of optimal control, the ODI mechanism, which

may be used to study pattern formation in optimal control problems.

3.1 Linear-Quadratic Approximations and the Separation Principle

In order to provide precise results for the mechanism through which diffusion may result in destabilization of the stable manifold of a FOSS with the local saddle point property, we use a linear quadratic (LQ) approximation of original problem (1)-(2). If (x^*, q^*) is a FOSS for the problem (1)-(2), then under certain regularity assumptions problem (1)-(2) can be approximated by problem:

$$\max_{u(t,z)} \int_0^\infty \int_0^L e^{-\rho t} \left[-\frac{A}{2} x(t,z)^2 - \frac{B}{2} u(t,z)^2 + Nx(t,z)ux(t,z) \right] dzdt \quad (10)$$

$$A, B, \rho > 0, \quad AB - N^2 > 0 \quad (11)$$

$$\text{s.t. } \frac{\partial x(t,z)}{\partial t} = Sx(t,z) - Gu(t,z) + D \frac{\partial^2 x(t,z)}{\partial z^2} \quad S, G > 0 \quad (12)$$

$$x(0, z) = x_0(z) \text{ given, } z \text{ in a circle } z \in [0, L] \quad (13)$$

$$x(t, 0) = x(t, L) \text{ for all } t \quad (14)$$

where, by a slight abuse of notation, $x(t, z)$, $u(t, z)$ are now deviations from the FOSS values (x^*, u^*) , and the costate variable $q(t, z)$ associated with the Hamiltonian of the problem (10)-(14) is measured in deviations from the FOSS value, q^* . See appendix for the details of the LQ approximation.

For the control problem (10)-(14) we restrict ourselves to a set of controls U which is the set of all controls that have Fourier series expansions with piecewise continuous coefficients in t , or⁹

$$U = \left\{ u(t, z) : u(t, z) = \sum_n \left[u_{1n}(t) \cos\left(\frac{2\pi n z}{L}\right) + u_{2n}(t) \sin\left(\frac{2\pi n z}{L}\right) \right], n = 0, 1, 2, \dots \right\} \quad (15)$$

In this case the solution of (12) for any $u(t, z) \in U$ will have a Fourier series expansion with piecewise continuously differentiable coefficients in t , or

$$x(t, z) = \sum_n \left[x_{1n}(t) \cos\left(\frac{2\pi n z}{L}\right) + x_{2n}(t) \sin\left(\frac{2\pi n z}{L}\right) \right], n = 0, 1, 2, \dots \quad (16)$$

⁹As shown in Priestley (1981, p. 189), if a function $f(z, t)$ is of a bounded variation on a circle, then its Fourier series converges.

Substituting the $x(t, z)$ and $u(t, z)$ in (12) we obtain the following set of transition equations parametrized by $n = 0, 1, 2, \dots$

$$\dot{x}_0(t) = Fx_0(t) - Gu_0(t), n = 0 \quad (17)$$

$$\dot{x}_{1n}(t) = S(k)x_{1n}(t) - Gu_{1n}(t), n = 1, 2, \dots \quad (18)$$

$$\dot{x}_{2n}(t) = S(k)x_{2n}(t) - Gu_{2n}(t), n = 1, 2, \dots \quad (19)$$

$$S(k) = F - D \left(\frac{2\pi n}{L} \right)^2 = F - Dk^2, k = \frac{2\pi n}{L} \quad (20)$$

In (17) $n = 0$ corresponds to the transition equation for a spatially independent problem, while (18) corresponds to the cosine part of the expansion and (19) corresponds to the sine part of the expansion. The set of functions $\left\{ \cos\left(\frac{2\pi n z}{L}\right), \sin\left(\frac{2\pi n z}{L}\right) \right\}, n = 0, 1, 2, \dots$ is a complete orthogonal set over $[0, L]$ (Priestley, 1981, p. 187). Thus, the inner product of these functions is:

$$\int_0^L \varphi_n(z) \varphi_m(z) dz = \|\varphi_n(z)\|^2 \delta_{n,m} = \frac{L}{2} \delta_{n,m}, n, m = 0, 1, 2, \dots$$

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\varphi_n(z) = \cos\left(\frac{2\pi n z}{L}\right) \text{ or } \sin\left(\frac{2\pi n z}{L}\right)$$

From the Fourier series representation of the state and control variables, and the orthogonality of the $\varphi_n(z)$ functions, Parseval's formula (Priestley 1981, p. 195) implies:

$$\int_0^L x(t, z)^2 dz = \frac{L}{2} \sum_n [x_{1n}(t)]^2 + \frac{L}{2} \sum_n [x_{2n}(t)]^2 \quad (21)$$

$$\int_0^L u(t, z)^2 dz = \frac{L}{2} \sum_n [u_{1n}(t)]^2 + \frac{L}{2} \sum_n [u_{2n}(t)]^2 \quad (22)$$

$$n = 0, 1, 2, \dots \quad (23)$$

Furthermore the orthogonality of the $\cos(\cdot)$ and $\sin(\cdot)$ functions and the Fourier series expansion of the state and control variables implies that

$$\int_0^L x(t, z) u(t, z) dz = \frac{L}{2} \sum_n x_{1n}(t) u_{1n}(t) + \frac{L}{2} \sum_n x_{2n}(t) u_{2n}(t)$$

Substituting (21) and (22) into (10) we obtain, omitting t to simplify

notation, the following countable set of independent optimal control problems for each $n = 0, 1, 2, \dots$

$$\max_{u_0(t)} \int_0^\infty e^{-\rho t} \left[-\frac{A}{2}x_0^2 - \frac{B}{2}u_0^2 + Nx_0u_0 \right] dt \quad (24)$$

$$\text{subject to } \dot{x}_0 = Fx_0 - Gu_0, \quad n = 0 \quad (25)$$

$$\max_{u_{1n}(t)} \frac{L}{2} \int_0^\infty e^{-\rho t} \sum_n \left[-\left(\frac{Q}{2}x_{1n}^2 + \frac{R}{2}u_{1n}^2 \right) + Nx_{1n}u_{1n} \right] dt \quad (26)$$

$$\text{subject to } \dot{x}_{1n} = S(k)x_{1n} - Gu_{1n}, \quad n = 1, 2, \dots \quad (27)$$

$$\max_{u_{2n}(t)} \frac{L}{2} \int_0^\infty e^{-\rho t} \sum_n \left[-\left(\frac{Q}{2}x_{2n}^2 + \frac{R}{2}u_{2n}^2 \right) + Nx_{2n}u_{2n} \right] dt \quad (28)$$

$$\text{subject to } \dot{x}_{2n} = S(k)x_{2n} - Gu_{2n}, \quad n = 1, 2, \dots \quad (29)$$

$$S(k) = F - D \left(\frac{2\pi n}{L} \right)^2 = F - Dk^2, \quad k = \frac{2\pi n}{L} \quad (30)$$

where (24)-(25) corresponds to the flat problem, (26)-(27) corresponds to the cosine part of the expansion, and (28)-(29) corresponds to the sine part of the expansion. Thus, the use of the LQ approximation and the use of controls with Fourier series expansions implies the above separation principles. The separation transforms the problem of the optimal control of a system governed by a partial differential equation to a countable set of independent optimal control problems which are governed by an ordinary differential equation.¹⁰

3.2 Optimal Diffusion-Induced Instability

The maximized current value Hamiltonian or pre-Hamiltonian for the flat LQ system, where we drop subscripts and superscripts to simplify notation, is

$$H^0(x, q) = \max_u \left\{ -\frac{A}{2}x^2 - \frac{B}{2}u^2 + Nxu + q[Fx - Gu] \right\} \quad (31)$$

¹⁰This transformation is based on the *proper orthogonal decomposition*. See, for example, Padhi and Balakrishnan (2002).

The Jacobian of the MHDS at the flat optimal steady state (x^*, q^*) is defined as:¹¹

$$J^0(x^*, q^*) = \begin{bmatrix} H_{xq}^0(x^*, q^*) & H_{qq}^0(x^*, q^*) \\ -H_{xx}^0(x^*, q^*) & \rho - H_{qx}^0(x^*, q^*) \end{bmatrix} = \begin{bmatrix} F - \frac{GN}{B} & \frac{G^2}{B} \\ A - \frac{N^2}{B} & \rho - F + \frac{GN}{B} \end{bmatrix} \quad (32)$$

The eigenvalues of $J^0(x^*, q^*)$, for $\rho > 0$, are either positive or they have opposite signs.¹² Assume that the FOSS (x^*, q^*) has the saddle point property. In order to examine whether this state can be locally destabilized by diffusion, we need to find sufficient conditions for the negative eigenvalue of the linearization of the state-costate equations to become positive under perturbations induced by diffusion. The result of this instability could be the emergence of a regular stable patterned distribution of the state and the costate variables across the spatial domain. The theorem below gives one set of sufficient conditions for diffusion-induced instability of optimal control.

Theorem 1 *Assume that in the linear quadratic approximation (10)-(14) of problem (1)-(2) with $D = 0$, the optimal flat steady state (x^*, q^*) associated with the Jacobian matrix $J^0(x^*, q^*)$ has the local saddle point property. Then if*

$$2H_{xq}^0(x^*, q^*) > \rho \quad (33)$$

$$\frac{\rho^2}{4} > -H_{xx}^0(x^*, q^*) H_{qq}^0(x^*, q^*) \quad (34)$$

there is a $D > 0$, such that the negative eigenvalue of the linearization

$$\mathbf{w}_t = J^0 \mathbf{w} + \tilde{D} \mathbf{w}_{zz}, \quad \tilde{D} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \quad (35)$$

$\mathbf{w} = (x(t, z) - x^*, q(t, z) - q^*)$ becomes positive. That is, both eigenvalues of the Jacobian matrix in (35) have positive real parts. Thus diffusion locally destabilizes the optimal flat steady state. To put it another way, if problem (1)-(2) has a quadratic payoff and linear dynamics and conditions (33) and (34) are satisfied, then the optimal dynamics are unstable.

For proof see appendix.

¹¹Subscripts associated with functions denote partial derivatives.

¹²In our simple scalar case the eigenvalues are always real. In vector cases there are more possibilities (Kurz, 1968).

As shown in the appendix to the above theorem, the eigenvalues associated with the spatial optimal control problem are given by

$$\lambda_1(k^2) = \frac{1}{2} \left(\rho + \sqrt{\rho^2 - 4h(k^2)} \right) \quad (36)$$

$$\lambda_2(k^2) = \frac{1}{2} \left(\rho - \sqrt{\rho^2 - 4h(k^2)} \right) \quad (37)$$

$$h(k^2) = -D^2k^4 + D [2H_{xq}^0(x^*, q^*) - \rho] k^2 + \det J^0(x^*, q^*) \quad (38)$$

For a spatial domain which is a circle of length $[0, L]$, $k = 2n\pi/L$, $n = \pm 1, \pm 2, \dots$. The quantity $1/k = L/2n\pi$ is a measure of the wave-like pattern. The quantity k is called the *wavenumber*. Thus, $1/k$ is proportional to the wavelength ω ; $\omega = 2\pi/k = L/n$. If the conditions of the theorem are satisfied, then $\lambda_2 > 0$ and the conditional stability, in the saddle point sense, is lost due to diffusion. It follows from (37) that destabilization of the negative eigenvalue by diffusion is equivalent to $h(k^2) > 0$. Following Murray (2003), we shall call $h(k^2)$, the *dispersion relationship*. If $h(k_{\max}^2) > 0$, then there exist two positive roots $k_1^2 < k_2^2$ such that $h(k^2) > 0$ and $\lambda_2(k^2) > 0$ for $k^2 \in (k_1^2, k_2^2)$. As shown in the appendix, $h(k_{\max}^2) = \rho^2/4 + H_{xx}^0 H_{qq}^0$, with $k_{\max}^2 = (2H_{xq}^0 - \rho)/2D$ and

$$k_{1,2}^2 = \frac{(2H_{xq}^0 - \rho) \pm \sqrt{(\rho^2 + 4(H_{xx}^0 H_{qq}^0))}}{2D} \quad (39)$$

The interval (k_1, k_2) determines the range of the unstable modes associated with the spatially heterogeneous solution. Thus diffusion driven instability in the optimally controlled system emerges if the maximum of the dispersion relationship $h(k^2)$, is in the positive quadrant. We call the set \mathcal{U} of parameters, for which the conditions of theorem 1 are satisfied, the set of *optimal diffusion-induced instability*.

It can easily be seen from (38) that for $D = 0$, $h(k^2) = \det J^0(x^*, \lambda^*) < 0$ by the saddle point property of the flat steady state, and no diffusion-induced instability is possible. However, for $D > 0$ and provided that the conditions of theorem 1 are satisfied, we have $h'(0) = D(2H_{xq}^0 - \rho) > 0$. Thus the concave dispersion relationship is increasing at $h(0) = \det J^0 < 0$, and the two positive roots (k_1^2, k_2^2) emerge. It can be seen from (39) that the smaller D is, the larger these roots are. A dispersion relationship is shown in figure 4 for the application analyzed in section 5.1. The horizontal line corresponds to the flat case $D = 0$, but for $D > 0$, $h(k_{\max}^2) > 0$, since the conditions of theorem 1 are satisfied. It is also clear that for $\rho = 0$, the ODI space is empty and diffusion-driven instability does not emerge. However, for higher discount

rates and for appropriate values of parameters in the Jacobian matrix (32), the ODI space need not be empty, and we show that it is not empty for the economic examples discussed later on.

The same condition for ODI can be obtained by using the transformed problem obtained by the separation principle. The Jacobian matrix corresponding to the state-costate system of problem (26)-(27) or (28)-(29) is

$$J^0(k) = \begin{bmatrix} S(k) & G^2/B \\ A - \frac{N^2}{B} & \rho - S(k) \end{bmatrix}, S(k) = F - Dk^2, k = \frac{2\pi n}{L}$$

with characteristic roots (36), (37). The dispersion relationship can be written as

$$h(k^2) = S(k) \left[\hat{A} - S(k) \right] - \frac{G^2}{B}, \hat{A} = A - \frac{N^2}{B} \quad (40)$$

Thus, around the FOSS, both the original LQ problem and its distributed parameter transformation imply the same dispersion relationship and the same condition for optimal diffusion induced instability.

3.3 State-Costate Paths

To get a clearer picture of the state-costate paths when the flat steady state is linearly destabilized by diffusion, we start again by analyzing the solution of the linearized MHDS for the flat case.

We know that for the optimally controlled system with $D = 0$ the optimal first approximation (linear) solution is derived from the linearized MHDS in the neighborhood of the flat steady state as:

$$\begin{pmatrix} \tilde{x}(t, z) \\ \tilde{q}(t, z) \end{pmatrix} = C_2 \mathbf{v}_2 e^{\lambda_2 t}, \text{ for all } z \quad (41)$$

$$\tilde{x}(t, z) = x(t, z) - x^*, \tilde{q}(t, z) = q(t, z) - q^* \quad (42)$$

where C_2 is a constant determined by initial conditions on x and transversality conditions, and \mathbf{v}_2 is the eigenvector corresponding to λ_2 . For the linearized system the transversality condition at infinity, (8), forces the constant C_1 associated with positive root λ_1 to be zero.

The corresponding optimal first approximation to a spatially heterogeneous solution, for a linear approximation in the neighborhood of the FOSS, with the spatial domain being a circle $[0, L]$, can be written as the sum of unstable modes. As time t increases, the dominant contribution will come from those modes where $\lambda_2(k^2) > 0$, since modes with $\lambda_2(k^2) < 0$ will fade

away in influence as t increases. The relevant expression, which satisfies the circle boundary conditions, is given by

$$\begin{pmatrix} \tilde{x}(t, z) \\ \tilde{q}(t, z) \end{pmatrix} \sim \sum_{n_1}^{n_2} \mathbf{B}_n \exp [\lambda_2(k^2) t] [\cos(kz) + \sin(kz)], \quad k = \frac{2n\pi}{L} \quad (43)$$

where the x -component of the vector \mathbf{B}_n , call it a_n , is determined by initial conditions on x at date $t = 0$ ¹³ and the q -component of the vector \mathbf{B}_n , call it $P_n a_n$, is given by the requirement that the vector $\mathbf{B}_n := (a_n, P_n a_n)$ lie on the one-dimensional eigenspace spanned by the eigenvector with the smallest real part of the two eigenvalues corresponding to mode n . Furthermore, $\lambda_2(k^2) > 0$ for $k^2 \in (k_1^2, k_2^2)$, n_1 is the smallest integer greater or equal to $Lk_1/2\pi$ and n_2 is the largest integer less than or equal to $Lk_2/2\pi$, and the wavenumbers k_1 and k_2 are such that $h(k^2) > 0$ for $k^2 \in (k_1^2, k_2^2)$. Since $\lambda_2(k^2) > 0$ for $k^2 \in (k_1^2, k_2^2)$, only these modes grow with time; all the remaining modes for which $\lambda_2(k^2) < 0$ tend to zero exponentially. It is important to note the significance of the size L of the spatial domain in the emergence of diffusion-induced instability. For example the smallest wavenumber corresponds to $n = 1$. This implies that $n_1 = n_2 = 1$ and the length L should satisfy $2\pi/k_2 \leq L \leq 2\pi/k_1$. In general the size of the spatial domain should be such that allowable wavenumbers exist, and pattern formation emerges. This is made clearer in the application part, section 5.1. Assume that the spatial domain is such that there is only one unstable wavenumber, or $n = 1$. Then the growing instability is determined by¹⁴

$$\begin{pmatrix} \tilde{x}(t, z) \\ \tilde{q}(t, z) \end{pmatrix} \sim \mathbf{B}_1 \exp [\lambda_2(k^2) t] [\cos(kz) + \sin(kz)] \quad (44)$$

where the vector of constants \mathbf{B}_1 is determined by initial conditions on x and by the appropriate eigenspace as above. Since the instability occurs on the stable manifold of the linearized MHDS, it would be natural to choose initial conditions on this manifold and close to the FOSS. Take a small $\mathbf{B}_1 = (\epsilon_x, \epsilon_q)$, then using the definition of $(\tilde{x}(t, z), \tilde{q}(t, z))$ from (42), the optimal spatially

¹³Since our initial conditions on x are of the form $x(0, z) = x_0(z)$, we restrict ourselves to the set of initial condition functions $x_0(z)$ which have Fourier series expansion with piecewise continuously differentiable coefficients in t .

¹⁴Initial and terminal conditions on the circle are satisfied at $z = 0$ and $z = L$.

heterogeneous first approximation solution evolves approximately as:

$$x(t, z) \sim \epsilon_x \exp \left[\lambda_2 \left(\frac{4\pi^2}{L^2} \right) t \right] \left[\cos \left(\frac{2\pi z}{L} \right) + \sin \left(\frac{2\pi z}{L} \right) \right] + x^*, \quad \frac{4\pi^2}{L^2} = k^2 \quad (45)$$

$$q(t, z) \sim \epsilon_p \exp \left[\lambda_2 \left(\frac{4\pi^2}{L^2} \right) t \right] \left[\cos \left(\frac{2\pi z}{L} \right) + \sin \left(\frac{2\pi z}{L} \right) \right] + q^* \quad (46)$$

Since $\lambda_2 \left(\frac{4\pi^2}{L^2} \right) > 0$, the solution which is characterized by (45)-(46) does not decay exponentially with the passage of time, in order to converge to the FOSS as in the no diffusion ($D = 0$) case, but rather moves away from the flat state in a wave-like spatial pattern. This of course is spatial heterogeneity, since the state variable and its shadow value - the costate variable - will have different values in different spatial points at any given point in time. In this case the path for the optimal control $u^*(t, z)$ in the neighborhood of the flat steady state will be determined by (5). The first approximation solution (45) is depicted in figure 5 for the application of section 5.1.

In this case, with a non-empty ODI space, spatially heterogeneous solutions for the first approximation similar to (45)-(46) grow exponentially. This, however, cannot be valid for all t , since then exponential growth would imply that $(x, q) \rightarrow \infty$ at $t \rightarrow \infty$. However, the kinetics of the Hamiltonian system (6)-(7) and the transversality condition at infinity (8) should bound the solution. This suggests that the growing solution of the MHDS might settle to a spatial pattern as $t \rightarrow \infty$.¹⁵ This implies that in the given spatial domain, the state, the costate, and the control variables will be different from the flat optimal steady-state levels. In this case an ultimate spatially Heterogeneous Optimal Steady-State solution (HOSS) for the optimally controlled system will emerge. This HOSS will satisfy

$$0 = \rho q(z) - H_x^0(x(z), q(z)) - D \frac{\partial^2 q}{\partial z^2} \quad (47)$$

$$0 = H_q^0(x(z), q(z)) + D \frac{\partial^2 x}{\partial z^2} \quad (48)$$

with the appropriate spatial boundary conditions. Setting $v = \frac{\partial x}{\partial z}$, $u = \frac{\partial q}{\partial z}$,

¹⁵See Murray (2003, vol II, chapter 2.4, pp. 93-94) for similar arguments.

we obtain the first-order system

$$\frac{\partial v}{\partial z} = -\frac{1}{D} H_q^0(x, q), \quad \frac{\partial x}{\partial z} = v \quad (49)$$

$$\frac{\partial u}{\partial z} = \frac{1}{D} (\rho q - H_x^0(x, q)), \quad \frac{\partial q}{\partial z} = u \quad (50)$$

The system can be solved with the appropriate spatial boundary conditions in order to determine the HOSS. More precise solutions, showing that a pattern will actually emerge at the steady state, are presented in the application part. However a further clarification might be possible here with the help of the *heterogeneity function* (Berding, 1987; Murray, 2003), which is defined as:

$$\mathcal{H} = \int_Z [(x_z(z))^2 + (q_z(z))^2] dz \geq 0$$

Integrating by parts, using circle boundary and transversality conditions, and substituting from (47)-(48), we obtain¹⁶

$$\begin{aligned} \mathcal{H} &= [xx_z + qq_z]_0^L - \int_0^L (xx_{zz} + qq_{zz}) dz \\ &= \frac{1}{D} \int_0^L \{ [x(z) H_q^0(x(z), q(z))] - [q(z) (\rho q(z) - H_x^0(x(z), q(z)))] \} dz \end{aligned}$$

It is clear that at a FOSS, $\mathcal{H} = 0$, since $H_q^0(x, q) = 0$, $\rho q - H_x^0(x, q) = 0$. The HOSS will exist if $\mathcal{H} > 0$.¹⁷

3.4 Diffusion as a Stabilizing Force

The main focus of this paper is the analysis of diffusion-induced instability as a means of spatial patterning. Diffusion, however, may act as a stabilizing force for an unstable system. To show this we examine now the case where diffusion can act as a stabilizing force when the FOSS is unstable, which im-

¹⁶ $[xx_z + qq_z]_0^L = 0$ with circle boundary conditions.

¹⁷Murray (2003, vol. II, chapter 2.9) uses \mathcal{H} as a Lyapunov function to show that large diffusion prevents spatial patterning in reaction diffusion mechanisms. This entails showing that for large diffusion, $d\mathcal{H}/dt$ could be negative, which implies that $\mathcal{H} \rightarrow 0$ as $t \rightarrow \infty$, which in turn implies spatial homogeneity and points to the potentially stabilizing forces of strong diffusion. We conjecture that a similar result might hold in some of the settings we examine in our applications. We suspect that such a result can be formulated and proved in the descriptive setting presented in section 5.2.2, where there is a fixed harvesting rule in a Solow type model of harvesting a diffusing resource, but this is outside the scope of the present paper.

plies that there exist positive eigenvalues $\lambda_{1,2} > 0$ for $k^2 = 0$ in (36). Positive eigenvalues $\lambda_{1,2}$ means that, since $\text{tr}J^0 = \rho > 0$, $\det J^0 > 0$. Diffusion can stabilize the system in the sense of producing a negative eigenvalue. For the smallest eigenvalue to turn negative, or $\lambda_2 < 0$, it is sufficient, from (36), that $h(k^2) < 0$. For the quadratic (38), we have that $h(0) = \det J^0 > 0$, which is the instability condition for the FOSS, and furthermore $h(k^2)$ is concave, and therefore has a maximum. Therefore there exists a root $k_2^2 > 0$, such that for $k^2 > k_2^2$, we have $(h(k^2), \lambda_2) < 0$. The solutions for $x(t, z)$ and $q(t, z)$ will be determined by the sum of exponentials of λ_1 and λ_2 . Since we want to stabilize the system, we set the constant associated with the positive root λ_1 equal to zero. Then the solution will depend on the sum of unstable and stable modes associated with λ_2 .

Following the previous procedure, the solutions for x and q will be of the form:

$$\begin{pmatrix} \tilde{x}(t, z) \\ \tilde{q}(t, z) \end{pmatrix} \sim \sum_0^{n_2} \mathbf{C}_{\hat{n}} \exp \left[\lambda_2 \left(\frac{4n^2\pi^2}{L^2} \right) t \right] [\cos(kz) + \sin(kz)] + \sum_{n_2}^N \mathbf{C}_n \exp \left[\lambda_2 \left(\frac{4n^2\pi^2}{L^2} \right) t \right] [\cos(kz) + \sin(kz)] \quad (51)$$

where n_2 is the smallest integer greater or equal to $Lk_2/2\pi$ and $N > n_2$. Since $\lambda_2 \left(\frac{4n^2\pi^2}{L^2} \right) < 0$ for $n > n_2$, all the modes of the second term of (51) decay exponentially. Thus to converge to the steady state, we need to set $\mathbf{C}_{\hat{n}} = 0$, and have the coordinates of the vector \mathbf{C}_n lie on the one-dimensional eigenspace spanned by the the eigenvector with the negative real part of the two eigenvalues corresponding to mode n . Then the spatial patterns corresponding to the second term of (51) will die out with the passage of time and the system will converge to the spatially homogeneous steady state (x^*, q^*) .

4 Diffusion Induced Instability of Optimal Control and Economic Interpretations

The results of the previous section suggest that diffusion can destabilize a FOSS and eventually drive the MHDS to a new patterned steady state. In the general model used above, we have seen that diffusion driven instability is indicated by conditions on the Hamiltonian function of the problem and its derivatives. In this section we present some economic intuition behind these conditions.

Theorem 1 presents two sufficient conditions for diffusion driven instability. Satisfaction of the second condition (34) implies the violation of the *curvature condition* derived by Brock and Scheinkman (1976) for the GAS of infinite horizon optimal control in the time domain only. This condition is related to the curvature matrix Q :

$$Q(x^*(t), q^*(t)) = \begin{pmatrix} -H_{xx}^0(x^*(t), q^*(t)) & \rho/2 \\ \rho/2 & H_{qq}^0(x^*(t), q^*(t)) \end{pmatrix}$$

where $(x^*(t), q^*(t))$ are solutions of the MHDS. In the scalar case if $H^0(x, q)$ is concave in x and convex in q , then $(-H_{xx}^0, H_{qq}^0) > 0$. The curvature condition implies that Q is positive definite, or

$$\frac{\rho^2}{4} < -H_{xx}^0(x^*, q^*) H_{qq}^0(x^*, q^*) \quad (52)$$

When the curvature condition is satisfied, it implies that for bounded solutions $(x^*(t), q^*(t))$ of the MHDS with $D = 0$, $(x^*(t), q^*(t)) \rightarrow (x^*, q^*)$ as $t \rightarrow \infty$, where (x^*, q^*) is the FOSS. Thus the FOSS is GAS. If we assume that $H^0(x, q)$ is α -concave $-\beta$ -convex, then the FOSS is GAS (Rockafellar, 1976) provided that $\rho^2 < 4\alpha\beta$. It is clear that (34) also violates the condition α -concavity $-\beta$ -convexity.

We can obtain a number of insights from purely formal manipulations and using the intuition behind the role of the matrix Q in analogy to Brock and Scheinkman's results in n -dimensional control systems. For example, if the Brock and Scheinkman (1976) curvature condition is satisfied, that is,

$$-(\dot{x}, \dot{q}) Q \begin{pmatrix} \dot{x} \\ \dot{q} \end{pmatrix} \leq 0 \quad (53)$$

then $\rho^2/4 < -H_{xx}^0 H_{qq}^0$, or equivalently the matrix Q is positive definite. In this case the local diffusive instability condition (34) is not satisfied. We might usefully conjecture that the FOSS is GAS by an adaptation of Brock and Scheinkman's n -dimensional arguments to our infinite dimensional setting here. That is, for a bounded solution, we might conjecture that $(x^*(t, z), q^*(t, z)) \rightarrow (x^*, q^*)$ as $t \rightarrow \infty$. Furthermore, as has been shown by Feinstein and Oren (1983) in the finite dimensional case, if Q is positive definite, then the FOSS has the saddle point property. So, in terms of local analysis, satisfaction of (53) suggests that the FOSS cannot be destabilized locally by diffusion-induced perturbations. It is beyond the scope of this paper to extend the results of Brock-Scheinkman (1976), Cass-Shell (1976), Rockafellar (1976) to the infinite horizon problems being treated here. We

simply content ourselves with a purely local analysis of the first-order conditions that is parallel to Murray's (2003) treatment.¹⁸

Hence, in a space time framework we conjecture that spatial heterogeneity might emerge with the violation of (53) along with the satisfaction of (33). We examine the economics behind this conjecture using the following approach.

1. Apply the envelope theorem to the maximized Hamiltonian (31) to obtain

$$H_q^0 = g(x, u(x, q)), \quad H_{qq}^0 = -g_u \frac{\partial u(x, q)}{\partial q} \quad (54)$$

It is a standard result of optimal control theory (Arrow and Kurz, 1970) that the optimal control $u^* = u(x, q)$ defined through (5) describes short-run equilibrium and can be interpreted most of the times as the short-run demand for the control u as a function of the state variable x and its shadow value, q (the costate variable), which are both treated as fixed in the short run. Thus $\partial u / \partial q$ can be interpreted as the slope of the short-run demand function along the optimal solution.

2. Take the total derivative of (6) at the FOSS to obtain

$$\rho dq + q^* d\rho - H_{xx}^0(x^*, q^*) dx - H_{xq}^0(x^*, q^*) dq = 0 \quad \text{or} \quad (55)$$

$$\frac{dx}{d\rho} = \frac{q^*}{H_{xx}^0} \quad (56)$$

In (56) $dx/d\rho$ is negative. This is the *capital deepening response* discussed by Burmeister and Turnovsky (1972). To put it another way, a FOSS is regular in their sense. In Burmeister and Turnovsky's "regular" case, the shadow value of capital stock falls as the interest rate increases. Here the discount rate, ρ , plays the role of the interest rate and the state variable x plays the role of capital stock. Condition (56) implies that, as the discount rate on the future increases, the steady state x falls. Similarly we may interpret $dx/d\rho$ as the slope of the steady-state demand curve for the state variable (stock) at the FOSS.

Combining (54) and (56) we obtain

$$-H_{xx}^0 H_{qq}^0 = \frac{\partial u / \partial q}{\partial x / \partial \rho} (g_u q^*) \quad \text{or} \quad -H_{xx}^0 H_{qq}^0 = \frac{\varepsilon_{uq} \rho u^*}{\varepsilon_{x\rho} x^*} \quad (57)$$

¹⁸Our results for the LQ problem suggest that instability of the LQ problem at some mode signals local instability of the original nonlinear problem. We do not provide a complete rigorous statement and proof of this conjecture here. Our specific numerical examples seem to support this conjecture.

where ε_{uq} and $\varepsilon_{x\rho}$ are short- and long-run elasticities of demand with respect to the state variable shadow value and the discount rate respectively, weighted by the interest charged on the control variable per unit of the state variable. If the numerical values of the elasticities are such that the elasticities condition

$$\frac{\rho^2}{4} > \frac{\varepsilon_{uq} \rho u^*}{\varepsilon_{x\rho} x^*} \quad (58)$$

is satisfied, then the FOSS is destabilized through diffusion-induced instability and a pattern emerges.

Another important factor in destabilizing the FOSS is the size of the cross partial H_{xq}^0 . From (36) and (38) it is clear that since destabilization, requires $h(k^2) > 0$, H_{xq}^0 should be sufficiently high. From (55) it follows that $dq/d\rho = q^*/H_{xq}^0$. Thus high H_{xq}^0 means that the sensitivity of the costate variable at the FOSS to the discount rate is small. Here is another way to view the size of the cross partial H_{qx} . Brock and Malliaris (1989, chapter 5) point out how concepts of Hamiltonian curvature (*e.g.* the Q -condition) do not capture the underlying dynamics when control is *not* applied, whereas the quantity H_{qx} is closely related to the underlying dynamics of the system when *no* control is applied. Hence, intuitively, one expects that the optimally controlled system is *more* likely to be stable if the underlying dynamics without control is stable. This effect is not captured by Hamiltonian curvature conditions like the Q -condition, but it is captured (at least in part) by the term H_{qx} . Given this background we can make two observations about conditions (33) and (34) of theorem 1. First, condition (33) is close to saying that the uncontrolled system must not be "too" stable or must even be "unstable" relative to the size of ρ . Putting it another way, it says that the system must be more "unstable" the larger is ρ . Second, equation (34) essentially asserts that ρ must be larger than Hamiltonian curvature. The conclusion is that one should look out for the possibility of diffusion-induced instability in systems that are slightly "unstable" when no control is applied and where Hamiltonian curvature is weak relative to ρ . We explain in more detail below.

Since all of the parameters that appear in the key dispersion relation, $h(k^2)$, are summarized in the matrix J^0 as well as by the diffusion parameter D and the size of the space L , we provide more discussion of comparative statics by using subsets of these parameters and stressing the role of D and of L . The first point to discuss is the forces that determine whether $h(k^2)$ is positive for some k^2 . That is whether there is an interval $I = (a, b)$ such that $h(k^2) > 0$ for $k^2 \in I$. The second point to discuss is the set of values of L that allows existence of a positive integer n such that $[(2n\pi/L)]^2$ is in I . And, finally, the third point to discuss is bifurcational loss of local

stability of the FOSS and where the HOSS is likely to appear. Following Fujita, Krugman, and Venables (1999), hereafter FKV, we shall call a HOSS an *agglomeration*. However, note that our agglomeration (if it exists) is an optimal agglomeration. FKV do not treat optimization as we do here. Hence there is no role for ρ , for example, to play in their discussion. In our treatment a zero value of ρ (or a small enough value of ρ) eliminates any agglomeration. As far as we know, discussion of this role of ρ (as well as theorem 1) in allowing (or preventing) agglomeration is new in our paper.

By graphing $h(k^2)$ against k^2 , we see the following: (i) $h(0) = \det J^0$, (ii) $h'(0) = 2H_{xq}^0 - \rho$, and (iii) the maximum value of $h(k_{\max}^2)$ is equal to $\rho^2/4 + H_{xx}^0 H_{qq}^0$ which is the negative of the determinant of the Brock and Scheinkman curvature matrix. Thus we see that there is an interval I where $h(k^2)$ is positive, provided that $\det J^0 > 0$, or $2H_{xq}^0 - \rho > 0$ and the anti Brock and Scheinkman condition holds. Of course we also need $D > 0$. If $D = 0$, $h(k^2)$ is the constant, $\det J^0$, for all values of k^2 . As D goes to zero from a positive value, the function $h(k^2, D)$ becomes horizontal at the value $\det J^0$. But notice that for each positive D , the value of $\max h(k^2, D)$ (which is the negative of the determinant of the Brock and Scheinkman curvature matrix) is the same independently of $D > 0$. As D decreases towards zero, the maximum k_{\max}^2 and the interval I shifts off towards plus infinity. But mere existence of an interval I such that $h(k^2)$ is positive for k^2 in I is not enough for existence of ODI. The size of the space L is crucial.

Suppose there is an interval $I(D)$ such that $h(k^2, D) > 0$ on $I(D)$ for a fixed positive value of D . The size L must be such that there is a positive integer n such that $[(2n\pi/L)]^2$ is in I . Notice that

$$\left(\frac{2\pi}{L}\right)^2 < \left(\frac{2\pi \cdot 2}{L}\right)^2 < \dots < \left(\frac{2n\pi}{L}\right)^2 < \left(\frac{2(n+1)\pi}{L}\right)^2 < \dots, \forall n = 1, 2, \dots$$

Thus we see that as L decreases, the entire set of values above shifts to the right. Therefore if L is small enough, there will be no positive integer k such that $[2\pi k/L]^2$ is in I for a fixed finite interval I . In conclusion we see that a larger space L and a smaller D make ODI more likely. Now let us think of L and D as very slowly changing variables relative to the dynamics of the optimization problem. As L slowly increases and D slowly decreases, a FOSS becomes locally unstable provided that $2H_{xq}^0 - \rho > 0$ and $\rho^2/4 + H_{xx}^0 H_{qq}^0 > 0$. That is, in our optimization system, much like FKV (1999), we have a bifurcation of a flat earth optimum to form an *optimal agglomeration*, i.e. a HOSS in our jargon. The same type of discussion of path dependence on this very slow time scale as in FKV (1999, chapter 17) applies here too. We believe there is a potentially very fruitful research

program in extending the theory in our paper to optimized versions of FKV systems, but that task is way beyond the scope of the current paper.

Finally it should be noticed that to create a pattern in the optimal control model, there is no need to assume increasing returns. We mention this because it is common to assume zones of increasing returns in the more recent literature on spatial economics and agglomeration. Zones of increasing returns will be assumed in some of the applications below.

A clearer economic interpretation of diffusion-induced instability in optimal control can be obtained by using explicitly the LQ approximation and the separation principle.

To simplify, normalize without loss of generality, $A = 1$ and assume $N = 0$ for the cross product term.¹⁹ Then the instability conditions of theorem 1 become:

$$2F > \rho \quad (59)$$

$$\rho^2 > \frac{4}{\theta}, \quad \theta = \frac{B}{G^2} \quad (60)$$

To explore the implication of these parameter restrictions, consider the distributed parameter transformation (24)-(29) of the LQ problem. Assuming linear feedback controls $u = Mx$, the state equation can be written, dropping subscripts to simplify notation, as $\dot{x} = S(k)x - Gu$, or $\dot{x} = C(k)x$, $C(k) = S(k) - GM$, with solution $x = x_0 \exp(C(k)t)$. Substituting this solution, and using $u = Mx$ in the objective functional we can compute the benefit integral as:

$$BI = -\frac{x_0^2 [1 + \theta [S(k) - C(k)]]}{2 [\rho - 2C(k)]} \quad (61)$$

The original control problem can be regarded now as choosing C for each $S = S(k)$ to maximize (61). Maximizing (61) with respect to C , disregarding the constant $(x_0^2/2)$, and choosing the smallest root to satisfy transversality conditions at infinity, we obtain the optimal C as a function of $S = S(k)$, as

$$C(S) = \frac{\rho}{2} - \frac{1}{2\theta} \sqrt{(\rho\theta)^2 + 4\theta + 4\theta^2 S(S - \rho)} \quad (62)$$

The following characteristics of (62) can be noted: $C(0) = C(\rho) < 0$, while $C(S)$ takes its maximum value for $S = \rho/2$, with $C(\rho/2) = \rho/2 - 1/\sqrt{\theta}$. However, if $C(\rho/2) > 0$, then the instability condition (60) is satisfied. The

¹⁹If $N \neq 0$ the problem can be reduced to the case $N = 0$ by the change in units $\hat{u} = u + (N/B)x$ (Brock and Malliaris, 1989).

$C(S)$ curve with $C(\rho/2) > 0$ is shown in figure 1.

Figure 1

Thus the emergence of ODI can be reduced, for the LQ approximation, to the following problem:

Let $S(k) = F - Dk^2$, $k = 2\pi n/L$. For sufficiently large F so that (59) is satisfied, $D > 0$ and n integer, find a $\theta = B/G^2$, such that there exists an interval \mathcal{I} such that $C(S(k)) > 0$ for $S(k) \in \mathcal{I}$. The interval \mathcal{I} is the set between the roots of $C(S) = 0$, or the set between S_1 and S_2 in figure 1. Since $C(0) = C(\rho) < 0$, the interval \mathcal{I} is in the positive real line.

Thus using the distributed parameter transformation of the LQ problem, we see that for fixed D , fixed circle length L , and fixed $F > 0$, ODI emerges if a positive n exists such that

$$S(D, L; n) = F - D \left(\frac{2\pi n}{L} \right)^2 \text{ is in } \mathcal{I}$$

Then, there exist D and L such that ODI emerges, and the set \mathcal{U} of parameters for which optimal diffusion-induced instability is possible is not empty. It can be noted, as was discussed earlier in this section, that it is easier to fit an unstable mode into a larger circle $[0, L_1]$ than a smaller one $[0, L_2]$, $L_1 < L_2$.²⁰

We can further analyze the economic intuition by setting $S(k) = \rho/2$ to simplify further, and writing the derivative of (61) with respect to C as

$$BI'(C) = BI'_1(C) + BI'_2(C) = -\frac{2}{(\rho - 2C)^2} + \frac{\theta}{2}$$

A reduction in C will reduce benefits by $\theta/2$ through the $BI'_2(C)$ term, but will increase benefits a lot, through the $BI'_1(C)$ term, if C is close to $\rho/2$. So the reduction is profitable. At $C = 0$ a further reduction in C will increase benefits by $2/\rho^2$ and reduce them by $\theta/2$. This reduction, which in a sense reduces the state disequilibrium, is not profitable if θ is large enough. Thus the optimal C will be positive if θ is large enough. The size of θ that corresponds to the optimal positive C is determined by the instability condition (60). Thus if θ is high enough relative to ρ , there is a zone of F where it is not optimal to push down C to negative values. To summarize, the LQ approximation suggests that ODI occurs, provided that

²⁰If there is more than one positive integers n such that $S(D, L, n) \in \mathcal{I}$, then the n^* such that $C(S(D, L; n^*))$ is the largest is likely to be where the HOSS is for the original nonlinear system that gave rise to this LQ problem at the FOSS.

the circle $[0, L]$ is large enough to accommodate the potentially unstable nodes, when the discount rate on the future is larger than the critical value $\rho_{cr} = 2/\sqrt{\theta}$, $\theta = B/G^2$. This ODI emerges as the relative marginal benefits that are associated with the optimal control of a state variable in space-time change between a spatially homogeneous and a spatially heterogeneous solution, as the strength of diffusion across sizes increases and the size of the space increases.

5 Applications

The previous sections developed a general theoretical framework of optimal control in time space under diffusion. We explored conditions under which optimal control in time and space could lead to pattern formation and spatial heterogeneity at a steady state. In this section we present some economic applications. These applications are very abstract and highly stylized. But we think they are useful in illustrating the potential usefulness of analysis of diffusion driven spatial stability or instability in dynamic economics.

5.1 Optimal Management and Spatial Pattern Formation in Shallow Lakes

Our first example is a stylized version of the *lake* problem (Carpenter et al., 1999; Brock and Starrett, 2003; Mäler et al., 2003), but where the dynamics of the lakes are connected by diffusion of pollutants, *e.g.*, inflow of unwanted nutrients for algae, across lakes. We will call this problem the *shallow lake problem* because damaging changes to alternative stable states have actually been documented by experiments with shallow lakes (cf. references to the limnological literature in Carpenter et al. (1999)). Here the problem is to maximize the discounted stream of services from an ecosystem, *e.g.*, a stylized lake, whose dynamics are nonlinear where there is a trade-off between “loaders” and those who enjoy the amenities of an unspoiled ecosystem. The loaders include industry, agriculture, green lawn owners, leaking septic tanks of cottage owners, etc., and the enjoyers of the amenities provided by the lake include fishermen, swimmers, sail boaters, birders, hikers, other types of viewers, etc. The mechanism of damage to the lake is nutrient inflow from loaders which causes algae to multiply which causes noxious blooms, fish kills, unpleasant smells, etc. Following Carpenter et al. (1999), we use a stylized one-dimensional state variable called *phosphorous sequestered in algae* to summarize the state of each lake. The dynamics of each lake allow the possibility of alternative stable states, one of which is called *oligotrophic*

and which is characterized by low phosphorous sequestered in algae and is “good”; the other is called *eutrophic* and is characterized by low phosphorous sequestered in algae, and is “bad”.

Carpenter and Brock (2003) study a system of spatially connected lakes where the collapse of one lake stimulates negative spillover effects into other nearby lakes. Their system is much more realistic, their mechanism of spillover is different from ours, and their model is much more complicated than the stylized system of spatially connected lakes that we study here. For example the only type of spillover we allow here is the spillover of unwanted nutrients from one lake into neighboring lakes. Furthermore we represent "space" as a ring of lakes which are connected like a chain of beads on that ring. Nevertheless we can obtain some understanding of how the interaction of the discount rate with the curvature of the Hamiltonian function can lead to diffusion-induced instability, *i.e.* a "domino" effect, using the methods developed in our paper, whereas Carpenter and Brock (2003) used "movies" and numerical methods to uncover the causes of potential domino effects in their system. We believe understanding is enhanced using both types of methods, that is: (i) Numerical analysis of more realistic, but analytically intractable spatially connected dynamical systems, and (ii) Strategically chosen severe abstractions of spatially connected dynamical systems which are still analytically tractable using methods like those developed in this paper.

The spatial optimal management shallow lake problem can be stated as:

$$\begin{aligned} & \max_{a(t,z)} \int_0^\infty \int_0^L e^{-\rho t} [B(a(t,z)) - C(x(t,z))] dz dt \\ & \text{s.t. } \frac{\partial x(z,t)}{\partial t} = a(t,z) - bx(t,z) + h(x(t,z)) + D \frac{\partial x^2(z,t)}{\partial z^2} \quad (63) \\ & x(t,0) = x(t,L) \end{aligned}$$

where $a(t, z)$ is phosphorous loading at time t and spatial point $z \in [0, L]$ of a one-dimensional spatial domain, and $x(t, z)$ is the corresponding stock of phosphorus measured as phosphorous sequestered in algae as in Carpenter et al. (1999), taking non-negative values in a compact set \mathcal{X} . Here $B(a(t, z))$ denotes concave benefits to the loaders and $D(x(t, z))$ denotes convex damages to those enjoying the lake's amenities. In (63), which describes the evolution of the phosphorous stock, b is the rate of loss per unit stock (*e.g.* sedimentation, outflow) and the function $h(x(t, z))$ is a positive feedback representing internal loading which is assumed to be S-shaped. The general-

ized current value Hamiltonian of this problem is

$$\tilde{H}(a, x, \lambda) = B(a) - C(x) + \lambda \left[a - bx + h(x) + D \frac{\partial x^2(z, t)}{\partial z^2} \right]$$

Using the necessary conditions implied by the Maximum Principle under diffusion we obtain

$$B'(a) = -\lambda \Rightarrow a = \bar{a}(\lambda), \quad \frac{da}{d\lambda} = -\frac{1}{B''} > 0$$

$$\frac{\partial x(z, t)}{\partial t} = H_\lambda^0(x, \lambda) = \bar{a}(\lambda) - bx + h(x) + D \frac{\partial x^2(z, t)}{\partial z^2} \quad (64)$$

$$\frac{\partial \lambda(z, t)}{\partial t} = \left[\rho + b - h'(x) \right] \lambda + C'(x) - D \frac{\partial \lambda^2(z, t)}{\partial z^2} \quad (65)$$

The FOSS (x^*, λ^*) is determined by the solution of the system

$$0 = \bar{a}(\lambda) - bx + h(x) = H_\lambda^0(x, \lambda)$$

$$0 = \left[b + \rho - h'(x) \right] \lambda + C'(x) = \rho \lambda - H_x^0(x, \lambda)$$

$$H^0(x, \lambda) = \max_a H(a, x, \lambda)$$

So the FOSS is characterized by:

$$\bar{a}(\lambda) = bx - h(x)$$

$$\lambda|_{\dot{\lambda}=0} = \frac{-C'(x)}{[b + \rho - h'(x)]}$$

$$\bar{a} \left(\frac{-C'(x^*)}{[b + \rho - h'(x^*)]} \right) \Big|_{\dot{x}=0} = bx^* - h(x^*) \Rightarrow x^* = X(\rho, b)$$

$$\lambda^* = \frac{-C'(X(\rho))}{[b + \rho - h'(X(\rho, b))]} = \Lambda(\rho, b)$$

The stability of the FOSS depends upon the Jacobian evaluated at the FOSS, or

$$J^0 = \begin{pmatrix} H_{\lambda x}^0 & H_{\lambda \lambda}^0 \\ -H_{xx}^0 & \rho - H_{x\lambda}^0 \end{pmatrix} = \begin{pmatrix} -b + h'(x^*) & -1/B'' \\ -h''(x^*) \lambda^* + C''(x^*) & b + \rho - h'(x^*) \end{pmatrix}$$

$$= \begin{pmatrix} S & G \\ Q & \rho - S \end{pmatrix}$$

We have $\text{tr} J^0 = \rho > 0$ and $\det J^0 = (-b + h'(x^*)) (b + \rho - h'(x^*)) -$

$$\frac{(-h''(x^*)\lambda^* + C''(x^*))}{B''}.$$

The sufficient conditions for local diffusion-induced instability according to theorem 1 are:

$$\det J^0 < 0 \quad (66)$$

$$2S - \rho > 0 \text{ or } 2(-b + h'(x^*)) - \rho > 0 \quad (67)$$

$$\frac{\rho^2}{4} - QG > 0 \text{ or } \frac{\rho^2}{4} - \frac{-(-h''(x^*)\lambda^* + C''(x^*))}{B''(a^*)} > 0 \quad (68)$$

It should be noted that if there is no feedback, that is, $h(x) \equiv 0$, then $\det J^0 = -b(b + \rho) + \frac{C''(x^*)}{B''} < 0$, and the FOSS is a saddle point. Since $2S - \rho = -2b - \rho < 0$, local diffusion-induced instability is not possible in this case, and the ODI space is empty.

With nonzero feedback the conditions for diffusion-induced instability can also be expressed in terms of marginal benefits and marginal damages. We can gain some intuition by expressing these conditions in terms of slopes of marginal benefits and slopes of marginal damages at the FOSS, which in turn can be expressed as:²¹

$$\begin{aligned} \rho - 2\frac{C'(x^*)}{B'(a^*)} &> 0 \\ \frac{\rho^2}{4} + \frac{[h''(x^*)B'(a^*) + C''(x^*)]}{B''(a^*)} &> 0 \end{aligned}$$

To clarify whether diffusion-induced spatial instability actually emerges in the optimal control of a stylized shallow lake, we examine the following specification for the problem:²²

$$\begin{aligned} B(a) &= \ln a, \quad C(x) = \frac{1}{2}cx^2, \quad h(x) = \frac{x^2}{1+x^2}, \quad c = 0.5 \\ \dot{x} &= a - bx + \frac{x^2}{1+x^2} + D\frac{\partial^2 x}{\partial z^2}, \quad b = 0.54, \quad \rho = 0.1273 \end{aligned}$$

The value of $b = 0.54$ corresponds, for the flat case ($D = 0$), to a reversible

²¹The conditions can be obtained by noting that from $0 = [\rho + b - h'(x^*)]\lambda^* + D'(x^*)$ and $B'(a^*) = -\lambda^*$, we have $D'(x^*) = [\rho + b - h'(x^*)]B'(a^*)$ or

$\frac{D'(x^*)}{B'(a^*)} = -S + \rho$ or $S = \rho - \frac{D'(x^*)}{B'(a^*)}$, $a^* = \bar{a}(\lambda^*)$.

²²See Mäler et al. (2003) for the functional forms.

shallow lake with hysteresis. This is shown in figure 2, which depicts, in the (x, a) space, the curve of the lake equilibria, defined by $\dot{x} = 0$ in (63), for fixed loadings a . For loadings below the loading corresponding to the local maximum of this curve, the lake remains in a locally stable oligotrophic state. A small increase in loading above the local maximum will cause the lake to flip to a locally stable eutrophic state. To bring the lake back to the oligotrophic state, the loading has to be reduced to a level below the local minimum of the lake equilibria curve. This is the hysteresis effect.

Figure 2

The FOSSs for this problem are characterized by

$$\begin{aligned} \frac{1}{a} &= -\lambda \\ \phi(x)|_{\dot{x}=0} &: \lambda = \frac{1}{-bx + h(x)} \\ \psi(x)|_{\dot{\lambda}=0} &: \lambda = \frac{-cx}{b + \rho - h'(x)} \end{aligned} \quad (69)$$

There are three FOSSs as shown in figure 3 below. The dashed line corresponds to the $\phi(x)|_{\dot{x}=0}$ curve, while the solid corresponds to the $\psi(x)|_{\dot{\lambda}=0}$ curve. Comparing to figure 2, the $\phi(x)|_{\dot{x}=0}$ curve corresponds to the lake equilibria curve when optimal loading is applied at the state-costate space (x, λ) . It can be seen that when the oligotrophic steady state is very close to the flip point, a small perturbation can move the system to the eutrophic steady state.

Figure 3

The Jacobian at the FOSSs corresponding to (69) is

$$J_s^0 = \begin{pmatrix} -b + h'(x^*) & 1/(\lambda^*)^2 \\ -h''(x^*)\lambda^* + c & b + \rho - h'(x^*) \end{pmatrix}$$

and FOSSs are characterized in table 1.

Table 1: FOSS for the Shallow Lake

| i | $(x^*, \lambda^*)_i$ | Eigenvalues | $\det J^0$ | Stability |
|-----|----------------------|-------------------|------------|--------------|
| 1 | (0.571, -16.019) | 0.13542, -0.00812 | -0.00109 | Saddle Point |
| 2 | (0.583, -16.350) | 0.11833, 0.00897 | 0.00106 | Unstable |
| 3 | (2.445, -2.157) | 0.62449, -0.49719 | -0.31049 | Saddle Point |

There are two locally stable basins of attraction: a eutrophic (high phosphorus stock) and an oligotrophic (low phosphorus stock), with the expected high and low (in absolute terms) shadow phosphorus cost reflected by λ .

Spatial perturbation due to movement of the phosphorous stock in the lake might destabilize, through diffusion-induced instability, the first or the third FOSS. The conditions for a non empty ODI space and the corresponding values for the chosen specification are presented in table 2.

Table 2: Emergence of Spatial Heterogeneity in the Shallow lake

| Diffusion-induced instability conditions | Value at $(x^*, \lambda^*)_i$ | |
|---|-------------------------------|-----------|
| | $i = 1$ | $i = 3$ |
| (67) and (68) | $i = 1$ | $i = 3$ |
| $2H_{x\lambda}^0 - \rho = 2 \left[-b + h'(x_i^*) \right] - \rho > 0$ | 0.091633 | -1.00648 |
| $\frac{\rho^2}{4} + H_{xx}^0 H_{\lambda\lambda}^0 = \frac{\rho^2}{4} + \left[\frac{h''(x_i^*)}{\lambda^*} - \frac{c}{(\lambda^*)^2} \right] > 0$ | 0.00100 | -0.057241 |

As shown by our results, the phosphorus movements can produce spatial patterns for the optimally managed lake only if the lake is in the oligotrophic state. The elasticity condition (58) implies that pattern formation requires that the ratio of the elasticity of the demand for phosphorus loading with respect to the phosphorus stock's shadow cost, to the elasticity of the phosphorus stock with respect to the discount rate, multiplied by the interest charge on loadings per unit of phosphorus stock be sufficiently small relative to the discount rate. If the lake has flipped to a eutrophic state, then even with movements of phosphorus the optimal strategy suggests a spatially homogenous outcome in the sense that the FOSS is locally asymptotically stable. We turn now to the case where the FOSS is oligotrophic.

The dispersion relationship (38) corresponding to the oligotrophic FOSS is $h(k^2) = -D^2k^4 + 0.091633Dk^2 - 0.00109891$; it is shown in figure 4 for values of $D = \{0, 0.025, 0.05, 1\}$. The dispersion relationship corresponding to $D = 0$ is the horizontal line at $h = -0.00109891$, and the concave curve closest to the vertical axis is the one corresponding to $D = 1$.

Figure 4

We choose $D = 0.5$ with $(k_1^2, k_2^2) = (0.168463, 0.393556)$. To have one unstable mode in the linear approximation solution (44) for the growing instability in the neighborhood of the oligotrophic FOSS, the length of the spatial domain should satisfy, as shown in section 3, $2\pi/k_2 \leq L \leq 2\pi/k_1$, which implies that $15.9652 \leq L \leq 37.2971$. Choosing $L = 8\pi$, the positive eigenvalue is $\lambda_2 = 0.00652502$, the growing instability is given by $x(t, z) \sim \epsilon_x \exp(0.00652502t) \left[\cos \frac{2\pi z}{L} + \sin \frac{2\pi z}{L} \right] + 0.571$ for small ϵ_x , and is shown in Figure 5.

Figure 5

Once the oligotrophic FOSS is destabilized through diffusion and its linear approximation evolves similar to the shape shown in Figure 5, the question is whether the nonlinear ecosystem will end up, bounded by the intertemporal transversality condition, in a new steady state characterized by spatial heterogeneity and a persistent spatial pattern. The HOSS is characterized by

$$\begin{aligned} D \frac{\partial x^2(z, t)}{\partial z^2} &= -\bar{a}(\lambda) + bx - h(x) \\ D \frac{\partial \lambda^2(z, t)}{\partial z^2} &= [b + \rho - h'(x)] \lambda + D'(x) \end{aligned}$$

or

$$\begin{aligned} \frac{dv(z)}{dz} &= \frac{1}{D} \left(\frac{1}{\lambda(z)} + bx(z) - h(x(z)) \right), \quad \frac{dx(z)}{dz} = v(z) \\ \frac{dw(z)}{dz} &= \frac{1}{D} \left[(b + \rho - h'(x(z))) \lambda(z) + cx(z) \right], \quad \frac{d\lambda(z)}{dz} = w(z) \end{aligned}$$

with boundary conditions derived by the circle and transversality conditions $x(0) = x(L) = 0.571396$, $\lambda(0) = \lambda(L) = -16.0194$, $v(0) = v(L)$, $w(0) = w(L)$ and $L = 8\pi$. The system was solved numerically by a multiple shooting approach using Mathematica. The spatial paths for phosphorus stock and its shadow cost have very flat U shapes.²³

5.2 Spatial Pattern Formation in Natural Resource Management

Let $x(t, z)$ denote the concentration of the biomass of a renewable resource (*e.g.* fish) at spatial point $z \in [0, L]$ of a one-dimensional spatial domain at time t , with x taking non-negative values in a compact set \mathcal{X} . As is well-known (*e.g.* Clark 1990), natural resources can be interpreted as capital assets since the resource's value is equal to the present value of the flow of net future expected benefits. We will exploit this capital-theoretic aspect of resource management to explore diffusion-induced instability and pattern formation in renewable management by using two approaches. The first uses a standard Schaefer structure that includes stock effects. The second mimics growth theory and explores pattern formation in resource models which have the structure of the traditional Ramsey and Solow models of growth theory,

²³A more profound pattern emerges in the next application.

without stock effects in the objective function. We use circle boundary conditions, since we want to study endogenous pattern formation. In both cases we identify conditions for diffusion-induced instability and pattern formation.

5.2.1 Optimal resource management with stock effects

Let the evolution of the biomass in space-time depend on resource growth, according to a standard concave growth function $f(x(t, z))$, dispersion in space with a constant diffusion coefficient D , and harvesting at a rate of $h(t, z)$. We assume that the movement of the biomass is from high concentration towards low concentration. Stock effects are introduced by specifying harvesting as $h(t, z) = qE(t, z)x(t, z)$, where q is the catchability coefficient and $E(t, z)$ is harvesting effort at time t and location z .²⁴ Therefore,

$$\frac{\partial x(t, z)}{\partial t} = f(x(t, z)) - h(t, z) + D \frac{\partial^2 x(t, z)}{\partial z^2}$$

We assume that net harvesting benefits at each point in space-time can be represented by a concave net benefit function $Q(qEx)$. The optimal harvesting problem in space-time is then defined as:

$$\max_{E(t, z)} \int_0^\infty \int_Z e^{-\rho t} Q(qE(t, z)x(t, z)) dz dt \quad (70)$$

$$\text{s.t. } \frac{\partial x(t, z)}{\partial t} = f(x(t, z)) - qx(t, z)E(t, z) + D \frac{\partial^2 x(t, z)}{\partial z^2} \quad (71)$$

$$x(t, 0) = x(t, L) \text{ and smooth pasting} \quad (72)$$

The generalized current value Hamiltonian for this problem is defined as:

$$\tilde{H} = Q(qEx) + \mu \left[f(x) - qEx + D \frac{\partial^2 x(t, z)}{\partial z^2} \right]$$

²⁴See for example Neubert (2003) for a fishery model with a similar structure or Sanchirico and Wilen (2005) for renewable resource management with metapopulation models in a patchy environment.

Following our results in section 2, the optimality conditions are:

$$qQ'x = q\mu x \Rightarrow Q' = \mu \Rightarrow E = E(x, \mu) \quad (73)$$

$$Q''x dE + Q''E dx = d\mu$$

$$\frac{\partial E}{\partial x} = -\frac{E}{x} < 0, \quad \frac{\partial E}{\partial \mu} = \frac{1}{Q''x} < 0$$

$$\dot{x} = f(x) - qE(\mu, x)x + Dx_{zz} \quad (74)$$

$$\dot{\mu} = \left[\rho - f'(x) + qE(x, \mu) \right] \mu - qQ'E(x, \mu) - D\mu_{zz} \text{ or} \quad (75)$$

$$\dot{\mu} = \left[\rho - f'(x) \right] \mu - D\mu_{zz} \text{ using } Q' = \mu$$

The FOSS is defined for $D = 0$ as:

$$0 = f(x^*) - qE(\mu^*, x^*)x^* = H_\mu^0(x, \mu) \quad (76)$$

$$0 = \rho - f'(x^*) = \rho - H_x^0(x, \mu) \quad (77)$$

$$H^0(x, \mu) = \max_E(x, \mu, E) \quad (78)$$

Linearization at the FOSS results in²⁵

$$J^0 = \begin{pmatrix} H_{\mu x}^0 & H_{\mu \mu}^0 \\ -H_{xx}^0 & \rho - H_{x\mu}^0 \end{pmatrix} = \begin{pmatrix} f'(x^*) & \frac{-q}{Q''} \\ -f''(x^*)\mu^* & \rho - f'(x^*) \end{pmatrix} =$$

$$\begin{pmatrix} S & G \\ Q & \rho - S \end{pmatrix} = \begin{pmatrix} \rho & \frac{-q}{Q''} \\ -f''(x^*)\mu^* & 0 \end{pmatrix} \quad (79)$$

Since $\text{tr} J^0 = \rho > 0$ and $\det J^0 = -(qf''(x^*)\mu^*)/Q'' < 0$, the FOSS is a local saddle point for $\mu^* > 0$. From (79), $S = \rho$ and (33) is satisfied for $\rho > 0$. Condition (34), the violation of the curvature assumption, requires $(\rho^2/4) > -H_{xx}^0 H_{\mu\mu}^0$, or

$$\frac{\rho^2}{4} > \left(-f''(x^*)\mu^* \right) \left(\frac{-q}{Q''} \right) \quad (80)$$

If (80) is also satisfied, the stable manifold of the saddle point FOSS is destabilized by biomass movement, through diffusion-induced instability and optimal control interactions and a spatial pattern for the biomass concentration emerges.

The elasticity condition for diffusion-induced instability implies in this case that if $\rho^2/4 > \frac{\varepsilon E \mu}{\varepsilon x \mu} q \rho E^*$, then optimal harvesting of the moving biomass

²⁵Since $H_{\mu x} = f'(x) - qE(\mu, x) - qx \frac{dE}{dx} = f'(x) - qE(\mu, x) - qx \left(-\frac{E}{x} \right) = f'(x)$.

leads to spatial patterns, where $\varepsilon_{E\mu}$ is the demand elasticity for effort, $\varepsilon_{x\mu}$ is the elasticity of demand for *in situ* biomass with respect to the discount rate, and $(q\rho E^*)$ is a constant evaluated at the steady state reflecting the interest charged on effort multiplied by the catchability coefficient. Thus if the effort demand elasticity is small, the biomass demand elasticity is large, the interest charged on effort weighted by the catchability coefficient is not high, and the discount rate is sufficiently high, then biomass movements might destabilize the FOSS and create spatial patterns.

Pattern formation and the heterogeneous optimal steady state To examine whether there is a parameter set satisfying (80), or to put it differently, to examine whether the ODI space is non-empty, we consider again a numerical example. Assume a standard logistic model, $f(x) = rx(1 - \frac{x}{K})$, where r is the intrinsic growth rate and K the carrying capacity, with $r = 0.08$ and $K = 400000$ whales.²⁶ Assuming a quadratic benefit function $Q(h) = A + \alpha h - (1/2)\beta h^2$, $h = qEx$, we obtain:

$$E(t, z) = \frac{\alpha - \mu(t, z)}{q\beta x(t, z)}$$

For the FOSS we obtain, using (76)-(77),

$$x^* = \frac{K(r - \rho)}{2\rho}, \quad \mu^* = \frac{r(2\alpha - K) + K\rho}{\beta r(r + \rho)}$$

Taking $\alpha = 80000$, $\beta = 10$, $q = 1$, the relationship $g(\rho) = \rho^2/4 - (-f''(x^*)\mu^*)(-1/Q'')$ is depicted in figure 6. Diffusion-induced instability emerges when $g(\rho) > 0$. Choosing $\rho = 0.05$, so that $g(0.05) > 0$, we obtain $(x^*, \mu^*) = (75000, 7692.31)$ for the FOSS.

Figure 6

The HOSS is characterized by

$$\frac{\partial^2 x}{\partial z^2} = \frac{1}{D} [E(\mu, x)x - f(x)] \quad (81)$$

$$\frac{\partial^2 \mu}{\partial z^2} = \frac{1}{D} [\rho - f'(x)] \mu \quad (82)$$

Making the substitutions $\frac{\partial x}{\partial z} = v(z)$, $\frac{\partial \mu}{\partial z} = w(z)$ and using the circle boundary conditions and the appropriate transversality conditions $x(0) = x(L) =$

²⁶These parameters correspond to the Antarctic fin-whale (Clark, 1990, p. 49).

75000, $\mu(0) = \mu(L) = 7692.31$, $v(0) = v(L)$, $w(0) = w(L)$, we solve the system by multiple shooting. figure 7 shows the spatial paths for x and μ at the HOSS for $L = 2\pi$. The U curve is the spatial path for the biomass stock, while the lower curve, which has a very flat inverted U shape, is the spatial path for the resource's user cost.

Figure 7

5.2.2 Ramsey/Solow Capital Theoretic Models of Renewable Resource Management

The capital theoretic structure of this model implies that the evolution of the biomass in space-time is described by

$$\frac{\partial x(z, t)}{\partial t} = f(x(z, t)) - \delta x(z, t) - h(z, t) + D \frac{\partial x^2(z, t)}{\partial z^2}$$

where $f(x(z, t))$ is a concave growth function, δ reflects mortality rate, $h(z, t)$ is harvest rate and dispersion of the resource in space is determined by the diffusion coefficient D . We analyze first a Ramsey-type optimal resource management problem.

Pattern formation in Optimal Resource Management Consider the spatial optimal resource management problem

$$\begin{aligned} & \max_{\{h(t,z)\}} \int_0^\infty \int_0^L e^{-\rho t} U(h(z, t)) dz dt \\ \text{s.t. } & \frac{\partial x(z, t)}{\partial t} = f(x(z, t)) - h(z, t) - \delta k(z, t) + D \frac{\partial x^2(z, t)}{\partial z^2} \\ & x(t, 0) = x(t, L), \text{ and smooth pasting} \end{aligned}$$

where $U(\cdot)$ is a standard utility function reflecting the flow of net benefits from harvesting. This problem has the structure of an optimal growth model and corresponds to an optimal fishery management problem without resource stock externalities affecting the objective function. Resource stock externalities are likely to be negligible if the size of the fish population is very large relative to the industry (Smith, 1969). Using again the necessary conditions for the maximum principle under diffusion, we obtain:

$$\tilde{H}(x, h, p) = U(x(z, t)) + p(z, t) \left[f(x(z, t)) - h(z, t) - \delta x(z, t) + D \frac{\partial x^2(z, t)}{\partial z^2} \right]$$

$$U_h(h(z, t)) = p(z, t) \Rightarrow h(z, t) = \bar{h}(p(z, t)), \quad \frac{dh}{dp} = \frac{1}{U_{hh}} < 0 \quad (83)$$

$$\begin{aligned} \frac{\partial x(z, t)}{\partial t} &= f(x(z, t)) - \bar{h}(p(z, t)) - \delta x(z, t) + D \frac{\partial x^2(z, t)}{\partial z^2} \\ \frac{\partial p(z, t)}{\partial t} &= (\rho + \delta - f_x(x(z, t)))p(z, t) - D \frac{\partial p^2(z, t)}{\partial z^2} \end{aligned} \quad (84)$$

The FOSS (x^*, p^*) is determined by the solution of the system

$$\begin{aligned} 0 &= f(x) - \bar{h}(p) - \delta x = H_p^0(x, p) \\ 0 &= (\rho + \delta - f_x(x))p = \rho - H_x^0(x, p) \\ H^0(x, p) &= \max_h H(x, h, p) \end{aligned}$$

and the FOSS is given by

$$\begin{aligned} x^* &: f_x(x^*) = \rho + \delta \\ p^* &: \bar{h}(p^*) = f(x^*) - \delta x^* \end{aligned}$$

Assuming a Cobb-Douglas growth function $f(x) = x^a$,²⁷ $0 < a < 1$, and $U(h) = \frac{h^{1-\theta}-1}{1-\theta}$, $\theta > 1$ we obtain

$$\begin{aligned} \bar{h}(p) &= p^{-1/\theta} \\ x^* &= \left(\frac{\rho + \delta}{a} \right)^{\frac{1}{a-1}} \\ p^* &= [(x^*)^a - \delta x^*]^{-\theta} \end{aligned}$$

The stability of the FOSS is determined by the Jacobian matrix

$$\begin{aligned} J^0 &= \begin{pmatrix} H_{px}^0 & H_{pp}^0 \\ -H_{xx}^0 & \rho - H_{xp}^0 \end{pmatrix} = \begin{pmatrix} f_x(x^*) - \delta & -\bar{h}'(p^*) \\ -f_{xx}(x^*)p^* & 0 \end{pmatrix} = \\ & \begin{pmatrix} \rho & \frac{1}{\theta} p^{-(\frac{1}{\theta}+1)} \\ -(a-1)a(x^*)^{(a-2)}p^* & 0 \end{pmatrix} = \begin{pmatrix} S & G \\ Q & \rho - S \end{pmatrix} \end{aligned}$$

It is clear that: $\text{trace} J^0 = \rho > 0$, $\det J^0 = \frac{-(a-1)a(x^*)^{(a-2)}(p^*)^{-1/\theta}}{\theta} < 0$. There-

²⁷Our Cobb-Douglas assumption could be a convenient parametrization under the assumption that steady states occur within the upward sloping part of the growth law. To have finite carrying capacity, we can paste an increasing and then a decreasing part onto the Cobb-Douglas function for large x , which would give a large carrying capacity. This part, however, does not enter our solution for the parameter constellation we are using for our illustrative example.

fore the FOSS is a saddle point.

To destabilize, according to theorem 1, the negative root through diffusion-induced instability via optimal control interactions, we need

$$\begin{aligned} 2S - \rho &> 0, \text{ or } \rho > 0 \\ \frac{\rho^2}{4} - QG &> 0 \text{ or} \end{aligned} \quad (85)$$

$$g(\rho) = \rho^2 + 4 \frac{(a-1)a(x^*)^{(a-2)}(p^*)^{-1/\theta}}{\theta} > 0 \quad (86)$$

For $\delta = 0.01, \theta = 3, a = 0.65$, relation (86) implies, as in the previous case, that diffusion-induced instability emerges when $\rho > 0.037$. Choosing $\rho = 0.25$, the corresponding FOSS is $(x^*, \mu^*) = (13.708, 0.0065)$.

To explore the economic intuition as before, the elasticity condition (58) implies

$$\frac{\rho^2}{4} > -H_{pp}^0 H_{xx}^0 = p \frac{d\bar{h}/dp}{dx/d\rho} = \frac{\varepsilon_{hp}}{\varepsilon_{x\rho}} \frac{\rho h^*}{x^*} \quad (87)$$

where ε_{xp} is the consumer demand elasticity for harvested biomass, $\varepsilon_{x\rho}$ is the elasticity of demand for *in situ* biomass with respect to the discount rate, and $(\rho h^*/k^*)$ is a constant evaluated at the steady state reflecting the interest charged on harvested biomass per unit of biomass remaining *in situ*. Actually ρh^* can be interpreted as forgone biomass returns due to the fact that the biomass did not remain *in situ*, but was removed by harvesting. Thus (87) implies that if the consumer demand elasticity is small, the biomass demand elasticity is large, the interest charged on harvested biomass per unit of *in situ* biomass at the FOSS is not high, and the discount rate is sufficiently high, then biomass movements might destabilize the FOSS and create spatial patterns.

The HOSS, approximated again by multiple shooting, implies a very flat U curve as the spatial path for the biomass and a very flat inverted U curve as the spatial path for its shadow value.

Pattern formation with a fixed harvesting rule Assume that fishery management is characterized by a fixed harvesting rule which states that a fixed proportion $1-s$, $s \in (0, 1)$ of the gross additions to biomass is harvested at any point in time. In this case biomass evolution is described by

$$\frac{\partial x(z, t)}{\partial t} = sf(x(z, t)) - \delta x(z, t) + D \frac{\partial^2 x(z, t)}{\partial z^2}$$

where the fixed harvesting rule is common to all locations.²⁸ It is a routine result, with the Cobb-Douglas growth function used above, that a nonzero flat steady state (FSS) is defined for $D = 0$ as: $x^* = (\delta/s)^{1/(a-1)}$, with $a \in (0, 1)$ for concavity which implies diminishing returns in biomass growth. It is also a standard result that the FSS is asymptotically stable for positive values of biomass.²⁹ Let us consider the impact of the diffusion-induced perturbation on this FSS. The linearized model is

$$\begin{aligned} \frac{\partial x(z, t)}{\partial t} &= \left[sf'(x^*) - \delta \right] (x(t, z) - x^*) + D \frac{\partial x^2(z, t)}{\partial z^2} \\ x(t, 0) &= x(t, L) \end{aligned} \quad (88)$$

We look for solution $(x(z, t) - x^*) \propto e^{\lambda t} \cos(2n\pi z/L)$, $n = 1, 2, \dots$. Substituting into (88) and canceling equal terms, we obtain

$$\lambda = [(a-1)\delta - D(2n\pi/L)^2] < 0 \quad (89)$$

Therefore $(x(z, t) - x^*) \rightarrow 0$ as $t \rightarrow \infty$, and diffusion can not produce spatial heterogeneity in the fixed harvesting rule model, with diminishing returns to biomass.

It should be noticed that, since the emergence of diffusion requires that $\lambda > 0$, this is possible only if $\alpha > 1$. Even with $\alpha = 1$, which corresponds to a linear growth function, spatial heterogeneity is not possible. So for the emergence of spatial heterogeneity in the fixed harvesting model, we need convex growth, that is increasing returns which should be sufficiently high to overcome the impact of diffusion. It follows from (89) that if D is large, then λ could be negative even with $a > 1$, in which case spatial patterns die out and the system converges to a FSS, despite increasing returns. In this case large diffusion slows down the impact of increasing returns and induces spatial homogeneity.

Considering that convex growth is unlikely in the typical fishery model, this result suggests that fixed harvesting rules tend to produce spatially homogeneous biomass distribution, while the optimal harvesting rule is likely, under a certain parameter constellation, to result in spatially heterogeneous biomass distribution. This heterogeneity result does not require increasing returns.

²⁸The structure of this model is similar to the Solow growth model with fixed savings ratio.

²⁹The linearization around the FSS results in $\dot{x} = [sf'(x^*) - \delta](x(t) - x^*)$. Since $sf'(x^*) - \delta = (a-1)\delta < 0$, the FSS is stable.

6 Concluding Remarks

This paper has developed recursive intertemporal infinite horizon optimal theory for a one-dimensional state variable for a continuum of spatial sites with spillovers across sites caused by state variable diffusion. We developed a local analysis of stability of optimal steady state in this setting and found a rather surprising connection between sufficient conditions for optimal diffusion-induced instability and quantities such as Hamiltonian curvature and the discount rate, which play key roles in global asymptotic stability analysis of infinite horizon multi-sectoral recursive optimal control models.

We would like to stress that the type of instability discussed in this paper, is *not a Turing phenomenon*, since Turing diffusion-induced instability requires at least two state variables with different diffusion coefficients (Murray 2003, vol. II, chapter 2). In our case we have a one-dimensional state variable, and diffusion-induced instability is *not* the result of inhibitor-activator interaction as in Turing's case, but the result of optimizing behavior when parameters are in the set of ODI. This is the set of parameters for which the trade-off between costs and benefits is such that it is optimal to allow the system to go unstable in some mode. This optimal diffusion-induced instability analyzed in this paper is a new version of diffusion-induced instability beyond Turing's.

Genuine Turing instability may arise in an uncontrolled two-dimensional system with inhibitor-activator characteristics. If, with a high discount rate, the benefits from stabilization of the controlled system are not large in the sense described in section 4, then it might not be optimal to stabilize and ODI is possible. Thus, the emergence of our ODI is possible in higher dimensional systems for which Turing instability may appear without optimal control. The Turing-unstable system may be stabilized under optimal control if benefits are sufficiently large. However, if benefits from stabilization are not sufficient, the system could be "optimally unstable" with the emergence of ODI. The extension of the methods developed in this paper in higher dimension systems potentially characterized by Turing instability, when no control is applied, is an area for further research.

Another key topic for future research is to generalize our results to more general forms of diffusion, for example, diffusion governed by a kernel as in Murray (2003, vol. II, chapter 12). Kernels are much more general than the localized diffusion treated in this paper because flexible kernels can represent spillovers from distant sites, not just the neighboring site diffusion we analyze here. Our method for LQ approximations extended to kernels can be used to study *spillover induced instability* in the context of new growth theory.

For example there are papers in Semmler (2005), especially the article

by Grune, Kato, and Semmler which studies optimal tax theory, that compute optimal solutions for non spatial ecological management problems and other non spatial problems that are closely related to the lake problem and renewable resource problem studied in our paper. . It would be valuable to extend their work to such interesting issues as the interaction between spatial spillovers, discounting, diffusion-induced instability, and optimal tax design.

Dasgupta and Mäler (2003) contains a collection of papers that explores nonconvex ecosystems and their management. The models include lakes, rangelands, and boreal forests. It would be valuable to generalize this type of work to incorporate spatial spillovers and study potential diffusion impacts on stability in a setting like ours. Our example of spatially connected lakes gives an indication of what such a generalization would look like, but it merely scratches the surface of potential work that could be done.

We close this paper with one final suggestion for future research which could potentially be very important. Magill (1977a,b) exposed the key role of curvature matrices like the Q matrix in proving local stability theorems for optimized systems in one of his early papers referenced in Magill (1977a). He also developed local linear quadratic approximations for optimized systems forced by small stochastic shocks and showed how to compute the frequency spectra for such approximations in Magill (1977b) and even showed how to apply the approximations to business cycle analysis. We believe that a very important research project would be to develop "small noise" linear quadratic approximation theory by extending Magill's (1977a,b) work to optimized spatial systems. Linear quadratic approximations are sometimes quite accurate in practical work, and we believe that such approximations, as well as the general methods developed in this paper, could become very useful to economists in future applications. Indeed, we plan to work on many of these applications ourselves.

Appendix

The Linear - Quadratic Approximation

Consider the problem

$$\max_{\{u(t,z)\}} \int_{z_0}^{z_1} \int_0^\infty e^{-\rho t} f(x(t,z), u(t,z)) dt dz \quad (90)$$

$$\text{s.t. } \frac{\partial x(t,z)}{\partial t} = g(x(t,z), u(t,z)) + D \frac{\partial^2 x(t,z)}{\partial z^2}, x(t_0, z) \text{ given} \quad (91)$$

with spatial boundary conditions

$$(x(t, z_0) = x(t, z_1) = \bar{x}(t), \forall t, \text{ the space is a circle}) \quad (92)$$

with the associated current value Hamiltonian function

$$\tilde{H} = (x(t,z), u(t,z), q(t,z)) = f(x, u) + q \left[g(x, u) + D \frac{\partial^2 x}{\partial z^2} \right] \quad (93)$$

and define the performance functional as

$$J(u) = \int_{z_0}^{z_1} \int_0^\infty e^{-\rho t} f(x(t,z), u(t,z)) dt dz \quad (94)$$

Our purpose is to extend the method developed by Magill (1977b), by which a non-linear optimal stochastic control problem is replaced by a simpler linear quadratic optimal stochastic control problem, to the case of a deterministic control problem, such as (90)-(92) where the transition of the system is described by a partial differential equation with a diffusion term and not by a stochastic ordinary differential equation. Let $u^*(t, z)$ be the optimal control for problem (90)-(92) and $x^*(t, z)$ the corresponding optimal trajectory for the state variable. Let (x^*, u^*, q^*) be a FOSS obtained for $D = 0$ which satisfies the usual optimality conditions, and assume that the diffusion process (91) starts close to the steady state or that $x_0 = x(0, z)$ starts close to x^* for all $z \in \mathcal{Z} = [z_0, z_1]$.

Also let $\bar{u}(t, z) = u(t, z) - u^*(t, z)$, $\bar{x}(t, z) = x(t, z) - x^*(t, z)$, and perturb the control $u^*(t, z)$ by letting

$$u(t, z) = u^*(t, z) + \varepsilon \bar{u}(t, z) \quad (95)$$

Following Athans and Falb (1966, p. 261) we suppose that the state variable trajectory generated by (95) can be written as

$$x(t, z) = x^*(t, z) + \varepsilon \bar{x}(t, z) + \varepsilon^2 \xi(t, z) + o(\varepsilon^2, t, z) \quad (96)$$

where \bar{x} and ξ are first and second order state perturbations respectively and $o(\varepsilon^2, t, z) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly in (t, z) . Athans and Falb (1966, pp. 254-265) show that control perturbations of the form (95) lead to state perturbations of the form (96) under appropriate regularity conditions for the case where \mathcal{Z} is one point. We proceed heuristically here, we leave to future research to develop a completely rigorous argument. Substituting (96) and (95) into (91), the $g(x, u)$ function describing the kinetics of the state variable can be written as:

$$g(x^*(t, z) + \varepsilon\bar{x}(t, z) + \varepsilon^2\xi(t, z) + o(\varepsilon^2, t, z), u^*(t, z) + \varepsilon\bar{u}(t, z)). \quad (97)$$

Substituting also for $x(t, z)$ into the derivatives $\frac{\partial x(t, z)}{\partial t}$ and $\frac{\partial^2 x(t, z)}{\partial z^2}$, using (96) and expanding as a Taylor series around $(x^*(t, z), u^*(t, z))$, we obtain³⁰

$$\begin{aligned} \varepsilon \frac{\partial \bar{x}(t, z)}{\partial t} + \varepsilon^2 \frac{\partial \xi(t, z)}{\partial t} &= g(x^*(t, z), u^*(t, z)) + g_x(\varepsilon\bar{x} + \varepsilon^2\xi + o(\varepsilon^2, t, z)) \\ &+ g_u(\varepsilon\bar{u}) + w'Ww + \varepsilon D \frac{\partial^2 \bar{x}(t, z)}{\partial z^2} + \varepsilon^2 D \frac{\partial^2 \xi(t, z)}{\partial z^2} + \text{h.o.t.} \quad (98) \\ w &= (\varepsilon\bar{x} + \varepsilon^2\xi, \varepsilon\bar{u})', W = \begin{pmatrix} g_{xx} & g_{xu} \\ g_{ux} & g_{uu} \end{pmatrix} \end{aligned}$$

Divide (98) by ε and then take the limit as $\varepsilon \rightarrow 0$, evaluate all functions at the FOSS (x^*, u^*) and note that $g(x^*, u^*) = 0$ because (x^*, u^*) is a steady state, to obtain the linear approximation of (91) around the optimal flat steady state as

$$\frac{\partial \bar{x}(t, z)}{\partial t} = g_x \bar{x}(t, z) + g_u \bar{u}(t, z) + D \frac{\partial^2 \bar{x}(t, z)}{\partial z^2} \quad (99)$$

$$\text{with } \bar{x}(t_0, z) = 0 \text{ for all } z. \quad (100)$$

$$\bar{x}(t, z) = x(t, z) - x^*, \quad \bar{u}(t, z) = u(t, z) - u^* \quad (101)$$

If, using the equality of the ε -terms in (98) we cancel these terms, divide by ε^2 and then take the limit $\varepsilon^2 \rightarrow 0$, we obtain a differential equation in the second-order state perturbation

$$\begin{aligned} \frac{\partial \xi(t, z)}{\partial t} &= g_x \xi(t, z) + \\ &g_{xx} \xi(t, z) + g_{uu} \bar{u}(t, z) + 2g_{xu} \xi(t, z) \bar{u}(t, z) + D \frac{\partial^2 \xi(t, z)}{\partial z^2} \quad (102) \end{aligned}$$

$$\text{with } \xi(t_0, z) = 0 \text{ for all } z. \quad (103)$$

³⁰Subscripts denote derivatives.

Write the performance functional (94) using the Hamiltonian function (93) with $x(t, z)$ and $u(t, z)$ given by the perturbations (96) and (95) and with $q(t, z)$ evaluated along the optimal path $q^*(t, z)$ as

$$J(u) = \int_{z_0}^{z_1} \int_0^\infty e^{-\rho t} \left[H(x(t, z), u(t, z), q^*(t, z)) - q^*(t, z) \frac{\partial x(t, z)}{\partial t} \right] dt dz \quad (104)$$

Write the performance functional along an optimal path as

$$J(u^*) = \int_{z_0}^{z_1} \int_0^\infty e^{-\rho t} \left[H(x^*(t, z), u^*(t, z), q^*(t, z)) - q^*(t, z) \frac{\partial x^*(t, z)}{\partial t} \right] dt dz \quad (105)$$

then

$$\begin{aligned} J(u) - J(u^*) = & \\ & \int_{z_0}^{z_1} \int_0^\infty e^{-\rho t} \left[H(x(t, z), u(t, z), q^*(t, z)) - H(x^*(t, z), u^*(t, z), q^*(t, z)) \right. \\ & \left. - q^*(t, z) \frac{\partial (x(t, z) - x^*(t, z))}{\partial t} \right] dt dz \end{aligned} \quad (106)$$

Expanding in a Taylor series we obtain³¹

$$\begin{aligned} J(u) - J(u^*) = & \int_{z_0}^{z_1} \int_0^\infty e^{-\rho t} \left[H_{x\varepsilon} \bar{x}(t, z) + H_{u\varepsilon} \bar{u}(t, z) + \frac{1}{2} \mathbf{v}' \mathbf{Q} \mathbf{v} \right. \\ & \left. + o(\varepsilon^2, t, z) - q^*(t, z) \frac{\partial (x(t, z) - x^*(t, z))}{\partial t} \right] dt dz \end{aligned} \quad (107)$$

$$\mathbf{v} = (\varepsilon \bar{u}(t, z), \varepsilon \bar{x}(t, z))' \quad (108)$$

$$\mathbf{Q} = \begin{pmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xu} \\ f_{ux} & f_{uu} \end{pmatrix} + q^* \begin{pmatrix} g_{xx} & g_{xu} \\ g_{ux} & g_{uu} \end{pmatrix} \quad (109)$$

In (107) integrating by parts the term $e^{-\rho t} q^*(t, z) \frac{\partial (x(t, z) - x^*(t, z))}{\partial t}$ we obtain

$$\begin{aligned} & (-1) \int_{z_0}^{z_1} \int_0^\infty e^{-\rho t} q^*(t, z) \frac{\partial \bar{x}(t, z)}{\partial t} dt dz = \\ & \int_{z_0}^{z_1} \left[- [e^{-\rho t} q^*(t, z) \bar{x}(t, z)]_0^T + \int_0^\infty \bar{x}(t, z) \frac{\partial (e^{-\rho t} q^*)}{\partial t} dt \right] dz = \\ & \int_{z_0}^{z_1} \left[- [e^{-\rho t} q^* \bar{x}]_0^T + \int_0^\infty e^{-\rho t} \bar{x} \left(\frac{\partial q^*}{\partial t} - \rho q^* \right) dt \right] dz \end{aligned} \quad (110)$$

³¹See Athans and Falb (1966, pp. 261-263) for such an expansion in the context of deriving necessary conditions for standard control problems without diffusion.

In the first term of (110) the one term in T goes to zero as $T \rightarrow \infty$ by the intertemporal transversality conditions, while the other term is zero, for $T = 0$ by initial conditions on the state perturbation. The second term under the integral can be written, using (96) and the optimality conditions, as:

$$-H_x [x^*(t, z) + \varepsilon \bar{x}(t, z) + \varepsilon^2 \xi(t, z) + o(\varepsilon^2, t, z) - x^*(t, z)] \quad (111)$$

Furthermore $H_u = 0$ by the optimality conditions. Substituting into (107) dividing by ε taking limits, and evaluating all function at the optimal flat steady state we obtain

$$J(u) - J(u^*) = \int_{z_0}^{z_1} \int_{t_0}^{t_1} \frac{1}{2} \mathbf{v}' \mathbf{Q} \mathbf{v} dt dz \quad (112)$$

Therefore a “good approximation” of problem (90)-(92) can be obtained if we replace the function $f(x(t, z), u(t, z))$ in problem (90)-(92) with $\frac{1}{2} \mathbf{v}' \mathbf{Q} \mathbf{v}$ and the transition equation (91) with the linearized diffusion equation (99).

It should be noted that since the diffusion coefficient D is independent of the state and the control variables, this term drops out from the approximation of the objective, but enters the problem through the linearized diffusion equation. It is clear that extra terms including the diffusion coefficient should be added into the approximating matrix \mathbf{Q} in the general case where $D = D(x, u)$.

Abusing notation by denoting with $x(t, z)$, $u(t, z)$, $q(t, z)$ deviations from the FOSS values (x^*, u^*, q^*) , setting

$$\frac{1}{2} \mathbf{v}' \mathbf{Q} \mathbf{v} = \frac{1}{2} \begin{bmatrix} x & u \end{bmatrix} \begin{bmatrix} -A & N \\ N & -B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \quad (113)$$

$$\frac{1}{2} [-Ax^2 + 2Nxu - Bu^2] \quad (114)$$

and

$$S = g_x(x^*, q^*) \quad , \quad -G = g_u(x^*, q^*)$$

we obtain the LQ approximation (10)-(14) in the neighborhood of the FOSS.

Proof of Theorem 1

Define deviations from the FOSS $\mathbf{w} = (x(t, z) - x^*, q(t, z) - q^*)$, denote partial derivatives by subscripts and write the linearization of the full MHDS

$$\mathbf{w}_t = J^0 \mathbf{w} + \tilde{D} \mathbf{w}_{zz}, \quad \tilde{D} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \quad (115)$$

Following Murray (2003) we consider the time-independent solution of the

spatial eigenvalue problem, with appropriate boundary conditions

$$\mathbf{W}_{zz} + k^2 \mathbf{W} = 0 \quad (116)$$

where k is the eigenvalue. For the one-dimensional domain $[0, L]$ we have solutions for (116) which are of the form

$$\mathbf{W}_k(z) = \mathbf{A}_{1n} \cos\left(\frac{2n\pi z}{L}\right) + \mathbf{A}_{2n} \sin\left(\frac{2n\pi z}{L}\right), \quad n = \pm 1, \pm 2, \dots, \quad (117)$$

where \mathbf{A}_n are arbitrary constants. Solution (117) satisfies circle boundary conditions at $z = 0$ and $z = L$. The eigenvalue is $k = 2n\pi/L$, and $1/k = L/2n\pi$ is a measure of the wave-like pattern. The eigenvalue k is called the *wavenumber* and $1/k$ is proportional to the wavelength $\omega : \omega = 2\pi/k = L/n$. Let $\mathbf{W}_k(z)$ be the eigenfunction corresponding to the wavenumber k . We then look for solutions to (115) of the form

$$\mathbf{w}(t, z) = \sum_k c_k e^{\lambda t} \mathbf{W}_k(z) \quad (118)$$

Substituting (118) into (115), using (116) and canceling $e^{\lambda t}$, we obtain for each k or equivalently each n , that $\lambda \mathbf{W}_k = J^0 \mathbf{W}_k - Dk^2 \mathbf{W}_k$. Since we require non-trivial solutions for \mathbf{W}_k , λ must solve

$$\left| \lambda I - J^0 + \tilde{D}k^2 \right| = 0$$

Then the eigenvalue $\lambda(k)$ as a function of the wavenumber is obtained as the roots of

$$\lambda^2 - \rho\lambda + h(k^2) = 0 \quad (119)$$

$$h(k^2) = -D^2 k^4 + D(2H_{xq}^0 - \rho)k^2 + \det J^0 \quad (120)$$

where the roots are given by:

$$\lambda_{1,2}(k^2) = \frac{1}{2} \left(\rho \pm \sqrt{\rho^2 - 4h(k^2)} \right)$$

It should be noted that the flat ($D = 0$) case corresponds to $k^2 = 0$, so that $h(k^2 = 0) = \det J^0$, and $\lambda_{1,2} = \frac{1}{2} \left(\rho \pm \sqrt{\rho^2 - 4 \det J^0} \right)$. In this case Kurz's (1968) result holds, either $(\lambda_2, \lambda_1) > 0$ and the FOSS is unstable or $\lambda_2 < 0 < \lambda_1$ and the FOSS is saddle point stable, and there is a one-dimensional stable manifold containing the FOSS. Solutions of the MHDS

on this manifold are GAS.

We consider now the impact of a perturbation induced by diffusion on a saddle point FOSS. Under diffusion the smallest root λ_2 is given by

$$\lambda_2(k^2) = \frac{1}{2} \left(\rho - \sqrt{\rho^2 - 4h(k^2)} \right) \quad (121)$$

Then,

- If $0 < h(k^2) < \rho^2/4$ for some k , then λ_2 becomes real and positive.
- If $h(k^2) > \rho^2/4$ for some k , then both roots corresponding to λ are complex with positive real parts.

In both cases above, the linearly stable steady state on the stable manifold becomes unstable to *spatial* disturbances. Therefore if $h(k^2) > 0$ for some k , then $\lambda_2(k^2) > 0$ and the optimally controlled Hamiltonian system becomes unstable to spatial perturbations, in the neighborhood of the flat steady state and along the stable manifold. From (120) the quadratic function $h(k^2)$ is concave, and therefore has a maximum. Furthermore, $h(0) = \det J^0 < 0$ and $h'(0) = (2H_{xq}^0 - \rho)$. Then $h(k^2)$ has a maximum for

$$k_{\max}^2 : h'(k_{\max}^2) = 0, \text{ or } k_{\max}^2 = \frac{(2H_{xq}^0 - \rho)}{2D} > 0, \text{ for } (2H_{xq}^0 - \rho) > 0 \quad (122)$$

If $h(k_{\max}^2) > 0$ or $-D^2k_{\max}^4 + D(2H_{xq}^0 - \rho)k_{\max}^2 + \det J^0 > 0$, and $2H_{xq}^0 - \rho > 0$, then there exist two positive roots $k_1^2 < k_2^2$ such that $h(k^2) > 0$ and $\lambda_2(k^2) > 0$ for $k^2 \in (k_1^2, k_2^2)$. Using (122) the existence of two positive roots $k_1^2 < k_2^2$ requires

$$\frac{(2H_{xq}^0 - \rho)^2}{4} + \det J^0 > 0 \quad (123)$$

$$\text{which is equivalent to } \frac{\rho^2}{4} > -H_{xx}^0 H_{qq}^0 \quad (124)$$

$$k_{1,2}^2 = \frac{(2H_{xq}^0 - \rho) \pm \sqrt{(\rho^2 + 4(H_{xx}^0 H_{qq}^0))}}{2D} > 0 \quad (125)$$

The interval (k_1, k_2) determines the range of the unstable modes associated with the spatial heterogeneous solution, while $h(k^2)$ is the dispersion relationship associated with the optimal control problem. Diffusion driven instability in the optimally controlled system emerges if the maximum of the dispersion relationship is in the positive quadrant. These conditions are

summarized below.

$$2H_{xq}^0(x^*, q^*) > \rho \quad (126)$$

$$\frac{\rho^2}{4} > -H_{xx}^0(x^*, q^*) H_{qq}^0(x^*, q^*) \quad (127)$$

■

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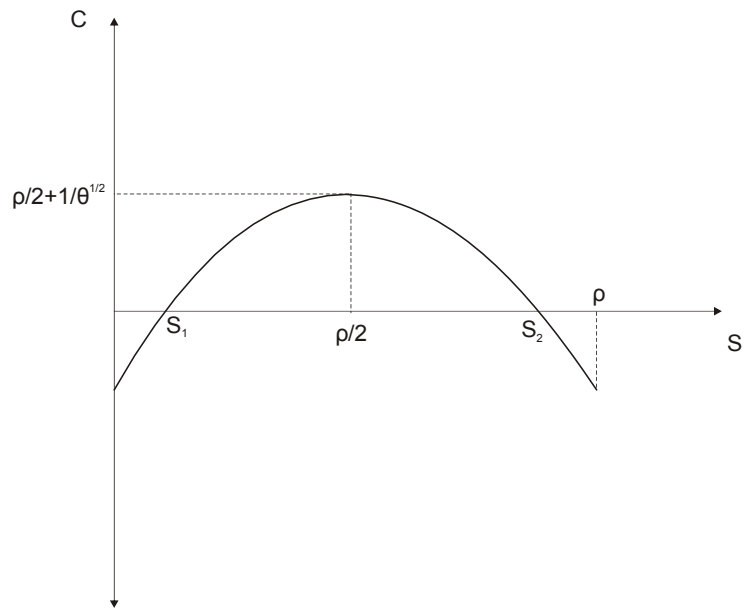


Figure 1: The $C(S)$ curve

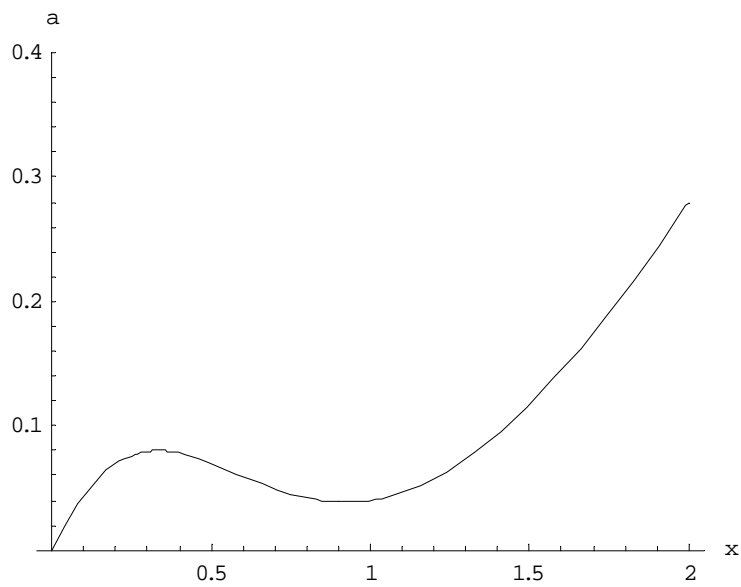


Figure 2: Shallow lake: Reversibility and hysteresis

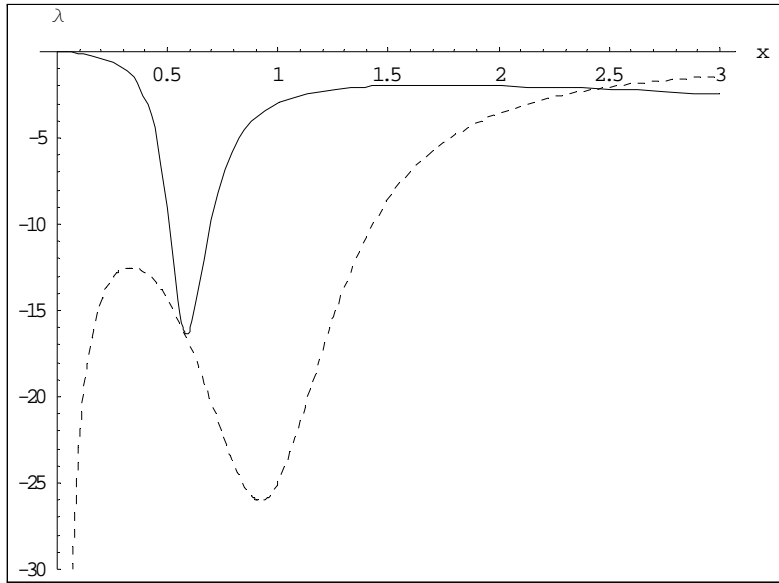


Figure 3: Flat optimal steady states for the shallow lake.

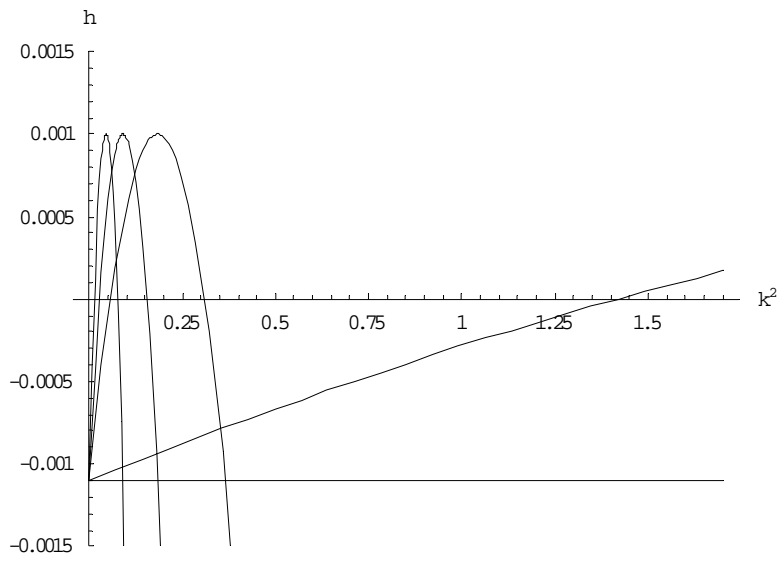


Figure 4: The dispersion relationship

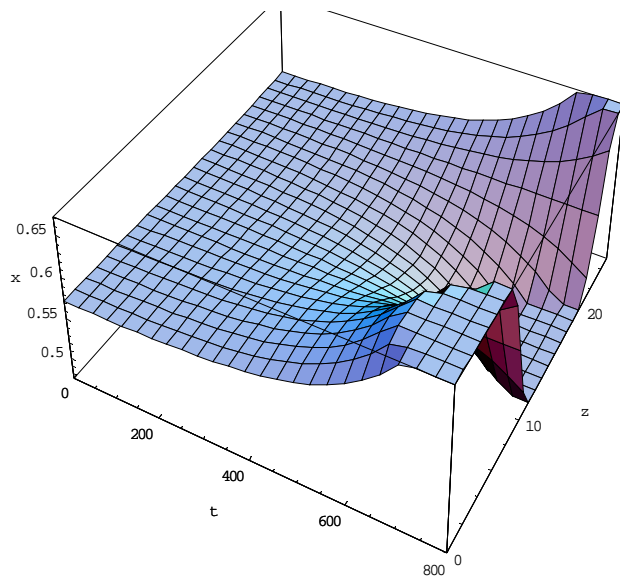


Figure 5: The growing spatial instability

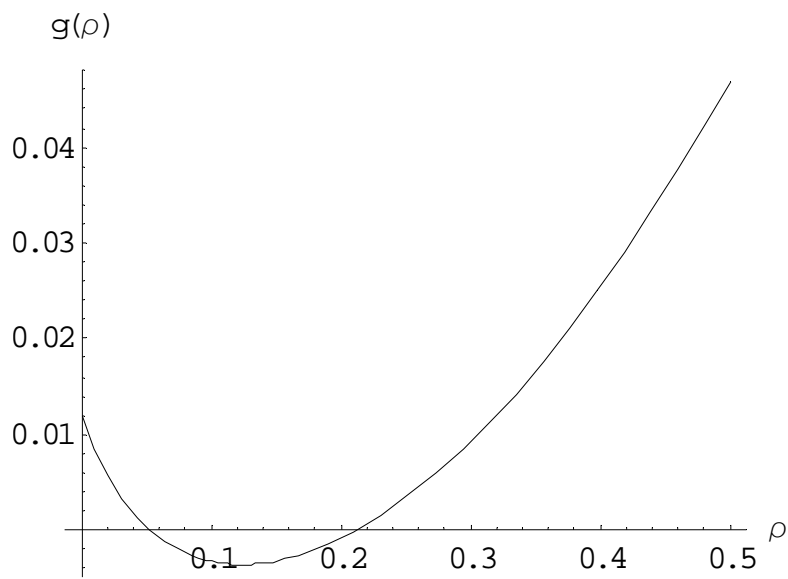


Figure 6: The $g(\rho)$ relationship: Optimal resource management with stock effects

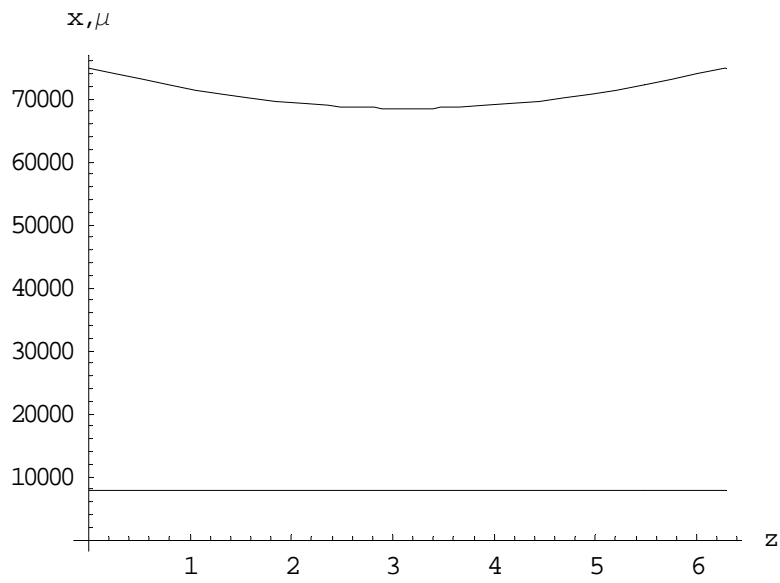


Figure 7: Spatial biomass pattern at the HOSS for optimal resource management with stock effects